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Title: Spectral properties of the Möbius function and a random Möbius model.

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This is a paper that revolves around Sarnak's conjecture stating that the Möbius function $\mu(n)$ of Number Theory, defined as $(-1)^r$ if the positive integer n is the product of r distinct primes and zero otherwise, is orthogonal to any deterministic sequence a_n:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)\, a_n = 0.$$

A sequence a_n is called deterministic if there is a dynamical system (X, T) with zero topological entropy, where X is a compact space, $T : X \to X$ a continuous map, and a continuous function $f : X \to \mathbb{C}$ such that $a_n = f(T^n x)$ for some $x \in X$.

A probabilistic proxy μ_{rand} for μ has been defined by Ng as follows: Let ϵ_n, $n \in \mathbb{N}$, be independent and identically distributed random variables with $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = 1/2$ and set $\mu_{\mathrm{rand}}(n) = \epsilon_n$ if n is the product of distinct primes and zero otherwise. One of the results of the paper under review is that Sarnak's statement holds true, with probability one, for μ_{rand}. More precisely, it is shown that

$$\mathbb{P}\left(\forall f \in C(X)\, \forall x \in X \ \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu_{\mathrm{rand}}(n) f(T^n x) = 0\right) = 1.$$

Another result states that, under the assumption that Sarnak's conjecture holds, and under another technical assumption, the spectral measure of the Möbius function is absolutely continuous with respect to the Lebesgue measure.

Elliott's conjecture states that the spectral measure of the Möbius function is precisely the Lebesgue measure up to a multiplicative constant. Equivalently, this conjecture says that the correlation function of the Möbius function is zero except at the origin:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)\,\mu(n+h) = 0, \quad h \neq 0.$$

As a kind of converse to the above result the authors prove that if Elliott's conjecture holds true then the Möbius function is orthogonal to any sequence generated by a uniquely ergodic dynamical system with singular spectrum.

Lastly, the authors notice that the Cellarosi-Sinai theorem, stating that $\mu(n)^2$ is orthogonal to any sequence generated by a weakly mixing dynamical system, is directly provable by use of a theorem by Mirsky (1949).