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Title: Predicting the ultimate supremum of a stable Lévy process with no negative jumps.

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Review text:

Consider a zero-mean α-stable Lévy process $(X_t)_{0\le t\le T}$ on a finite time interval $[0,T]$ with Lévy measure supported on $(0,\infty)$, and let $S_T := \sup_{0\le t\le T} X_t$. The goal of the paper is the study of stopping times τ, with values in $[0,T]$, such that X_τ is as close as possible to the overall supremum S_T, as measured by an L_p-norm:

$$V := \inf E(S_T - X_\tau)^p,$$

where the infimum is taken over all stopping times τ. Assume $1 < p < \alpha$. The latter inequality ensures that p-th moments of X are finite. This is not a standard optimal stopping problem because it involves the whole path. However, by a series of clever transformations the problem is reduced to standard one. These transformations are based on the following key properties: (i) reflection on the maximum, and (ii) the fact that

$$Y := (S_t - X_t)_{0\le t\le T},$$

the so-called Skorokhod reflection, is a strong Markov process. The facts that X has no negative jumps (spectrally positive Lévy process) and that the law of S_t has been studied by the authors in a recent paper [Bernyk *et al.* 2008], also play an important role in the solution of the problem. Take $T = 1$, without loss of generality (X is self-similar). Property (i) is used to give

$$E[(S_1 - X_t)^p \mid \mathcal{F}_t^X] = (1-t)^{p/\alpha} E\left[\left(\frac{Y_t}{(1-t)^{1/\alpha}} \vee \widetilde{S}_1\right)^p \mid \mathcal{F}_t^X\right]$$

where $\mathcal{F}_t^X$ is the natural history of X and $\widetilde{S}_1$ is an independent copy of S_1, and this shows that V can be obtained as the solution of the optimal stopping problem

$$V = \inf EF(\tau, Y_\tau),$$

for the deterministic function F determined by the previous display. This, being a time-varying optimal stopping problem, is further reduced by the deterministic (strictly increasing) time change

$$[0, \infty) \ni s \mapsto t \in (0, 1]; \quad (1-t)^{p/\alpha} = e^{-s},$$

resulting into

$$V = \inf E\big\{e^{-p\sigma} E\big[(e^\sigma Y_{t(\sigma)} \vee \widetilde{S}_1)^p \mid Y_{t(\sigma)}\big]\big\} \equiv \inf E\big\{e^{-p\sigma} G(e^\sigma Y_{t(\sigma)})\big\},$$

where the infimum is taken over all stopping times σ with values in the new time axis $[0, \infty)$, and where G is a deterministic function completely specified by the law of S_1. Noticing further that the strong Markov process $e^s Y_{t(s)}$ is a member of the family

$$Z_s^z := e^s(z \vee S_{t(s)} - X_{t(s)}), \quad z \geq 0,$$

of (time-homogeneous) strong Markov processes ($Z_s^0 = e^s Y_{t(s)}$), the problem becomes a standard one in the area of optimal stopping (see, for instance, Kyprianou 2006, Chapter 9, for the Lévy case):

$$V(z) = \inf E\big\{e^{-p\sigma} G(Z_\sigma^z)\big\}.$$

The rest of the paper is concerned by solving this difficult problem by solving a fractional free-boundary problem of Riemann-Liouville type. The optimal stopping time is the first entrance of Z into a set D, which translates into an optimal stopping time

$$\tau_* = \inf\{0 \leq t \leq 1 : \; S_t - X_t \geq z_*(T-t)^{1/\alpha}\}$$

for the original problem. It is shown that there is $\alpha_* \in (1, 2)$ and a strictly increasing function $p_* : (\alpha_*, 2) \to (1, 2)$ satisfying $p_*(\alpha_*+) = 1$, $p_*(2-) = 2$ and $p_*(\alpha) < \alpha$ for $\alpha \in (\alpha_*, 2)$ such that, for every $\alpha \in (\alpha_*, 2)$ and $p \in (1, p_*(\alpha))$, τ^* is optimal, and z_* is the unique root to a transcendental equation which depends on α and p. Moreover, if either $\alpha \in (1, \alpha_*)$ or $p \in (p_*(\alpha), \alpha)$ then it is not optimal to stop when $S_t - X_t$ is sufficiently large. The authors remark that this in sharp contrast to the Brownian motion case (formally obtained by setting $\alpha = 2$) and that this is due to how the transition from light to heavy tails is balanced by the parameter p.

References used in the review:

V. Bernyk, R.C. Dalang and G. Peskir (2008). The law of the supremum of a stable Lévy process with no negative jumps. *Annals Probab.* **36**, 1777-1789.

A.E. Kyprianou (2006). *Introductory Lectures on Fluctuations of Lévy processes with Applications.* Springer-Verlag, Berlin.