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Applied probability has a long tradition in studying stochastic models that are useful in applications such as computer performance, networking, engineering, risk analysis, finance, economics, biology, physics, etc. There is always a trade-off between a model's simplicity and usefulness. Everyone knows that the ubiquity of the exponential distribution in applied probability models (such as queueing systems) stems from its memoryless property and is thus useful because it leads to explicit computations, just as the normal distribution goes hand-in-hand with linear systems. This book is partly about the study of generalizations of the exponential distribution that do eventually lead to closed-form formulas or explicit computational algorithms. These are the matrix exponential (ME) distributions and a variety of subclasses of them, notably the phase-type ones.

The title of the book does not quite reflect its wider scope. In my opinion, it should have been "applied probability and ME distributions" rather than the other way around because a large number of topics, familiar to applied probabilists, are developed almost from scratch in this book. Thus, the book is useful not only for those who want to learn about ME distributions but also about renewal theory, random walks, Markov chains, etc, along with a variety of fundamental results such as the law of iterated logarithm and the key renewal theorem. Of course, the emphasis is, indeed, on ME distributions; their theory, in a rather explicit manner, is developed in the book. The book should be accessible to a beginning graduate student and many portions of it to undergraduates as well. The first two chapters consist of purely standard probability/stochastic processes material, including martingales, as is often taught in standard courses. The book will also be useful to those who want to obtain an explicit formula for models of their real-world problems that are harder and yet computationally tractable than the traditional ones. For example, if, say, one wants to study the effect of the delay in a network where the packet sizes are correlated then the book offers inspiration and guidelines on how to do that, if simulation is not desirable or not possible. I believe that anyone working in a variety of "stochastic" fields will find the book useful and will be happy to know that the authors have gone long way in order to derive the often tedious algebraic results.

A phase-type (PH) distribution is the distribution of the time until absorption of a finite-state Markov jump process. For example, the sum of independent exponential random variables has PH distribution, and so does a mixture of them. A PH distribution has support the nonnegative real numbers and density that can be expressed in the form $\pi e^{tS} s$ where S is a substochastic rate matrix, π a probability vector and s is such that s_i is the sum of the i-th row entries of S. Clearly, a PH distribution has rational Laplace transform. A ME distribution is the convex combination of an atom at 0 and a density of the form $\alpha e^{tQ}\beta$ where α, Q, β are $n \times 1$, $n \times n$, $n \times 1$ matrices without further restrictions other than that the function $t \mapsto \alpha e^{tQ}\beta$ be nonnegative and integrable. PH distributions are analyzed in Chapter 3, and ME ones in Chapter 4. Operations preserving the types of distributions are discussed. It is seen that the set of PH measures is dense (in the sense of weak convergence) in the set of all Borel probability measures on $[0, \infty)$. This means that even heavy-tailed measures can be approximated by PH ones, but this may not be satisfying. On the other hand, it is shown that infinite mixtures of PH distributions inherit the heavy-tailedness of the mixing coefficients so if one allows for countable Markov chains then one directly obtains the desirable tail.

PH distributions are studied probabilistically through the underlying Markov chain. This is not directly feasible for ME distributions, so the authors develop the matrix algebra and calculus required in this case. Mixtures and convolutions of ME are ME but so are order statistics of independent random variables with ME distributions. This is developed in detail also. There is also a "flow interpretation" of ME distributions that is seen to be an attempt to provide a more "physical", less "algebraic" feel to them.

The reader who is familiar with linear systems theory will recognize a number of linear systems terms and topics in this book. For example, the problem of minimal dimension realizability is solved. Given a rational Laplace transform, the problem of constructing one representation via the companion matrix is also studied. Even controllability is detectable along the discussion of minimal dimension representation. One wonders, at this point, what is the most general type of dynamical systems with, roughly speaking, the property that if the inputs are ME then so are the outputs. The book provides may classes of such systems and this can be seen to be one of (if not the) main theme of it. For example, a Lindley recursion with ME inputs has ME stationary distribution.

Along with the study of ME and PH distributions, the authors provide a number of pleasing characterizations, both probabilistic and geometric ones. The analogs of ME and PH distributions when the support is $\{0, 1, 2, \dots\}$ instead of $[0, \infty)$ are called matrix geometric (MG) and discrete phase type (DPH), respectively. These are also discussed in the book.

Chapter 5 is a study of renewal theory, along with results specific to the PH or ME case: for example, the renewal measure is seen to be of the same form. Chapter 6 is about random walks. Again, many aspects of the general theory (including the Wiener-Hopf factorization) are developed before the case where random walk increments are ME distributed or, more importantly, are of difference of independent random variables, one of which is ME. In this case, the supremum of the random walk is also ME. Chapter 7 is devoted to a complete treatment of classical results on regenerative processes and Harris chains.

Multivariate ME distributions (MVME), that is, distributions of random vectors $(X_1, \dots, X_d)$ where each X_i is nonnegative and where $\mathbb{E} \exp \sum_{i=1}^{d} \theta_i X_i$ is the ratio of two polynomials in the variables $\theta_1, \dots, \theta_n$. This is done by studying first many subclasses and culminating in the proof of the a result stating that $(X_1, \dots, X_d)$ is MVME if and only if each X_i is ME.

The next two chapters, 9 and 10, are devoted to Markov additive processes and Markovian point processes. They are generalizations of processes with independent increments and point processes, respectively, allowing an underlying Markov chain to change their characteristics. The standard theory for both cases is developed. This includes explaining the power of these processes as stochastic models encompassing every one seen thus far in the book. Special cases include Markov renewal processes, Markov random walks and Markov-modulated fluid models. The effect of assuming that the driving are of ME type is seen in this chapter. Special attention is given to reflected fluid models, as they are natural in several application areas. Regarding point processes, there is a quick review of Palm theory. Emphasis then is given to the Markovian Arrival Process (MAP), followed by a study of the MAP/GI/1 queue. The rational arrival process provides a model of a point process whose interarrival times are ME and dependent.

Chapter 11 gives applications to risk theory by showing how ruin probabilities can be computed in generalizations of the classical Cramér-Lundberg and

Sparre-Andersen models. Here, the growth of the risk reserve process may be nonlinear and the claims can be ME distributed.

The last two chapters, 12 and 13, deal with statistical issues. Chapter 12 develops aspects of likelihood theory, estimation for multinomial distributions, and estimation theory of Markov chains in discrete and continuous time. The expectation-maximization (ME) algorithm is also presented and a practical and a useful section on simulation of conditional PH distributions closes Chapter 12. The last chapter is devoted to estimation of PH distributions in a number of special cases, such as when only absorption times are observable or when there is censoring.

The whole book is a self-contained approach to applied probability models and, perhaps, more than 50% of it is devoted to developing tools, methods, theorems, and approaches that any applied probabilist should be familiar with. The pure probabilist or mathematician may, at some points, detect a little lack of rigor (or occasional typos) but this can easily be remedied by the astute reader. There is, after all, a compromise between presentability and full rigor. At the same time, if one bears in mind that this is a book geared towards practitioners then it does a very good job in explaining some of the finest points of the theory.