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The eigenvalues of certain self-adjoint random matrices can be described as determinantal random point fields (also known as determinantal point processes–see, e.g., Hough *et al.*) through their correlation functions. This paper extends work by Tracy and Widom who introduced differential equations and determinantal processes for the study of eigenvalues of random matrices. Given an integral operator on $L^2(\mathbb{R})$ with kernel $K(x,y)$, Soshnikov showed that $\rho_n(x_1,\ldots,x_n) := \det[K(x_j,x_k)]_{j,k=1}^n$, $n=1,2,\ldots$, are correlation functions of a determinantal point process if $0 \le K \le 1$ and if the restriction of K on each rectangle $[a,b]\times[a,b]$ defines a trace class operator. In other words, there exists a point process ν such that

$$\mathbb{E}\big[(\nu(B_1))_{n_1}\cdots(\nu(B_k))_{n_k}\big] = \int_{B_1^{n_1}\times\cdots\times B_k^{n_k}} \rho_n(x_1,\ldots,x_n)dx_1\cdots dx_n,$$

where $\mathbb{E}$ denotes expectations, $\nu(B)$ denotes the number of random points on the Borel set B, and $(m)_n := m(m-1)(m-2)\cdots(m-n+1)$. In the above, $B_1,\ldots,B_k$ are pairwise disjoint Borel sets and $n = n_1+\cdots+n_k$. Furthermore, the probability generating function of the number of points $\nu(a,b)$ on the (possibly infinite) interval (a,b) is given by

$$\mathbb{E}[z^{\nu(a,b)}] = \det(I+(z-1)P_{(a,b)}KP_{(a,b)}),$$

where $P_{(a,b)}f(x) := f(x)1_{(a,b)}(x)$ is the natural projection operator.

For example, kernels K in random matrix theory [see Mehta] are typically of the form

$$K(x, y) = \frac{f(x)g(y) - f(y)g(x)}{x - y},$$

with the 2-tuple $(f(x), g(x))$ satisfying a system of linear differential equations with x-dependent (polynomial) coefficients; e,g. $K(x-y) = (x-y)^{-1}\sin(x-y)$ for the Gaussian unitary ensemble (GUE).

In this paper, the kernel K defining a determinantal point process ν on $(0, \infty)$ is constructed by means of, possibly infinite-dimensional, linear systems, as follows. Let H (the state space) and H_0 be (complex) separable Hilbert spaces; often, $H_0 = \mathbb{C}$. Consider a linear system whose trajectory X evolves in H, takes inputs from H_0 and gives outputs in H_0:

$$\frac{dX}{dt} = -AX + BU, \quad Y = CX, \quad t > 0, \quad X(0) = 0.$$

One says that the system is defined by the triple $(-A, B, C)$. It it is assumed that certain natural conditions hold, e.g., that the semigroup $(e^{-tA}, t \geq 0)$ is bounded C_0 on H, that the range of B is contained in the domain of $-A$, and that C is defined on this domain. The shifted system at time $x > 0$ is defined by the triple $(-A, e^{-xA}B, Ce^{-xA})$. Let $\phi(t) := Ce^{-tA}B$ be the input-output operator for $(-A, B, C)$ and $\phi_{(x)}(t)$ the input-output operator for the shifted system. Let the Hankel operator Γ_ϕ be defined by $\Gamma_\phi f(t) := \int_0^\infty \phi(t+s)f(s)ds$, where f belongs to some dense linear subspace of $L^2(\mathbb{R}_+, H)$ or $L^2(\mathbb{R}_+, H_0)$. The corresponding operator for the shifted system is $\Gamma_{\phi_{(x)}}$. Introduce the controllability and observability Gramians by $L_0 := \int_0^\infty e^{-tA}BB^\dagger e^{-tA^\dagger}dt$, $Q_0 = \int_t^\infty e^{-tA^\dagger}C^\dagger Ce^{-tA}dt$, respectively, for the original system, and let L_x, Q_x be the corresponding Gramians for the shifted system. Assume that they are of trace class and have operator norms strictly less than one.

The main theorem of the paper consists of three parts:

(i) In the self-dual case, i.e. when $A = A^\dagger$, $C = B^\dagger$, $\phi(t) = B^\dagger e^{-At}B$, there is a determinantal point process ν such that $g_x(z) := \mathbb{E}[z^{\nu(x,\infty)}] = \det(I + (z-1)\Gamma_{\phi_{(x)}})$, $x > 0$.

(ii) In the general case, there is a determinantal point process ν such that $g_x(z) := \mathbb{E}[z^{\nu(x,\infty)}] = \det(I + (z-1)\Gamma_{\phi_{(x)}}\Gamma^\dagger_{\phi_{(x)}})$.

(iii) In the real case, there is a determinantal point process ν such that $g_x(z) := \mathbb{E}[z^{\nu(x,\infty)}] = \det(I + (z-1)\Gamma^2_{\phi_{(x)}})$.

Furthermore, in each case, it is shown that $\frac{d}{dx}\log g_x(z)$ is related to the solution of a Gelfand-Levitan integral equation. E.g., in case (i), $\frac{d}{dx}\log g_x(z) = T_{z-1}(x, x)$, where the kernel $T_\lambda(x, y)$ satisfies

$$T_\lambda(x, y) + \lambda\phi(x+y) + \lambda\int_x^\infty T_\lambda(x, u)\phi(u+y)du = 0, \quad 0 < x \leq y, |\lambda| < 1.$$

Similar equations hold in cases (ii) and (iii) also. The theorem is stated in the Introduction and proved in Sections 5 (cases (i) and (iii)) and 6 (case (ii)).

Section 3 is concerned with showing how some interesting kernels K (e.g., spatial kernels associated to Hamiltonian systems) factorize as $\Gamma\Gamma^\dagger$ which allows the construction of determinantal point processes with these kernels. An application to the eigenvalue problem $-\psi'' + q\psi = \lambda\psi$, associated with Schrödinger's equation with smooth compactly-supported potential q, is considered in Section 4. Use is made of McKean's results on the spectrum of the Schrödinger operator on $L^2(\mathbb{R})$ which define the scattering map $q \mapsto \phi$. The inverse scattering problem (c.f. Zakharov and Shabat) considered here seeks a linear system compatible with the scattering data; from it, a Gelfand-Levitan kernel is obtained, via which q is recovered. Finally, Section 7 considers the case when q evolves under the Korteweg-de Vries flow $4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x}$ and examines how the determinantal point process evolves under the corresponding flow in the scattering data. It is also noted that when $q(x) = -2\,\mathrm{sech}^2\,x$ then the probability distribution function of the largest point of the corresponding determinantal point process is logistic.

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