This is a review text file submitted electronically to MR.

Reviewer: Konstantopoulos, Takis

Reviewer number: 68397

Address:
Department of Mathematics
Uppsala University
PO Box 480
SE-75106 Uppsala
SWEDEN
takis@math.uu.se,takiskonst@gmail.com

Author: Cohen, Guy; Lin, Michael

Short title: Almost sure convergence of weighted sums of independent random variables.

MR Number: 2553208

Primary classification: 60F15

Secondary classification(s): 60G50,60G42,62J05,37A05

Review text:

The main type of question dealt with in this paper is the following: Under what conditions on a sequence of (real or complex) weights (b_n) do we have almost sure convergence of the random sequence $\sum_{n=1}^{\infty} X_n/b_n$ *for all* centered integrable i.i.d. sequences (X_n) belonging to a certain integrability class. Compare this, for instance, with a classical result by Marcinkiewicz and Zygmund (1937) who show that if $E|X_1|\log^+|X_1| < \infty$ or if X is symmetric then $\sum_{n=1}^{\infty} X_n/n$ converges a.s. (and this, by Kronecker's lemma, is a strengthening of Kolmogorov's SLLN). However, this is a condition for a specific random sequence. On the other hand, Jamison, Orey and Pruitt (1965) showed that, given positive weights w_n with $\sum_1^{\infty} w_k = \infty$, the random sequence $\sum_{k=1}^{n} w_k X_k / \sum_{k=1}^{n} w_k$ converges to zero almost surely for all centered integrable i.i.d. sequences (X_n) provided that $\limsup_{t\to\infty} N(t)/t < \infty$ (and this condition is tight) where $N(\cdot)$ is the generalized inverse of the function $n \mapsto \sum_{k=1}^{n} w_k/w_n$. The method (i.e. the consideration of the function N) introduced by Jamison *et al.* is the main tool in the paper under review. The first main theorem states that, given weights b_n, and letting $N(\cdot)$ be the generalized inverse of $n \mapsto |b_n|$, then the condition $\limsup_{t\to\infty} N(t)/t^p < \infty$, for some $p \in [1, 2)$, implies that $\sum_{k=1}^{\infty} X_k/b_k$ converges a.s., if (i) $1 < p < 2$ and $E|X_1|^p < \infty$, or if (ii) $p = 1$ and $E|X_1|\log^+|X_1| < \infty$, or if (iii) $p = 1$ and X_1 has a symmetric distribution. This generalizes the Marcinkiewicz-Zygmund result. Several interesting corollaries and related facts are obtained, for instance, a generalization of the Jamison-Orey-Pruitt result for the symmetric case. Two central sections in the paper are concerned with extensions from i.i.d. sequences to sequences which are martingale differences and are in L_p or are uniformly bounded. Martingale differences were considered by Azuma (1967) who showed that the condition $W_n^{-1} w_n \log\log W_n \to 0$ (here $W_n = \sum_1^n w_k$) implies that $W_n^{-1} \sum_{k=1}^n w_k X_k \to 0$, as long as the X_n are uniformly bounded martingale differences. Azuma's result generalized the i.i.d. case considered by Tsuchikura (1951). The paper under review generalizes this condition a little and gives an alternative proof of Azuma's result. The paper also considers cases where the random variables are multiplied by "random weights" (typical realizations of dynamical systems–in the language of the authors). The last section considers an application to the strong consistency in a linear regression with i.i.d. noise.

Bibliography used in this review:

K. Azuma (1967) Weighted sums of certain dependent random variables. *Tôhoku Math. J.* **19** (3), 357-367.

B. Jamison, S. Orey and W. Pruitt (1965), Convergence of weighted averages of independent random variables. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **4**, 40-44.

J. Marcinkiewicz and A. Zygmund (1937), Remarque sur la loi du logarithme itéré. *Fund. Math.* **29**, 215-222.

T. Tsuchikura (1951), Notes on Fourier analysis XL; Remark on the Rademacher system. *Proc. Japan Acad.* **27**, 141-145.