This is a review text file submitted electronically to MR.

Reviewer: Konstantopoulos, Takis

Reviewer number: 68397

Address:

Department of Mathematics
Uppsala University
PO Box 480
SE-75106 Uppsala
SWEDEN
takiskonst@gmail.com,T.Konstantopoulos@hw.ac.uk

Author: Cuny, Christophe; Lin, Michael

Short title: Pointwise ergodic theorems with rate and application to the CLT for Markov chains.

MR Number: 2548500

Primary classification: 60F05

Secondary classification(s): 60J05,37A30,37A05,47A35,37A50

Review text:

amsfonts

This paper deals with conditions for the validity of the central limit theorem for additive functionals $S_n(f) := \sum_{k=1}^n f(X_k)$ of a stationary Markov chain $(X_n, n = 0, 1, \dots)$ in general state space, without "classical" conditions such as regeneration, strong mixing, or existence of solution to Poisson's equation. Rather, conditions on the growth of the expectation of $S_n(f)$ are the starting point.

The main result is as follows. Let f be a square-integrable (with respect to the marginal law m of the chain) real-valued function on the state space, such that $\mathbb{E}_m f(X_0) = 0$. Let P be the transition operator for the chain (so m is invariant for P). Suppose there exists $\tau > 1$ such that

$$\sup_{n\ge 3} \frac{(\log n)^{5/2}(\log\log n)^\tau}{\sqrt{n}} \left\| \sum_{k=1}^n P^k f \right\|_2 < \infty.$$

Then $\lim_{n\to\infty} \mathbb{E}_m S_n(f)^2/n =: \sigma(f)^2$ exists and is finite, and the $\mathbb{P}_x$–law of $S_n(f)/\sqrt{n}$ converges (weakly) to the normal distribution $\mathcal{N}(0, \sigma(f)^2)$ for m-a.e. x.

The paper generalizes previous work by Derrienic and Lin (2003) and Wu and Woodroofe (2004). The above CLT is referred to as "quenched CLT" which easily implies the "annealed" version, i.e. that the same weak limit holds under the measure $\mathbb{P}_m$ (which makes the Markov chain stationary). This kind of CLT was apparently considered first in a series of papers by Gordin and Lifshitz

(1978, 1981, 1995) and then generalized by Maxwell and Woodroofe (2000).

The authors compare $S_n(f)/\sqrt{n}$ to a properly constructed martingale for which the CLT can be obtained and then show that the difference converges to zero. To do this, they prove a rate of convergence result to the Dunford-Schwartz theorem of functional analysis (stating that if T is an operator mapping some $L^1(\Omega, \mathcal{F}, \mu)$ to itself such that $\|T\|_p \leq 1$ for all $1 \leq p \leq \infty$, then, for all f in L^1, then $(1/n)\sum_{k=1}^n T^k f$ converges μ-a.e.) The result proved in this paper states that if $p > 1$ and $f \in L^p(\Omega, \mathcal{F}, \mu)$, then $(1/n^p)\sum_{k=1}^n T^k f$ converges to zero μ-a.e., under a certain growth condition for $\|\sum_{k=1}^n T^k f\|_p$. It is then applied in the Markov chain case by taking Ω to be the path space of the chain, $\mu := \mathbb{P}_m$, and $Tf(\omega) := f(\vartheta\omega)$, ϑ be the one-step shift on the path space.

Bibliography used in this review:

Y. Derriennic and M. Lin. The central limit theorem for Markov chains started at a point. Probab. Theory Related Fields 125 (2003) 73–76. MR1952457 MR1952457 (2003m:60185)

M. Gordin and B. Lifshitz. A central limit theorem for Markov process. Soviet Math. Doklady 19 (1978) 392–394. MR0501277

M. Gordin and B. Lifshitz. A remark about a Markov process with normal transition operator. Proc. Third Vilnius Conf. Probab. Statist. 147–148. Akad. Nauk Litovsk., Vilnius, 1981 (in Russian).

M. Gordin and B. Lifshitz. The central limit theorem for Markov processes with normal transition operator, and a strong form of the central limit theorem. In Limit Theorems for Functionals of Random Walks Sections IV.7 and IV.8. A. Borodin and I. Ibragimov (Eds). Proc. Steklov Inst. Math. 195, 1994. (English translation Amer. Math. Soc., Providence, RI, 1995.)

W. B. Wu and M. Woodroofe. Martingale approximations for sums of stationary processes. Ann. Probab. 32 (2004) 1674–1690. MR2060314 MR2060314 (2005c:60037)