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**Review text:**

Consider a random element of $SO(n)$ defined by mapping some $v \in \mathbb{R}^n$ to another $v' \in \mathbb{R}^n$ obtained as follows: First pick two distinct indices $i, j$ from $\{1, \dots, n\}$ at random. Then, independently, pick an angle $0 \le \theta \le 2\pi$ uniformly at random, and let $(v_i', v_j')$ be the rotation of $(v_i, v_j)$ by an angle $\theta$, but leave every other coordinate intact.

Letting $R(0), R(1), \dots$ be i.i.d. copies of this random element. Kac's random walk is defined by $v(t) = R(t-1)\cdots R(0)v(0)$, $t \ge 1$, with $v(0)$ independent of $R(0), R(1), \dots$; it evolves in the sphere with radius the Euclidean norm of $v(0)$. This paper studies the Markov process in $SO(n)$ defined by $X(t) = R(t-1)\cdots R(0)X(0)$, where $X(0)$ is a given random element of $SO(n)$, independent of $(R(0), R(1), \dots)$ with law, say, $\mu$. Let $K_x(\cdot) := \mathbb{P}(R(0)x \in \cdot)$. Then the law of $X(t)$ is given by $\mu K^t(\cdot) = \int_{SO(n)} \mu(dx) K_x^t(\cdot)$.

To state the results of the paper we need the notion of the Wasserstein distance $W_{d,p}$, or transportation cost, between probability measures $\mu, \nu$ on a metric space $(M, d)$: given $p \ge 1$, define

$$W_{d,p}(\mu, \nu) = \inf(\mathbb{E}d(X,Y)^p)^{1/p},$$

where $X, Y$ are random elements of $M$, defined on a common probability space and having distributions $\mu, \nu$, respectively, and where the infimum is taken over all such pairs.

Let $D$ be the Riemannian metric on $SO(n)$, i.e. $D(a, b)$ is defined as the length of the shortest path from $a$ to $b$. Let $\mathcal{H}$ be the Haar measure on $SO(n)$. The first result of the paper states that, given $\varepsilon > 0$, the smallest $t$ such that

$W_{D,2}(\mu K^t, \mathcal{H}) \le \varepsilon$, for all probability measures $\mu$, is at most $\lceil n^2 \log(\pi\sqrt{n}\varepsilon^{-1})\rceil$. This improves previously known bounds.

On the other hand, for $a, b \in SO(n)$, let

$$\mathrm{hs}(a,b) := \sqrt{\mathrm{trace}((a-b)^\dagger(a-b))}$$

be the Hilbert-Schmidt distance between $a$ and $b$. The second result of the paper asserts the existence of positive constants $\varepsilon_0$ and $c$ such that the smallest $t$ satisfying $W_{\mathrm{hs},1}(\mu K^t, \mathcal{H}) \le \varepsilon_0$, for all $\mu$, is at least $cn^2$.

The proof of the upper bound is based on a theorem (proved in the paper) stating the following: if $P_x(\cdot)$ is a transition probability kernel on a Polish space $(M,d)$ which (i) has finite $p$th moments for all $x$ (i.e. $\int_M d(a,y)^p P_x(dy) < \infty$ for all $a$ and $x$) and (ii) is locally $C$-Lipschitz (as a map from $(M,d)$ into the space of probability measures on $M$ with the $W_{d,p}$–Wasserstein metric) then

$$W_{d,p}(\mu P, \nu P) \le C W_{d,p}(\mu, \nu),$$

for all probability measures $\mu, \nu$ on $M$.

The case where $M$ is a compact space with finite diameter $\mathrm{diam}_d(M)$ is important for the paper. In this case, if $x \mapsto P_x$ is locally Lipschitz with $C = 1 - \kappa < 1$ then the last inequality and the fixed point theorem tells us that there is a unique probability measure $\mu_*$ such that $\mu_* P = \mu_*$ and, moreover, $W_{d,p}(\mu P^t, \mu_*) \le e^{-\kappa t}\mathrm{diam}_d(M)$, which implies that the least $t$ such that $W_{d,p}(\mu P^t, \mu_*) \le \varepsilon$ is at most $\kappa^{-1}\log(\varepsilon^{-1}\mathrm{diam}_d(M))$.

The upper bound then is established by showing (i) that $x \mapsto K_x$ (as a mapping from $(SO(n), D)$ into the space of probability measures on $SO(n)$ with the $W_{D,2}$ metric) is locally $C$-Lipschitz with $C = (1 - \frac{1}{2}n(n-1))^{1/2}$, and (ii) that $\mathrm{diam}_D(SO(n)) \le \pi\sqrt{n}$. To show (i) a particular coupling of $K_x$ and $K_y$ is devised. To show (ii) the length of a particular curve connecting two elements of $SO(n)$ is computed.

As for the lower bound, first an inequality involving packing and covering numbers is shown and then these quantities are estimated for $SO(n)$.

The paper also discusses related random walks (Kac's walk with non-uniform angles and a walk on unitary matrices) and concludes by stating a number of conjectures and open problems.