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Reviewer Name: Konstantopoulos, Takis

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Address:

Department of Mathematics
Uppsala University
PO Box 480
SE-75106 Uppsala
SWEDEN
takiskonst@gmail.com

Author: Landim, C.

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Motivated by work on limits of continuous-time Markov chains, such as zero-range processes, random walks among random traps and polymer models, the author proposes a new topology that facilitates weak convergence of processes (functionals of the chains) that have very fast transitions between certain states, so fast that, in the limit, instantaneous states appear. This typically happens in processes exhibiting metastability. The usual (J_1) Skorokhod topology on the space of $D([0,T],\mathbb{N})$ (integer-valued functions which are right-continuous and have left limits) is too strong to capture this phenomenon. This topology is not suitable for proving weak convergence of processes with paths in $D([0,T],\mathbb{N})$. Instead, a weaker topology is introduced on a suitably constructed set of functions $E([0,T],\overline{\mathbb{N}})$ from $[0,T]$ into $\overline{\mathbb{N}} := \mathbb{N} \cup \{\mathfrak{d}\}$, via the notions of soft limits and soft continuity. Here, $\mathfrak{d} := \infty$ plays the role of the point that achieves one-point compactification of $\mathbb{N}$.

The idea is the notion of soft limits. For example, say that a function $x : [0,T] \to \overline{\mathbb{N}}$ has a soft left-limit at t if either it has a left limit or if the set of limit points from the left contains two points, a point in $\mathbb{N}$ and the point $\mathfrak{d}$. Similarly, one defines the notion of soft right-continuity. The set of all functions which are soft right-continuous and have soft left-limits is denoted by $\mathbb{E}([0,T],\overline{\mathbb{N}})$. These are basically the paths of Markov chains possessing instantaneous states. Then operators $\mathfrak{R}_m$, $m \in \mathbb{N}$, and $\mathfrak{R}_\infty$ are introduced. Roughly speaking, for any $x \in \mathbb{E}([0,T],\overline{\mathbb{N}})$, $\mathfrak{R}_m x(t)$ is the last state in $\{1,\dots,m\}$ visited by x before time t. Similarly, for $\mathfrak{R}_\infty$. There is a subtle difference, however. Whereas $\mathfrak{R}_m$

takes a path $x \in \mathbb{E}([0,T],\overline{\mathbb{N}})$ and gives an "ordinary" path $\mathfrak{R}_m x \in D([0,T],\mathbb{N})$, this is not the case with $\mathfrak{R}_\infty$. By carefully examining the range of $\mathfrak{R}_\infty$, a set $E([0,T],\overline{\mathbb{N}}) \subset \mathbb{E}([0,T],\overline{\mathbb{N}})$ is constructed. This set turns out to be the right one over which the soft topology can be introduced.

The soft topology on $E([0,T],\overline{\mathbb{N}})$ is introduced via a metric. Let d_S be a metric for the usual Skorokhod topology. Then for any $x, y \in \mathbb{E}([0,T],\overline{\mathbb{N}})$ we can define the symmetric function

$$\mathbf{d}(x,y) := \sum_{m=1}^{\infty} 2^{-m}\, d_S(\mathfrak{R}_m x, \mathfrak{R}_m y).$$

This fails to be a metric on $\mathbb{E}([0,T],\overline{\mathbb{N}})$ because there are examples of different paths x, y such that $\mathfrak{R}_m x = \mathfrak{R}_m y$. But it is a metric on the smaller space $E([0,T],\overline{\mathbb{N}})$. The space, equipped with this metric, enjoys several pleasant properties. For example, it is a Polish space and hence the usual theory of weak convergence goes through.

Two main theorems are proved. The first is a characterization of what it means for a sequence P_n of probability measures to be converge weakly, in the soft topology, to a probability measure P. It means that, for all $m \in \mathbb{N}$, the sequence $P_n \circ \mathfrak{R}_m^{-1}$ converges weakly, in the Skorokhod topology, to $P \circ \mathfrak{R}_m^{-1}$. This is a necessary and sufficient condition. The second theorem presents sufficient conditions for weak convergence in the soft topology.

Two main examples are presented in the paper. The first is that of a zero-range process and the second that of a random walk among traps. In both cases, there are natural processes that are proven to converge weakly in the soft topology by means of applying the methods developed in this paper.