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Title: Some sufficient conditions for the ergodicity of the Lévy transformation.

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There has been a long-standing question regarding the ergodicity of the Lévy transformation $\mathbf{T}$ on $C[0,\infty)$,

$$(\mathbf{T}\beta)_t := \int_0^t \operatorname{sign}(\beta_s)\, d\beta_s, \quad t \geq 0,$$

which preserves the Wiener measure. Although the question is not answered in this paper, conditions are given. (Prior work on this, has been done by M. Malric.) Let $\beta^{(n)} := \mathbf{T}^n\beta$ where β is a Brownian motion (that is, β is the identity map on $C[0,\infty)$ when the latter is equipped with the Wiener measure.) Let $Z_n := \min_{0\leq k<n} |\beta_1^{(k)}|$. One result states that if $\liminf_{n\to\infty} Z_{n+1}/Z_n < 1$ almost surely, then $\mathbf{T}$ is strongly mixing and hence ergodic. Another result states that if $\nu(x) := \inf\{n \geq 0 : |\beta_1^{(n)}| < x\}$ is such that $\{x\nu(x) : 0 < x < 1\}$ is a tight family of random variables then $\mathbf{T}$ is strongly mixing.

These types of results are obtained by generalizing the transformation to another transformation T on $C^d[0,\infty)$, the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^d$, defined by

$$(T\beta)_t := \int_0^t h(s,\beta)\, d\beta_s,$$

with $h(s,\beta)$ being an orthogonal matrix-valued process with suitable measurability properties. It is shown that T is strongly mixing if and only if $\int_0^t h(s,T^{n-1}\beta)\cdots h(s,T\beta)h(s,\beta)\,ds$ converges to 0 in probability, with β being a standard d-dimensional Brownian motion. This then means that the original transformation $\mathbf{T}$ is strongly mixing if

$E(h_s^{(n)} h_1^{(n)}) \to 0$ for almost every $0 < s < 1$, where $h_s^{(n)} := \text{sign}(\beta_s^{(n-1)}) \cdots \text{sign}(\beta_s)$. To check this condition, the author devices a coupling-type of argument showing that $\limsup_{n\to\infty} E(h_s^{(n)} h_1^{(n)}) \leq P(\sup_{0\leq t\leq 1} |\beta_t| < C)$, for all $C > 0$, if certain conditions (conditions (a), (b) and (c) of Prop. 2) are satisfied. These conditions are then transformed into fine properties of a certain subset (basically, the set defined by these conditions) of the real line. The existence of the fine properties this set is then transformed into the aforementioned conditions involving Z_n and $\nu(x)$.

The paper gives new ideas on how to proceed to in establishing ergodicity of the Lévy transformation which has been one of the favorite questions of the late Marc Yor (who is credited for some of the inspirations in this paper). Its organization (including grammatical mistakes) could have been improved, but, admittedly, this is a subtle paper and probably hard to write efficiently. Proving the stated sufficient conditions could, possibly, be a way to establish the ergodicity question.