This is a review submitted to Mathematical Reviews/MathSciNet.

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At first sight, this book (or, rather, monograph) may appear to be about Lévy processes and various applications of them in mathematical physics. However, only the first two chapters treat some aspects of Lévy processes. The remaining 9 chapters are all devoted to different topics on problems in approximation of functions, prediction and filtering of stationary stochastic processes, factorization of operators, the zeros of polynomials and analytic functions, relations between quantum and classical approaches in physics, the Boltzmann equation, and some others.

It is hard to say *what* this book is about. I would rather call it a collection of the works of the author. No doubt, there is a lot of interest here for the specialist, but if the reader is looking for a textbook or for a connection between the topics and methods, then this he or she will not find in the book, at least not without further research of his or her own. The book presumes that the reader is a seasoned researcher with knowledge of various aspects of mathematics, applied mathematics, theoretical physics and (some) probability. Actually, probability is not the main focus of the book, and probabilistic tools are almost entirely absent.

For the reviewer, the problem is what story to tell in order to convey the feeling of the book's topic. This is an almost impossible task, therefore I will try to convey some information about the contents of the almost disjoint parts of the book. I should, however, add that large part of the author's research has been motivated by the work of Marc Kac in mathematical physics and probability.

The first two chapters deal with Lévy processes in one dimension. The author's aim is to tell something about the distribution $p(t,\Delta) = P(T_\Delta > t)$ of the first exit time T_Δ of the process from a set Δ (e.g., an interval). The basic observation is that for a large class of Lévy processes, the infinitesimal generator L can be written in the form $L = DSD$, where D is the derivative operator and S is a convolution-type operator. Stable processes fall in this class. The analysis requires use of a potential operator B for L restricted on Δ, which is called "quasi-potential". Under certain conditions on B, the exit problem is solved and is amply exemplified for the case of stable processes (including Brownian motion). In particular, the asymptotic behavior is shown to be of the form $p(t,\Delta) = e^{-t/\lambda_1}[q(t) + o(1)]$, as $t \to \infty$, where λ_1 is the largest eigenvalue of B. The second chapter is devoted to the eigenvalues of B for the case of α-stable processes, and it is shown that the n-th largest eigenvalue drops like $n^{-\alpha}$, as $n \to \infty$. It is also shown that the density of the measure $P(\sup_{0\le s\le t}|X_s| \le a, X_t \in dy|X_0) = x)$ is asymptotically, as $t \to \infty$, the density of $P(X_t \in dy|X_0 = x)$ (with a rate of convergence $t^{-1/\alpha}$ at $y = x$).

The third chapter deals with the problem of approximating continuous positive functions f on an interval (or periodic functions) satisfying the condition that second differences $|f(x+h) - f(x-h) - 2f(x)|$ are bounded above by $2|h|^\alpha$, for some $0 < \alpha < 2$, by $L_n f$ where L_n are "Korovkin operators". The problem of finding the quality of approximation is addressed and is shown to be related to the quasipotential operator B.

Chapter 4 studies the prediction and matched filtering problem for generalized stationary stochastic processes, i.e., processes which are stochastic analogs of Schwartz distributions. In other words, processes indexed by suitable test functions. The problems are solved for the case where the correlation functional (a bilinear map on test functions) is given in terms of a bounded positive operator S_J on $L^2(a,b)$ of convolutional type. Example include white and power-law noises.

Chapter 5 considers the problem of factorization of a positive operator in a Hilbert space as the product of two lower triangular operators. In finite dimensions, a non-singular positive definite matrix always admits such a factorization. But this is not the case in infinite dimensions, a result due to Larson (1985). This chapter deals with conditions and concrete examples of non-factorizable operators.

The mood of the book changes again in Chapter 6, where comparisons between a classical mechanical system and its quantum version are considered. Wigner and Kirkwood showed how the quantum partition function approaches the classical one when Planck's constant h goes to zero. The problem the author addresses is that of non-asymptotic comparisons between the partition functions and between the energy expectations in the quantum and classical case. In particular, the author asks when it is the case that the classical mean energy is bounded above by the quantum mean energy, for fixed h and all temperatures. The answer is positive for the one-dimensional potential well (potential infinite outside a compact interval), for the harmonic oscillator (quadratic potential), and, under some conditions, and in some different sense, for general potential functions.

Chapter 7 generalizes the Kac-Krein notion of dual string equations for some classes of canonical continuous and discrete systems and illustrates this by means of concrete examples. The string equation is the differential equation $-D^2\phi = \lambda\rho(x)^2\phi$, for given function ρ, on an interval $0 \leq x \leq \ell$ with given boundary conditions. The "dual" of that refers to a change of variables from x to M where $dM/dx = \rho(x)^2$. The idea is generalized in this chapter to higher-dimensional differential equations of special form (matrix string equations).

In Chapter 8, operators of the form $Sf(x) = L(x)f(x) + \text{P.V.} \int_a^b \frac{D(x,t)}{x-t} f(t)dt$ (where $L(x)$ and $D(x,t)$ are matrices) are considered. It is assumed that inverse S^{-1} is of the same form. The solution to an associated canonical differential system and its connection to a Riemann-Hilbert problem is studied.

Chapter 9 goes back to a problem for which, stated in the language of physics, there are fixed energy levels $E_1 < E_2 < \cdots$ and un-normalized probabilities $p_1, p_2, \ldots$. The author defines a "compromise function" $F(\lambda, p_1, p_2, \ldots) = \lambda E + S$, where E is the mean energy, S the entropy, and λ a negative number and shows that the maximum of F is achieved when $p_n = e^{-\lambda E_n}$. The problem is repeated in the classical case, and then generalized for a different notion of entropy called Tsallis entropy.

Chapter 11 takes a look of the "compromise function" associated with the inhomogeneous Boltzmann equation. The author shows that the extremum occurs at the "Maxwellian function" and also studies the deviation the compromise function from its maximum. In the spatially homogeneous case, the deviation coincides with the Kullback-Leibler distance and then it is studied in the inhomogeneous case.

The last chapter considers an extension of the concept of Bezoutiant [sic] which is used to define the number of common zeros of two polynomials. The author asks the same question for analytic functions instead of polynomials and, through an operator version of this concept, he studies common zeros between two entire functions. A generalization of the Schur-Cohn theorem is also given.

The book is not an easy read because of the diversity of the topics involved, but also because many quantities are left undefined (good guesses are of course possible). In addition, there is a significant number of typos and mathematically imprecise symbols. E.g., $\|f(x)\|$ on pg. 4 should have been f, interval $[0, -\infty)$ on pg. 13 should have been $[0, +\infty)$; the function μ should be increasing on $\mathbb{R}$ without 0 (on the same page); the symbol $D(y)$ on pg. 15 is not defined; the symbol Δ on pg. 20 defined as "the set of segments $[a_k, b_k]$ such that..." should have been defined as "the union of intervals...", and the case $a_1 = -\infty$ should have been included; there is a spurious d in eq. (17.5) on pg. 27; there is an unfinished fraction on the last line of pg. 72; "power-low" on pg. 79 and elsewhere should have been "power-law"; the second appearance of "the" on the first line of pg. 90 is not needed; the function $Z_c(\beta, h)$ on pg. 101 is also is written as $Z_c(h, \beta)$; the word "asymptotic" at the end of pg. 105 is misspelled; the symbol S^{n-1} on pg. 182 is not defined; on pg. 201, and elsewhere, the word "Bezoutiant" should probably have been "Bezoutian"; the first two words of the title of Chapter 11 should have been in reverse order; there should have been an article "The" starting Corollary 11.7 on pg. 203, etc. Many of these things are the responsibility of the publisher (Birkhäuser) who, apparently, did nothing (no copy-editing, no typesetting) other than put the manuscript between two hard covers, which is a real shame.

There is a lot of material covered, at various levels of rigor and generality, and specialists will find interesting ideas. However, the connections, as mentioned in the title of the book, are not fully developed.