This is a review submitted to Mathematical Reviews/MathSciNet.

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Mathematical Reviews/MathSciNet Reviewer Number: 68397

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Title: Noise as a Boolean algebra of σ-fields.

MR Number: MR3161487

Primary classification:

Secondary classification(s):

Review text:

This article provides a theory of noise as a Boolean algebra of σ-fields and answers, in the positive, a nontrivial question by J. Feldman. Given a probability space $(\Omega, \mathcal{F}, P)$, the author considers Boolean algebras, i.e., collections of sub-σ-fields B of $\mathcal{F}$ which form a lattice (under the operations $x \vee y = \sigma(x, y)$ and $x \wedge y := x \cap y$, $x, y \in B$) with maximum element $1_\Lambda = \mathcal{F}$ and minimum element 0_Λ the σ-field generated by P-null sets of $\mathcal{F}$. It is assumed that $\vee$ distributes over $\wedge$ and that every element $x \in B$ is complemented by an $x' \in B$ in the sense that $x \vee x' = \mathcal{F}$ and $x \wedge x' = 0_\Lambda$. The objects of study are *noise-type* Boolean algebras B, i.e., those for which x, x' are independent under P.

If $H = L_2(\Omega, \mathcal{F}, P)$ is separable then the first chaos space $H^{(1)}(B)$ consists of those random variables $f \in H$ such that $f = \mathbb{E}(f|x) + \mathbb{E}(f|x')$, for all $x \in B$. Here, $\mathbb{E}$ denotes conditional expectation (projection in the Hilbert space sense). We say that B is classical if $\sigma(H^{(1)}(B)) = \mathcal{F}$ and black if $\sigma(H^{(1)}(B)) = 0_\Lambda$.

There are noise-type Boolean algebras which are neither classical nor black and a simple example, due to Vershik (but also discovered independently by others) is the following. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $P(\xi_1 = \pm 1) = 1/2$. Let B_n be the smallest lattice containing the σ-fields $\sigma(\xi_1\xi_2), \ldots, \sigma(\xi_n\xi_{n+1})$ and $\sigma(\xi_{n+1}, \xi_{n+2}, \ldots)$. Then $B := \bigcup_{n\geq 1} B_n$ is a noise-type Boolean algebra which is neither black nor classical.

Feldman's question concerns a characterization of classical noise-type Boolean algebras. To state it, we say that B is complete if for any $X \subset B$ we have $\bigwedge_{x\in X} x \in B$ and $\bigvee_{x\in X} x \in B$, that is, if the lattice operations can be "passed on to the limit". Feldman conjectured that a complete B is classical. In this paper, loosely speaking, the equivalence of the two is proved. More precisely, B is classical if and only if there exists a complete noise-type Boolean algebra $\hat{B}$ including B. Moreover, this is equivalent to $(\bigvee_n x_n) \wedge (\bigwedge_n x'_n) = \mathcal{F}$ for all increasing sequences x_n in B. This requires the introduction of the completion of a noise-type Boolean algebra containing complemented elements of the set of lower limits of all sequences of elements of B.

The paper provides a wealth of useful information, links to the literature and to the author's previous works, as well as some open questions.