ON GENERALIZED LAGUERRE POLYNOMIALS WITH REAL AND COMPLEX PARAMETER.

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Abstract

In this paper we consider families of polynomials arising as eigenfunctions of the confluent hypergeometric operator $T = Q_1(z)\frac{d}{dz} + Q_2(z)\frac{d^2}{dz^2}$ where the polynomial coefficients Q_1 and Q_2 are linear. We study the location and properties of zeros of individual eigenpolynomials. The classical Laguerre polynomials appear as a special case and some well-known results about these are recovered and generalized.

1 Introduction

The confluent hypergeometric operator studied in this paper is a special case of a wider class of operators which we are interested in. Namely, consider the differential operator

$$T_Q = \sum_{j=1}^{k} Q_j(z) \frac{d^j}{dz^j}$$

where the Q_j are polynomials in one complex variable satisfying the condition $\deg Q_j \leq j$ with equality for at least one j. In [3] we studied the eigenvalue problem

$$T_Q(p_n) = \lambda_n p_n$$

where T_Q is an operator of the above kind of order k and where in particular $\deg Q_k = k$ (we call this the *non-degenerate case*). We proved that for such an operator there exists a unique monic eigenpolynomial p_n of degree n for all sufficiently large integers n. The main topic in [3] was asymptotic properties of the zeros of p_n. Our main result was that when the degree n tends to infinity, the zeros of p_n are distributed according to a certain probability measure which is *compactly supported on a tree* and which depends *only* on the leading polynomial Q_k. Moreover, we proved that the zeros of p_n are all contained in the convex hull of the zeros of Q_k.

An operator of the above type of order k but with the condition $\deg Q_k < k$ for the leading term is referred to as a *degenerate operator*. In this paper we restrict our study to properties of zeros of eigenpolynomials of the the simplest degenerate operator, namely the confluent hypergeometric operator[1]

$$T = Q_1(z)\frac{d}{dz} + Q_2(z)\frac{d^2}{dz^2}$$

where $\deg Q_1 = \deg Q_2 = 1$. With $Q_1(z) = \alpha z + \beta$ and $Q_2(z) = \gamma z + \delta$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha, \gamma \neq 0$, one can show (see Lemma 2 in Section 2) that by an appropriate affine transformation of z any such operator can be rewritten as

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right],$$

where $\delta, \kappa \in \mathbb{C}$. In what follows T will denote this operator. The corresponding eigenvalue equation then becomes

$$zp_n^{''}(z) + (z+\kappa)p_n^{'}(z) = np_n(z), \tag{1}$$

since $\lambda_n = n$. One can prove, using the same method as in [3], that there exists a unique and monic eigenpolynomial p_n of degree n for *every* integer n in (1), see Lemma 1 in Section 2.

In this paper we study the location of zeros of individual eigenpolynomials p_n of the operator T as above. In the sequel we will extend this study to asymptotic properties of zeros of eigenpolynomials of arbitrary degenerate operators.

The *Laguerre polynomials* appear as solutions to the Kummer hypergeometric equation

$$zy^{''}(z) + (\alpha + 1 - z)y^{'}(z) + ny(z) = 0$$

when $\alpha \in \mathbb{R}$, $\alpha > -1$ and $n \in \mathbb{Z}$.[2] Making the transformation $z \to -z$ it is easy to see that this equation corresponds to our eigenvalue equation (1) when $\kappa \in \mathbb{R}$ and $\kappa > 0$. Thus the classical Laguerre polynomials appear normalized[3] as solutions to the eigenvalue equation (1). One of the most important properties of the Laguerre polynomials is that they constitute an orthogonal system with respect to the weight function $e^{-x}x^{\alpha}$ on the interval $[0, \infty)$. It is well-known that the Laguerre polynomials are *hyperbolic* - that is all roots are real - and that the roots of two consecutive Laguerre polynomials p_n and p_{n+1} are *interlacing*. For

[1]Various familiar functions of mathematical analysis such as Hermite polynomials, Laguerre polynomials, Whittaker functions, Bessel functions and cylinder functions, are confluent hypergeometric functions, that is solutions to confluent hypergeometric equations.

[2]Observe that this equation has a degree n polynomial solution if and only if n is an integer. Without the condition that n is an integer we obtain the *Laguerre functions*.

[3]The nth degree Laguerre polynomial becomes monic when multiplied by $n!(-1)^n$.

other choices of the complex parameter α in Kummer's equation the sequence $\{p_n\}$ is in general *not* an orthogonal system and it can therefore not be studied by means of the theory known for such systems.

One of the results in this paper is the characterization of the exact choices on α for which T has hyperbolic eigenpolynomials and also for which α two consecutive eigenpolynomials have interlacing roots. It turns out that these properties are not restricted to the Laguerre polynomials solely. Our study can therefore be considered as a generalization of the properties of zeros of Laguerre polynomials to any family of polynomials appearing as eigenfunctions of the operator T. We also recover some well-known results (Theorems 3 and 4) by another method.

In what follows p_n denotes the nth degree unique and monic eigenpolynomial of T. These are our results:

Theorem 1. *The following two conditions are equivalent:*
(i) *there exists a real affine transformation $z \to az + b$ such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa > -1$,
(ii) *p_n is hyperbolic for all n.*

Remark. Each p_n is actually *strictly* hyperbolic here, that is all roots are real *and* simple, see Corollary 2. Note that (i)⇒(ii) for $\kappa > 0$ also follows from the general theory of orthogonal polynomial systems, since then the p_n are normalized Laguerre polynomials.

Theorem $1^{'}$. *The following two conditions are equivalent:*
$(\mathrm{i})^{'}$ *there exists a real affine transformation $z \to az + b$ such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa > -1$ or $\kappa = -1, -2, -3, \ldots, -(n-1)$,
$(\mathrm{ii})^{'}$ *p_n is hyperbolic.*

Remark. Thus if κ is a negative integer then all p_n such that $n > |\kappa|$ are hyperbolic. Note that when the degree n tends to infinity, p_n is hyperbolic for all negative integer values of κ.

The above results imply the following corollaries:

Corollary 1. *The following two conditions are equivalent:*
(i) *there exists a complex affine transformation $z \to \alpha z + \beta$ such that our oper-*

ator can be written

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ *and* $\kappa > -1$,
(ii) *the roots of* p_n *lie on a straight line in* $\mathbb{C}$ *for all* n.

Corollary $1^{'}$**.** *The following two conditions are equivalent:*
$(\mathrm{i})^{'}$ *there exists a complex affine transformation* $z \to \alpha z + \beta$ *such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ *and* $\kappa > -1$ *or* $\kappa = -1, -2, -3, \dots, -(n-1)$,
$(\mathrm{ii})^{'}$ *the roots of* p_n *lie on a straight line in* $\mathbb{C}$.

Remark. Thus if κ is a negative integer, then the roots of every p_n such that $n > |\kappa|$ lie ona straight line in $\mathbb{C}$.

Theorem 2. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta, \kappa \in \mathbb{C}$. *Then all roots of* p_n *are simple, unless* $\kappa = -1, -2, \dots, -(n-1)$.

Combining Theorem 1 (hyperbolicity) and Theorem 2 (simplicity) we obtain the following

Corollary 2. *The eigenpolynomials of* T *are strictly hyperbolic (all roots are real and simple) for all* n *if and only if there exists a real affine transformation* $z \to az + b$ *such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ *and* $\kappa > -1$.

Corollary 3. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$ *and* $\kappa = 0, -1, -2, \dots, -(n-1)$. *Then the eigenpolynomial* p_n *has* $(n+\kappa)$ *distinct roots, all of which are simple except the root at the origin which has multiplicity* $(1-\kappa)$. *Note that for* $\kappa = 0$ *all roots are simple.*

Moreover, it is possible to count the exact number of real roots of p_n. Namely,

Theorem 3. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa < -(n-1)$. Then p_n has no real roots if n is even, and p_n has precisely one real root if n is odd.

Theorem 4. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $-(n-1) < \kappa < -1$ such that κ is not an integer. Let $[\kappa]$ denote the integer part of κ. Then the number of real roots of p_n equals

$$\begin{cases} n + [\kappa] + 1, & \text{if } [\kappa] \text{ is odd} \\ n + [\kappa], & \text{if } [\kappa] \text{ is even.} \end{cases}$$

It is a classical fact that the roots of any two consecutive Laguerre polynomials interlace along the real axis. These polynomials arise (normalized) as eigenfunctions to our operator T when $\kappa > 0$. Here we extend this result and prove that the interlacing property also holds for polynomials arising as eigenfunctions of T when $\kappa = 0, -1, -2, \ldots, -(n-1)$. We have the following

Theorem 5. *Assume that our operator, after some complex affine transformation, can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}, \kappa \in \mathbb{R}$. Then the roots of any two consecutive eigenpolynomials p_n and p_{n+1} are interlacing if $\kappa = 0, -1, -2, \ldots, -(n-1)$.

Remark. If the eigenpolynomials are hyperbolic then the meaning of this is obvious, while if they are not hyperbolic the roots interlac along a straight line in the complex plane (see Corollary $1^{'}$).

Recent results on zero asymptotics. When α is arbitrary and real the polynomial solutions to Kummer's equation are referred to as *generalized Laguerre polynomials*. Some properties of the zeros when $\alpha \leq -1$ have been studied in [18]. In [23] similar results and several others are derived by considering the Laguerre polynomials as a limiting case of the Jacobi polynomials. In this paper we recover some of these results using yet another method. The asymptotic zero distribution for the generalized Laguerre polynomials (and several others) with real and degree dependent parameter α_n ($\alpha_n/n \to \infty$) have been studied in [6] using a continued fraction technique, and the same results are derived in [11] via a differential equation approach. It is known that when $\alpha \leq -1$ the zeros accumulate along certain interesting contours in the complex plane. More recent results on this can be found in [14] where a Riemann-Hilbert formulation for the Laguerre polynomials together with the steepest descent method (introduced in [6]) is used to obtain asymptotic properties of the zeros. The asymptotic location of the zeros depends on $A = \lim_{n\to\infty} -\frac{\alpha}{n} > 0$, and the results show a great sensitivity of the zeros to α_n's proximity to the integers. For $A > 1$ the contour is an open arc. For $0 < A < 1$ the contour consists of a closed loop together with an interval on the positive real axis. In the intermediate case $A = 1$ the contour is a simple closed contour. The case $A > 1$ is well-understood (see [21]), and uniform asymptotics for the Laguerre polynomials as $A > 1$ were obtained more recently, see [9], [15] and [26]. For fixed n interesting results can be found in [7] and [8].

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2 Proofs

We start with the following preliminary result:

Lemma 1. *Let*

$$T_Q = \sum_{j=0}^{k} Q_j(z)\frac{d^j}{dz^j}$$

be a linear differential operator where the polynomial coefficients satisfy $\deg Q_0 = 0$, $\deg Q_j = j$ *for exactly one* $j \in [1, k]$, *and* $\deg Q_m < m$ *if* $m \neq 0, j$. *Then* T_Q *has a unique and monic eigenpolynomial* p_n *of degree* n *for every integer* n. *Also, using the notation* $Q_m = \sum_{j=0}^{m} q_{m,j} z^j$, *we have*

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\dots(n-j+1)} = q_{j,j}$$

where λ_n *is the eigenvalue.*

Proof. In [3] we proved that for any operator T_Q as above but with the weaker restriction $\deg Q_j \leq j$ for all $j \in [0, n]$, the eigenvalue equation can be expressed as follows:

For $n \geq 1$ *the coefficient vector* X *of* $p_n = (a_{n,0}, a_{n,1}, \dots, a_{n,n-1})$ *satisfies the linear system* $MX = Y$, *where* Y *is a vector and* M *is an upper triangular matrix, both with entries expressible in the coefficients* $q_{m,j}$ *of the* Q_j.

We then used this to prove that there exists a unique monic eigenpolynomial of degree n for all sufficiently large integers n. Here we use the same method to prove that for the operator in Lemma 1 we actually obtain a unique and monic eigenpolynomial for *every* degree n.

If we compute the matrix M with respect to the basis of monomials $1, z, z^2, \ldots$, a diagonal element $M_{i+1,i+1}$ of M at the position $(i+1, i+1)$ (where $0 \le i \le n-1$) is given by

$$M_{i+1,i+1} = \sum_{0 \le m \le min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n$$

where

$$\lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!},$$

where we have used the notation $Q_m = \sum_{j=0}^{m} q_{m,j} z^j$. For the operator T_Q in Lemma 1 we have $\deg Q_m < m$ if $m \neq 0, j$, and so $q_{m,m} = 0$ for all $m \neq 0, j$. Inserting this in the expression for λ_n we obtain

$$\begin{aligned}\lambda_n &= \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!} = q_{0,0} + q_{j,j} \cdot \frac{n!}{(n-j)!} = \\ &= q_{0,0} + q_{j,j} \cdot n(n-1)\ldots(n-j+1),\end{aligned}$$

and thus

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\ldots(n-j+1)} = \lim_{n\to\infty} \left(\frac{q_{0,0}}{n(n-1)\ldots(n-j+1)} + q_{j,j} \right) = q_{j,j}.$$

To prove the uniqueness of p_n we calculate the determinant of the matrix M and since it is upper triangular this equals the product of the diagonal elements. Thus, if we prove that all diagonal elements are nonzero for every n, then M is invertible for every n and the system $MX = Y$ has a unique solution for every n and we are done. Inserting $q_{m,m} = 0$ for $m \neq 0, j$ we get

$$\begin{aligned}M_{i+1,i+1} &= \sum_{0 \le m \le min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n = \\ &= \sum_{0 \le m \le min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \left(q_{0,0} + q_{j,j} \cdot \frac{n!}{(n-j)!} \right) = \\ &= q_{j,j} \cdot \left(\frac{i!}{(i-j)!} - \frac{n!}{(n-j)!} \right) \neq 0\end{aligned}$$

where $q_{j,j} \neq 0$ since $\deg_{Q_j} = j$ and $i < n$. Note that for $i < j$ one sets $i!/(i-j)! = 0$. □

Remark. The operator

$$T = Q_1(z) \frac{d}{dz} + Q_2(z) \frac{d^2}{dz^2}$$

which we are interested in here is a special case of the operator T_Q in Lemma 1.

Lemma 2. *Any operator*

$$T = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *can be written*

$$T = \delta\left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

for some $\delta, \kappa \in \mathbb{C}$.

Proof. Dividing $T = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$ by β we obtain

$$T^* = T/\beta = \left(\frac{\alpha}{\beta}z + \frac{\gamma}{\beta}\right)\frac{d}{dz} + \left(z + \frac{\delta}{\beta}\right)\frac{d^2}{dz^2}$$

and making the translation $\tilde{z} = z + \frac{\delta}{\beta}$ we have

$$\begin{aligned}\tilde{T}^* &= \left(\frac{\alpha}{\beta}\left(\tilde{z} - \frac{\delta}{\beta}\right) + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \left(\tilde{z} - \frac{\delta}{\beta} + \frac{\delta}{\beta}\right)\frac{d^2}{d\tilde{z}^2} = \\ &= \left(\frac{\alpha}{\beta}\tilde{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \tilde{z}\frac{d^2}{d\tilde{z}^2}.\end{aligned}$$

Finally with $\bar{z} = \frac{\alpha}{\beta}\tilde{z} \Leftrightarrow \tilde{z} = \frac{\beta}{\alpha}\bar{z}$ we have $d\bar{z}/d\tilde{z} = \alpha/\beta$ and so

$$\begin{cases} \frac{d}{d\tilde{z}} = \frac{\alpha}{\beta}\frac{d}{d\bar{z}} \\ \frac{d^2}{d\tilde{z}^2} = \frac{d}{d\bar{z}}\left(\frac{d}{d\tilde{z}}\right)\frac{d\bar{z}}{d\tilde{z}} = \frac{d}{d\bar{z}}\left(\frac{\alpha}{\beta}\frac{d}{d\bar{z}}\right)\frac{\alpha}{\beta} = \frac{\alpha^2}{\beta^2}\frac{d^2}{d\bar{z}^2} \end{cases}$$

and we get

$$\begin{aligned}\bar{\tilde{T}}^* &= \left(\frac{\alpha}{\beta}\tilde{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \tilde{z}\frac{d^2}{d\tilde{z}^2} = \\ &= \left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}\bar{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{\alpha}{\beta}\frac{d}{d\bar{z}} + \frac{\beta}{\alpha}\bar{z}\cdot\frac{\alpha^2}{\beta^2}\frac{d^2}{d\bar{z}^2} = \\ &= \frac{\alpha}{\beta}\left[\left(\bar{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\bar{z}} + \bar{z}\frac{d^2}{d\bar{z}^2}\right] = \delta\left[(\bar{z} + \kappa)\frac{d}{d\bar{z}} + \bar{z}\frac{d^2}{d\bar{z}^2}\right]\end{aligned}$$

where $\delta = \frac{\alpha}{\beta}$ and $\kappa = -\frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}$. $\square$

Note that if $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ then $\delta, \kappa \in \mathbb{R}$.

We will now study hyperbolicity of the eigenpolynomials of T in detail. Note that performing the transformations in Lemma 2 above with all coefficients real does not affect hyperbolicity of the polynomial eigenfunctions.

Proof of Theorems 1 and $1^{'}$ and their corollaries. We first need the following well-known corollary (see [2]):

Corollary of Sturm's Theorem. *All roots of a monic and real polynomial are real if and only if the nonzero polynomials in its Sturm sequence have positive leading coefficients.*

Here the Sturm sequence is defined as follows. Let $p = p_0$ be a given real polynomial. Define $p_1 = p^{'}$ (the derivative of p) and choose the p_i to satisfy

$$\begin{aligned} p_0 &= p_1 q_1 - p_2, \quad \deg p_2 < \deg p_1 \\ p_1 &= p_2 q_2 - p_3, \quad \deg p_3 < \deg p_2 \\ p_2 &= p_3 q_3 - p_4, \quad \deg p_4 < \deg p_3 \\ &\vdots \end{aligned}$$

where the q_i are polynomials, and so on until a zero remainder is reached. That is, for each $i \geq 2$, p_i is the negative of the remainder when p_{i-2} is divided by p_{i-1}. Then the sequence $(p_0, p_1, p_2, \ldots)$ is called the *Sturm sequence* of the polynomial p.

We now calculate the Sturm sequence for a monic and real degree n eigenpolynomial $p = p_n$ of the operator $T = \delta\big[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\big]$, where $\delta \in \mathbb{C}$ and $\kappa \in \mathbb{R}$. Note that p is real if $\kappa \in \mathbb{R}$ and any two operators differing by a complex constant have identical polynomial eigenfunctions. Since our eigenpolynomials by assumption are monic, the first two elements in the Sturm sequence, p and $p^{'}$, clearly have positive leading coefficients, namely 1 and n. Define $R(i) = p_{i+1}$ in the Sturm sequence above. Then $R(1)$ is the negative of the remainder when p is divided by $p^{'}$. With $\deg p = n$ we have $\deg R(i) = n - i - 1$. The last element in the Sturm sequence (if it has not already stopped) will be the constant $R(n-1)$. We now claim that for every n and every $i \geq 1$ we have

$$\begin{cases} R(i) = A \cdot \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \frac{(\kappa+n-i-1)!}{(\kappa+j)!} z^j & \text{if } i \text{ is } \textit{odd} \\ R(i) = B \cdot \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \frac{(\kappa+n-i-1)!}{(\kappa+j)!} z^j & \text{if } i \text{ is } \textit{even} \end{cases} \tag{2}$$

where

$$\begin{cases} A = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)\ldots(n-i)(\kappa+n-i), \\ B = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)\ldots(n-i)(\kappa+n-i). \end{cases}$$

It is obvious that with $\kappa = 0$ the leading coefficients of all the $R(i)$ are positive and p will be hyperbolic. For $\kappa \in \mathbb{R}$ and $\kappa \neq 0$ we have the following conditions for the leading coefficients $R(i)_{lc}$ of the $R(i)$ to be positive:

$$\begin{cases} R(1)_{lc} > 0 \quad \Rightarrow \quad \kappa > 1-n \\ R(2)_{lc} > 0 \quad \Rightarrow \quad \kappa > 2-n \\ R(3)_{lc} > 0 \quad \Rightarrow \quad \kappa > 3-n \\ \vdots \\ R(i)_{lc} > 0 \quad \Rightarrow \quad \kappa > i-n \\ \vdots \\ R(n-1)_{lc} > 0 \quad \Rightarrow \quad \kappa > -1. \end{cases}$$

These conditions together yield $\kappa > -1$. Note that if some factor $(\kappa+n-j) = 0$, then not only the leading coefficient is zero, but the whole polynomial $R(i)$ is zero. So for $\kappa = j-n$ with $j \in [1, n-1]$ we also get hyperbolic p_n since the Sturm sequence by definition stops when a zero remainder is reached, and thus the leading coefficients of the previous components of the Sturm sequence are positive. So by the corollary of Sturm's Theorem, p_n is hyperbolic for *all* n *if and only if* $\kappa > -1$, and p_n is hyperbolic for a *particular* n *if and only if* $\kappa > -1$ *or* $\kappa = -1, -2, \ldots, -(n-1)$. One can prove by induction that the Sturm sequence polynomials are of the form claimed in (2) (see Appendix) . Moreover, it is obvious that if the roots of p_n lie on a straight line they can be transformed to the real axis by some complex affine transformation, and thus T must be on the form claimed by Theorem 1 or $1^{'}$, and so Corollaries 1 and $1^{'}$ follow. □

To prove Theorem 2 we need the following

Lemma 3. *Let* $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ *be the nth degree monic polynomial eigenfunction of* $T = \delta\big[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\big]$ *where* $\delta, \kappa \in \mathbb{C}$. *Note that* T *and* δT *have identical eigenpolynomials. Then the coefficients* $a_{n,j}$ *of* p_n *are given by*

$$a_{n,j} = \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!}, \quad \forall j \in [0, n].$$

Proof. Inserting $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ in $(z+\kappa)p_n^{'} + zp_n^{''} = np_n$ we have

$$(z+\kappa)\sum_{j=1}^{n} j a_{n,j} z^{j-1} + z\sum_{j=2}^{n} j(j-1) a_{n,j} z^{j-2} = n\sum_{j=0}^{n} a_{n,j} z^j$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{n} j a_{n,j} z^j + \sum_{j=1}^{n} \kappa j a_{n,j} z^{j-1} + \sum_{j=2}^{n} j(j-1) a_{n,j} z^{j-1} = \sum_{j=0}^{n} n a_{n,j} z^j$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{n} j a_{n,j} z^j + \sum_{j=0}^{n-1} \kappa(j+1) a_{n,j+1} z^j + \sum_{j=1}^{n-1} j(j+1) a_{n,j+1} z^j = \sum_{j=0}^{n} n a_{n,j} z^j.$$

Comparing coefficients we obtain

$$ja_{n,j} + \kappa(j+1)a_{n,j+1} + j(j+1)a_{n,j+1} = na_{n,j}$$
$$\Leftrightarrow$$
$$a_{n,j} = \frac{(j+1)(\kappa+j)}{(n-j)} \cdot a_{n,j+1}$$

Applying this iteratively and using $a_{n,n} = 1$ (by monicity of p_n) we arrive at

$$a_{n,j} = \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!}, \quad \forall j \in [0,n].$$

□

Proof of Theorem 2. Let $\alpha \neq 0$ be a root of p_n that is not simple. Then, by repeatedly differentiating our eigenvalue equation $zp_n^{''} + (z+\kappa)p_n^{'} = np_n$ and inserting $z = \alpha$, we get $p_n^{(j)}(\alpha) = 0\,\forall j$, which means the multiplicity of α is infinite, which is absurd. Thus, for all $\kappa \in \mathbb{C}$, any non-zero root α of p_n is simple[4]. Next we prove that if $\kappa \neq -1, -2, \ldots, -(n-1)$ and if $\alpha = 0$ is a root of p_n then it must be simple too. Let $\alpha = 0$ be a root of p_n of multiplicity m and write $p_n(z) = z^m q(z)$ where $\alpha = 0$ is not a root of $q(z)$. Then $p_n^{'}(z) = mz^{m-1}q(z) + z^m q^{'}(z)$ and $p_n^{''}(z) = m(m-1)z^{m-2}q(z) + mz^{m-1}q^{'}(z) + mz^{m-1}q^{'}(z) + z^m q^{''}(z)$. Inserting this in our eigenvalue equation we obtain

$$\begin{aligned}
\lambda_n p_n(z) =& z p_n^{''}(z) + (z+\kappa)p_n^{'}(z) \\
\Leftrightarrow& \\
z^{m-1}[\lambda_n z q(z)] =& m(m-1)z^{m-1}q(z) + mz^m q^{'}(z) \\
&+ mz^m q^{'}(z) + z^{m+1}q^{''}(z) + mz^m q(z) + z^{m+1}q^{'}(z) \\
&+ \kappa m z^{m-1}q(z) + \kappa z^m q^{'}(z) \\
=& z^{m-1}[m(m-1)q(z) + mzq^{'}(z) + mzq^{'}(z) \\
&+ z^2 q^{''}(z) + mzq(z) + z^2 q^{'}(z) + \kappa m q(z) + \kappa z q^{'}(z)].
\end{aligned}$$

Equating the expressions in the brackets and setting $z = 0$ we arrive at the relation $m(m-1)q(0) + \kappa m q(0) = 0 \Leftrightarrow m(m-1+\kappa) = 0$. Thus $m = 0$ or $m = 1-\kappa$ for the multiplicity m of the root $\alpha = 0$. But if $m = 0$ then $\alpha = 0$ is not a root of p_n whence all roots of p_n are simple and we are done. If $\kappa = 0$ then $m = 0$ or $m = 1$ (it will soon be proved that the latter is true, see below). If $\kappa \neq 0, -1, -2 \ldots, -(n-1)$ then either $m = 0$ and we are done, or $m = 1-\kappa$. Since m is the multiplicity of the root it must be a non-negative integer, and therefore $m = 1-\kappa$ is impossible unless $\kappa = 0, -1, -2, \ldots, -(n-1)$. Thus $\alpha = 0$ is not a root of p_n if $\kappa > -1$ and $\kappa \neq 0$. Also, $m = 1-\kappa$ is absurd if $\kappa \notin \mathbb{Z}$, and thus $m = 0$ for $\kappa \notin \mathbb{Z}$. Now consider the case $\kappa \in \mathbb{Z}$ with $\kappa \leq -n$.

[4] This also follows from the uniqueness theorem for a second order differential equation.

Then either $m = 0$ or $m = 1 - \kappa$. By Lemma 3 the constant term $a_{n,0}$ of p_n equals

$$a_{n,0} = \frac{(\kappa - 1 + n)!}{(\kappa - 1)!} = (\kappa - 1 + n)(\kappa - 2 + n)(\kappa - 3 + n)\dots(\kappa + 2)(\kappa + 1)\kappa.$$

and this cannot be zero if $\kappa \in \mathbb{Z}$ and $\kappa \leq -n$ - hence there is no zero at the origin ($m = 0$). Finally we prove that for $\kappa = -1, -2, \dots, -(n-1)$ the multiplicity of the root $\alpha = 0$ is $m = 1 - \kappa > 1$ and so in this case not all roots of p_n are simple. Recall that $m(m - 1 + \kappa) = 0$, so if $m \neq 0$ then $m = 1 - \kappa$ and we are done. Thus we have to prove that we do have a root at the origin for $\kappa = 0, -1, -2, \dots, -(n-1)$. But this is only possible if $a_{n,0} = 0$, and this is indeed the case if $\kappa = 0, -1, -2, \dots, -(n-1)$ and we can conclude that all roots of p_n are simple for all $\kappa \in \mathbb{C} \setminus \{-1, -2, \dots, -(n-1)\}$. □

Using Lemma 3 we also obtain the following

Proposition 1. *Let $p_n(\kappa, z)$ denote the* n*th degree monic eigenpolynomial of $T = \delta\left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$ where $\delta, \kappa \in \mathbb{C}$. Then, using the explicit representation of p_n in Lemma 3, we obtain the identity*

$$p_n^{(m)}(\kappa, z) = \frac{n!}{(n-m)!}p_{n-m}(\kappa + m, z), \quad n = 0, 1, \dots; m = 1, 2, \dots$$

and the recurrence formula

$$p_n(\kappa, z) = (z + 2n + \kappa - 2)p_{n-1}(\kappa, z) - (n-1)(n + \kappa - 2)p_{n-2}(\kappa, z),$$

where $p_0(\kappa, z) = 1$ and $p_1(\kappa, z) = z + \kappa$.

Proof of Corollary 3. By Theorem 2 all nonzero roots of p_n are simple, and from the proof of Theorem 2 we know that for $\kappa = 0, -1, -2, \dots, -(n-1)$ the multiplicity of the root at the origin is $m = 1 - \kappa$. We have a total of n roots of p_n and thus there are $n - (1 - \kappa) + 1 = n + \kappa$ distinct roots. □

As stated in **Theorems 3 and 4**, it is possible to count the exact number of real roots of p_n if $\kappa \in \mathbb{R}$ in T. We use *Sturm's Theorem* to count the number of real roots in any interval:[5]

Sturm's Theorem. *Let $(p_0(t), p_1(t), p_2(t), \dots)$ be the Sturm sequence of a polynomial $p(t)$ (as defined in the proof of Theorems 1 and $1'$). Let $u < v$ be real numbers. Assume that U is the number of sign changes in the sequence $(p_0(u), p_1(u), p_2(u), \dots)$ and let V be the number of sign changes in the sequence $(p_0(v), p_1(v), p_2(v), \dots)$. Then the number of real roots of $p(t)$ between u and v (with each multiple root counted exactly once) is exactly $U - V$.*

[5] see [2].

Remark. Combining Sturm's Theorem with Theorem 2 and its Corollary 3 it is possible to recover Theorems 1 and $1^{'}$ in the direction $\Rightarrow$. Namely, we get $(i) \Rightarrow (ii)$ if $\kappa > -1$ and p_n is the nth degree monic polynomial eigenfunction of T, since then the Sturm sequence of p_n has $(n+1)$ nonzero elements, all with positive leading coefficients. With $u = -\infty$ and $v = \infty$ we then have $U = n$ and $V = 0$, and therefore the number of real roots of p_n is $U - V = n$, so p_n is hyperbolic (Theorem 1). And similarly $(i)' \Rightarrow (ii)'$ for $\kappa = -1, -2, \ldots, -(n-1)$, since the Sturm sequence stops as soon as the zero remainder is reached, and here it has $(n+\kappa+1)$ nonzero elements, all with positive leading coefficients. Therefore, with $u = -\infty$ and $v = \infty$, we have $U = n + \kappa$ and $V = 0$. By Corollary 3 all roots of p_n are simple except the root at the origin which has multiplicity $1 - \kappa$. Thus, counted with multiplicity, p_n has $U - V + (-\kappa) = n$ real roots and is therefore hyperbolic (Theorem $1^{'}$) .

We already know that if $\kappa \neq -1, -2, \ldots, -(n-1)$, then all roots of p_n are simple and no element in the Sturm sequence of p_n is identically zero. The leading coefficients of the elements of the Sturm sequence are (see the proof of Theorems 1 and $1^{'}$) given by

$$\begin{cases} p_{lc} = 1 \\ p^{'}_{lc} = n \\ R(1)_{lc} = (n-1)(\kappa+n-1) \\ R(2)_{lc} = n(n-2)(\kappa+n-2) \\ R(3)_{lc} = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3) \\ R(4)_{lc} = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4) \\ R(3)_{lc} = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)(n-5)(\kappa+n-5) \\ R(4)_{lc} = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)(n-6)(\kappa+n-6) \\ \vdots \\ R(n-1)_{lc} = \ldots \end{cases}$$

We now use Sturm's Theorem to prove Theorems 3 and 4:

Proof of Theorem 3. Let p_n be the monic degree n eigenpolynomial of T where $\kappa < -(n-1)$, i.e. $\kappa + n - 1 < 0$ and therefore $\kappa + n - j < 0$ for every $j \geq 1$. Thus we get the following for the leading coefficients of the Sturm sequence elements:

$$\begin{cases} p_{lc} = 1 > 0 \\ p^{'}_{lc} = n > 0 \\ R(1)_{lc} < 0 \\ R(2)_{lc} < 0 \\ R(3)_{lc} > 0 \\ R(4)_{lc} > 0 \\ R(5)_{lc} < 0 \\ R(6)_{lc} < 0 \\ \vdots \end{cases}$$

This pattern continues up to the last element $R(n-1)$ of the sequence. Inserting $v = \infty$ in the Sturm sequence of p_n we find that there is a sign change at every $R(i)$ where i is odd. Therefore the number of sign changes V in this sequence equals the number of $R(i)$ where i is odd. Thus:

$$V = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Inserting $u = -\infty$ in the Sturm sequence we find that there is a sign change between the first two elements in the sequence and then at every $R(i)$ where i is even. and hence the number of sign changes U equals 1+ [the number of $R(i)$ where i is even]. Thus:

$$U = \begin{cases} \frac{n-2}{2} + 1 = \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} + 1 = \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

By Theorem 2 all roots of p_n are simple and thus the number of real roots of p_n equals $U - V = \begin{cases} 0 & \text{if } n \text{ is even.} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$ □

Proof of Theorem 4. Let p_n be the monic eigenpolynomial of T where $\kappa \in \mathbb{R}$ and $j - n < \kappa < j - n + 1$ for $j \in [1, n-2]$. Then $(\kappa + n - j) > 0$ and $(\kappa + n - j - 1) < 0$ and $[\kappa] = j - n$. Again we consider the leading coefficients in the Sturm sequence of p_n. Clearly $p_{lc} = 1 > 0$, $p'_{lc} = n > 0$ and $R(i)_{lc} > 0$ $\forall i \in [1, j]$. For the remaining leading coefficients we have

$$\begin{cases} R(j+1)_{lc} < 0 \\ R(j+2)_{lc} < 0 \\ R(j+3)_{lc} > 0 \\ R(j+4)_{lc} > 0 \\ R(j+5)_{lc} < 0 \\ R(j+6)_{lc} < 0 \\ \qquad \vdots \end{cases}$$

and this pattern continues up to the last element $R(n-1)$ in the sequence. Consider the sequence we obtain by inserting $v = \infty$ in this Sturm sequence. We have sign changes at every $R(j+l)$ where l is odd. Our last element is $R(n-1) = R(j+(n-j-1))$. Also note that if $n-j-1 = n-n-[\kappa]-1 = -[\kappa]-1$ is even then $[\kappa]$ is odd, and if $n-j-1$ is odd then $[\kappa]$ is even. Thus the number of sign changes V in this sequence is

$$V = \begin{cases} \frac{n-j-1}{2} & \text{if } [\kappa] \text{ is odd,} \\ \frac{n-j}{2} & \text{if } [\kappa] \text{ is even.} \end{cases}$$

Now insert $u = -\infty$ in the Sturm sequence. The number of sign changes from the first element p in the sequence till the element $R(j)$ is $(1 + j)$. For the remaining $n - j - 1$ elements of this sequence we have a change of sign at every $R(j + l)$ where l is even. Thus the number of sign changes is $(n - j - 1)/2$ if

$(n-j-1)$ is even $\Leftrightarrow [\kappa]$ is odd, and the number of sign changes is $(n-j-2)/2$ if $(n-j-1)$ is odd $\Leftrightarrow [\kappa]$ is even. Thus for the total number of sign changes U in this sequence we get

$$U = \begin{cases} (1+j) + \frac{n-j-1}{2} = \frac{n+j+1}{2} & \text{if } [\kappa] \text{ is odd} \\ (1+j) + \frac{n-j-2}{2} = \frac{n+j}{2} & \text{if } [\kappa] \text{ is even.} \end{cases}$$

Therefore the number of real roots $U - V$ of p_n, counted with multiplicity, is precisely

$$U - V = \begin{cases} \frac{n+j+1}{2} - \frac{n-j-1}{2} = j+1 = n+[\kappa]+1 & \text{if } [\kappa] \text{ is odd} \\ \frac{n+j}{2} - \frac{n-j}{2} = j = n+[\kappa] & \text{if } [\kappa] \text{ is even.} \end{cases}$$

since all roots of p_n are simple in this case by Theorem 2. □

Proof of Theorem 5. The proof of the interlacing property consists of a sequence of five lemmas. Lemmas 4 and 8 are well-known. Lemmas 4,5 and 6 are used in the proof of Lemma 7, which is proved using an idea due to S. Shadrin presented in [20]. The five lemmas are the following:

Lemma 4. *If R_n and R_{n+1} are strictly hyperbolic polynomials of degrees n and $n+1$ respectively, then $R_n + \epsilon R_{n+1}$ is hyperbolic for any sufficiently small ϵ.*

Lemma 5. *Let p_n and p_{n+1} be two polynomial eigenfunctions of the operator $T = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ with $\kappa = 0, -1, -2, \dots, -(n-1)$. Then $p_n + \epsilon p_{n+1}$ is hyperbolic for any sufficiently small ϵ.*

Proof of Lemma 5. From Corollary 3 we know that p_n and p_{n+1} have all their roots simple except for the root at the origin which for both polynomials has multiplicity $1-\kappa$. Thus we can write $p_n + \epsilon p_{n+1} = z^{1-\kappa}(R_{n+\kappa-1} + \epsilon R_{n+\kappa})$, where $R_{n+\kappa-1}$ and $R_{n+\kappa}$ are strictly hyperbolic polynomials of degrees $n+\kappa-1$ and $n+\kappa$ respectively. By Lemma 4, $R_{n+\kappa-1} + \epsilon R_{n+\kappa}$ is hyperbolic for any sufficiently small ϵ, and then clearly $z^{1-\kappa}(R_{n+\kappa-1} + \epsilon R_{n+\kappa}) = p_n + \epsilon p_{n+1}$ is also hyperbolic for any sufficiently small ϵ. □

Lemma 6. *Let $T = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ with $\kappa = 0, -1, -2, \dots, -(n-1)$, and let p_n and p_{n+1} be two consecutive eigenpolynomials of T. Then letting T act on any linear combination $\alpha p_n + \beta p_{n+1}$ with $\alpha, \beta \in \mathbb{R}$ that is hyperbolic (i.e. has all its roots real) results in a hyperbolic polynomial.*

Proof of Lemma 6. Note that the operators $T = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ and $T = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ have identical eigenpolynomials. Let $f = \alpha p_n + \beta p_{n+1}$ be a hyperbolic linear combination with real coefficients of two consecutive eigenpolynomials of T. Then f' is a hyperbolic polynomial by Gauss-Lucas Theorem. By Rolle's Theorem f and f' have interlacing roots and so by the well-known Lemma 8 below, $(f + f^{'})$ is a hyperbolic polynomial. By Corollary 3 both p_n and p_{n+1} have a root at the origin of multiplicity $1 - \kappa$. Thus $f = \alpha p_n + \beta p_{n+1}$ has a root at the origin of multiplicity at least $1 - \kappa$, and $f^{'}$ has a root at the origin of multiplicity at least $-\kappa$. Thus the polynomial $(f + f^{'})$ has a root at the origin of multiplicity at least $(-\kappa)$ and we can write $(f + f^{'}) = z^{-\kappa} g$ for some hyperbolic polynomial g. Now $z^{\kappa}(f + f^{'}) = g$ is a hyperbolic polynomial. But $D[z^{\kappa}(f + f^{'})] = \kappa z^{\kappa-1}(f + f^{'}) + z^{\kappa}(f^{'} + f^{''}) = z^{\kappa-1}[\kappa f + (z+\kappa)f^{'} + z f^{''}] = z^{\kappa-1}T(f)$ where $T(f) = \kappa f + (z+\kappa)f^{'} + z f^{''}$. By Gauss-Lucas Theorem one has that $D[z^{\kappa}(f + f^{'})]$ is a hyperbolic polynomial and therefore $T(f) = z^{1-k} D[z^{\kappa}(f + f^{'})]$ is a hyperbolic polynomial. □

Lemma 7. *Let $T = \kappa + (z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$. Any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients of two consecutive eigenpolynomials of T with $\kappa = 0, -1, -2, \ldots, -(n-1)$ is a hyperbolic polynomial.*

Proof of Lemma 7. Applying to $\alpha p_n + \beta p_{n+1}$ some high power T^{-N} of the inverse operator one gets

$$T^{-N}(\alpha p_n + \beta p_{n+1}) = \frac{\alpha}{\lambda_n^N} p_n + \frac{\beta}{\lambda_{n+1}^N} p_{n+1} =$$

$$= \frac{\alpha}{\lambda_n^N}(p_n + \epsilon p_{n+1}),$$

where ϵ is arbitrarily small for the appropriate choice of N (since $0 < \lambda_n < \lambda_{n+1}$). Thus, by Lemma 5, the polynomial $T^{-N}(\alpha p_n + \beta p_{n+1})$ is hyperbolic for sufficiently large N. Assume that $\alpha p_n + \beta p_{n+1}$ is non-hyperbolic and take the largest N_0 for which $R_{N_0} = T^{-N_0}(\alpha p_n + \beta p_{n+1})$ is still non-hyperbolic. Then $R_{N_0} = T(R_{N_0+1})$ where $R_{N_0+1} = T^{-N_0-1}(\alpha p_n + \beta p_{n+1})$. Note that R_{N_0+1} is hyperbolic and that if $\kappa = 0, -1, -2, \ldots, -(n-1)$ then letting T act on any hyperbolic linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients results in a hyperbolic polynomial by Lemma 6. Contradiction. □

Lemma 8 [classical]. If R_n and R_{n+1} are any real polynomials of degrees n and $n+1$, respectively, then saying that every linear combination $\alpha R_n + \beta R_{n+1}$ with real coefficients is hyperbolic is equivalent to saying that
(i) both R_n and R_{n+1} are hyperbolic, and
(ii) their roots are interlacing.

We now **prove Theorem 5**. Consider the operator $T = (z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ where $\kappa = 0, -1, -2, \ldots, -(n-1)$, and let p_n and p_{n+1} be two consecutive eigenpolynomials of T. Recall that δT and T have identical eigenpolynomials and that by Corollary $1^{'}$ the roots in this case lie on straight lines in the complex plane. By Lemma 5 the linear combination $p_n + \epsilon p_{n+1}$ is hyperbolic for any sufficiently small ϵ. Using Lemmas 5 and 6 we can therefore apply Lemma 7 which says that any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients α and β is a hyperbolic polynomial. By Lemma 8 this implies that the roots of

p_n and p_{n+1} are interlacing and we are done. $\square$

Remark. Note that we can recover the interlacing property for the Laguerre polynomials using the same proof as in Theorem 5 but with a small modification of Lemma 6. Namely, if $\kappa > 0$, then the application of T to *any* hyperbolic polynomial results in a hyperbolic polynomial. For if f is a hyperbolic polynomial, then $f^{'}$ is hyperbolic by Gauss-Lucas Theorem, f and $f^{'}$ have interlacing roots by Rolle's Theorem, and by the well-known Lemma 8 the linear combination $(f + f^{'})$, and therefore $z^{\kappa}(f + f^{'})$, is a hyperbolic polynomial. Finally $D[z^{\kappa}(f + f^{'})] = z^{\kappa-1}T(f)$ is hyperbolic by Gauss-Lucas Theorem.

When suitably scaled, it is possible to find a limiting expansion for p_n when $n \to \infty$ that is closely related to a Bessel function. Because of the scaling however, the convergence to the Bessel function only gives information about the asymptotic behaviour of p_n in an infinitesimal neighbourhood of the origin. Although other methods must be used to get information elsewhere, it is interesting that on the infinitesimal scale our eigenpolynomials mimic the global behaviour of this particular Bessel function. We have the following theorem, where $J_{\kappa-1}$ denotes the Bessel function of the first kind of order $(\kappa - 1)$:

Proposition 2. *Let $p_n(\kappa, z)$ denote the unique and monic eigenpolynomial of the operator*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta, \kappa \in \mathbb{C}$ and κ is not a negative integer. We then have the following limit formula:

$$\lim_{n\to\infty} \frac{n^{1-\kappa}}{n!} p_n(\kappa, z/n) = (-z)^{(1-\kappa)/2} J_{\kappa-1}(2i\sqrt{z}),$$

where the convergence holds for all $z \in \mathbb{C}$ and uniformly on compact z-sets.

The *Bessel function of the first kind* of order κ is defined by the series

$$J_{\kappa}(z) \equiv \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(z/2)^{\kappa+2\nu}}{\nu!\Gamma(\kappa+\nu+1)}$$

where $z, \kappa \in \mathbb{C}$ and $|z| < \infty$. Clearly $z^{-\kappa}J_{\kappa}(z)$ is an entire analytic function for all $z \in \mathbb{C}$ if κ is not a negative integer. This Bessel function is a solution to *Bessel's equation* [6] *of order* κ, which is the second-order linear differential equation given by

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \kappa^2)y = 0.$$

From now on we adopt the notational convention $\Gamma(n+\kappa) = (n+\kappa-1)!$ for $\kappa \in \mathbb{C}$, where Γ is the Gamma function. In order to prove Proposition 2, we

[6] Bessel's equation is encountered in the study of boundary value problems in potential theory for cylindrical domains. The solutions to Bessel's equation are referred to as *cylinder functions*, of which the Bessel functions are a special kind.

will need the following technical

Lemma 9.
$$\lim_{n\to\infty}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa-\nu}=\frac{1}{\Gamma(\kappa+\nu)}$$
where $n,\nu\in\mathbb{R}$ *and* $\kappa\in\mathbb{C}\setminus\{-1,-2,\ldots\}$.

Proof of Lemma 9. Using the following well-known asymptotic formula:

Corollary of the Stirling formula. [7]

$$\lim_{n\to\infty}\frac{\Gamma(n+\alpha)}{\Gamma(n)}n^{-\alpha}=1$$

where $\alpha\in\mathbb{C}$ *and* $n\in\mathbb{R}$,

we get

$$\begin{aligned}\lim_{n\to\infty}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa-\nu}&=\frac{1}{\Gamma(\kappa+\nu)}\lim_{n\to\infty}\frac{\Gamma(n+\kappa)}{\Gamma(n-\nu+1)}n^{1-\kappa-\nu}\\&=\frac{1}{\Gamma(\kappa+\nu)}\lim_{n\to\infty}\frac{\Gamma(n+\kappa)}{\Gamma(n)}n^{-\kappa}\lim_{n\to\infty}\frac{\Gamma(n)}{\Gamma(n-\nu+1)}n^{1-\nu}\\&=\frac{1}{\Gamma(\kappa+\nu)}.\end{aligned}$$

Proof of Proposition 2. By Lemma 3 our eigenpolynomials have the following explicit representation:

$$p_n(\kappa,z)=\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(\kappa+n-1)!}{(\kappa+\nu-1)!}z^\nu=\sum_{\nu=0}^{n}\binom{n+\kappa-1}{n-\nu}\frac{n!}{\nu!}z^\nu$$

where $\kappa\in\mathbb{C}$.

Thus, with the scaling $z\to z/n$ and using Lemma 9, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{n^{1-\kappa}}{n!}p_n(\kappa,z/n)&=\lim_{n\to\infty}\sum_{\nu=0}^{n}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa}\frac{1}{\nu!}\left(\frac{z}{n}\right)^\nu\\&=\lim_{n\to\infty}\sum_{\nu=0}^{n}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa-\nu}\frac{z^\nu}{\nu!}\\&=\sum_{\nu=0}^{\infty}\frac{z^\nu}{\Gamma(\kappa+\nu)\nu!}=(-z)^{(1-\kappa)/2}J_{\kappa-1}(2i\sqrt{z}).\end{aligned}$$

□

[7] see e.g.[25]

Remark. It is easy to prove that the power series $\sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\kappa+\nu)\nu!}$ does indeed satisfy the differential equation $zu^{''} + \kappa u^{'} - u = 0$, which arises by the limiting procedure $n \to \infty$ from our eigenvalue equation $zu^{''} + (z+\kappa)u^{'} - nu = 0$, after scaling the variables.

Appendix: Proof of (2) in Section 2.

Note that we have adopted the notational convention $\Gamma(n+\kappa) = (n+\kappa-1)!$ for $\kappa \in \mathbb{C}$, where Γ denotes the Gamma function. I start by calculating $R(1)$ and $R(2)$ and so the hypothesis (actually there are two hypotheses, one for even i and one for odd i) is true for one case of even i and one case of odd i. With the nth degree eigenpolynomial $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ we have by Lemma 3 that

$$a_{n,j} = \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} \quad \Rightarrow \quad p_n = \sum_{j=0}^{n} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^j.$$

Calculation of $R(1)$ = [the negative of the remainder when the eigenpolynomial p_n is divided by $p_n^{'}$]:

$$\begin{array}{r|l}
 & \dfrac{z}{n} + \dfrac{(n-1+\kappa)}{n} \\
\hline
\displaystyle\sum_{j=1}^{n} j\binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^{j-1} & \displaystyle\sum_{j=0}^{n} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^j \\
\hline
 & \displaystyle -\Big[\sum_{j=1}^{n} \frac{j}{n} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^j\Big] \\
\cline{2-2}
 & \displaystyle = \sum_{j=0}^{n-1} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} \Big[1 - \frac{j}{n}\Big] z^j \\
 & \displaystyle -\Big[\sum_{j=0}^{n-1} \frac{j+1}{n} \binom{n}{j+1} (\kappa+n-1) \frac{(\kappa+n-1)!}{(\kappa+j)!} z^j\Big] \\
\cline{2-2}
 & \displaystyle = \sum_{j=0}^{n-2} \Big[\binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} \Big(1 - \frac{j}{n}\Big) - \frac{(j+1)}{n} \binom{n}{j+1} (\kappa+n-1) \frac{(\kappa+n-1)!}{(\kappa+j)!}\Big] z^j
\end{array}$$

and it remains to prove that the negative of this remainder equals

$$R(1) = (n-1)(\kappa+n-1) \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(\kappa+n-2)!}{(\kappa+j)!} z^j.$$

Developing the coefficient in front of z^j in our remainder we obtain

$$\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}\left(1-\frac{j}{n}\right)-\frac{(j+1)}{n}\binom{n}{j+1}(\kappa+n-1)\frac{(\kappa+n-1)!}{(\kappa+j)!}$$

$$=\frac{n!}{(n-j)!j!}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}-\frac{n!}{(n-j)!j!}\frac{j}{n}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}-\frac{(j+1)}{n}\frac{n!(\kappa+n-1)}{(j+1)!(n-j-1)!}\frac{(\kappa+n-1)!}{(\kappa+j)!}$$

$$=\frac{n(n-1)(n-2)!}{(n-j-2)!(n-j-1)(n-j)j!}\frac{(\kappa+n-2)!(\kappa+n-1)(\kappa+j)}{(\kappa+j)!}$$

$$-\frac{(n-1)(n-2)!(\kappa+n-2)!(\kappa+n-1)j(\kappa+j)}{(n-j-2)!(n-j-1)(n-j)(\kappa+j)!j!}-\frac{(n-1)(n-2)!(\kappa+n-1)^2(\kappa+n-2)!}{j!(n-j-2)!(n-j-1)(\kappa+j)!}$$

$$=(n-1)(\kappa+n-1)\frac{(n-2)!}{j!(n-j-2)!}\frac{(\kappa+n-2)!}{(\kappa+j)!}\left[\frac{n(\kappa+j)}{(n-j-1)(n-j)}-\frac{j(\kappa+j)}{(n-j-1)(n-j)}\right.$$

$$\left.-\frac{(\kappa+n-1)(n-j)}{(n-j-1)(n-j)}\right]$$

$$=(n-1)(\kappa+n-1)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}\left[\frac{n\kappa+nj-j\kappa-j^2-\kappa n+\kappa j-n^2+nj+n-j}{n^2-nj-nj+j^2-n+j}\right]$$

$$=-(n-1)(\kappa+n-1)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!},$$

and we are done.

Calculation of $R(2)=$ [the negative of the remainder when $p_n^{'}$ is divided by $R(1)$]:

$$\frac{nz}{(n-1)(\kappa+n-1)}+\frac{n(2n-3+\kappa)}{(n-1)(\kappa+n-1)}$$

$$\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}(n-1)(\kappa+n-1)z^j \;\Big)\; \sum_{j=0}^{n-1}(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}z^j$$

$$-\left[\sum_{j=1}^{n-1}n\binom{n-2}{j-1}\frac{(\kappa+n-2)!}{(\kappa+j-1)!}z^j\right]$$

$$=\sum_{j=0}^{n-2}\left[(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}-n\binom{n-2}{j-1}\frac{(\kappa+n-2)!}{(\kappa+j-1)!}\right]z^j$$

$$-\left[\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}n(2n{-}3{+}\kappa)z^j\right]$$

$$=\sum_{j=0}^{n-3}\left[\frac{(\kappa+n-2)!}{(\kappa+j-1)!}\left((j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}-n\binom{n-2}{j-1}\right)-n(2n-3+\kappa)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}\right]z^j$$

and it remains to prove that the negative of this remainder equals

$$R(2)=n(n-2)(\kappa+n-2)\sum_{j=0}^{n-3}\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!}z^j.$$

Developing the coefficient in front of z^j in our remainder we have

$$\frac{(\kappa+n-2)!}{(\kappa+j-1)!}(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)}{(\kappa+j)} - \frac{(\kappa+n-2)!}{(\kappa+j-1)!}n\binom{n-2}{j-1} - \frac{(\kappa+n-2)!}{(\kappa+j)!}\binom{n-2}{j}n(2n-3+\kappa)$$

$$= \frac{(\kappa+n-2)!}{(\kappa+j-1)!}\frac{n!}{j!(n-j-1)!}\frac{(\kappa+n-1)}{(\kappa+j)} - \frac{(\kappa+n-2)!}{(\kappa+j-1)!}\frac{n(n-2)!}{(j-1)!(n-j-1)!}$$

$$- \frac{(\kappa+n-2)!}{(\kappa+j)!}\frac{(n-2)!}{(j!(n-j-2)!}n(2n-3+\kappa)$$

$$= \frac{(\kappa+n-3)!(\kappa+n-2)(n-3)!(n-2)(n-1)n(\kappa+n-1)}{(\kappa+j)!j!(n-j-3)!(n-j-2)(n-j-1)}$$

$$- \frac{(\kappa+n-3)!(\kappa+n-2)n(n-2)(n-3)!j(\kappa+j)}{j!(n-j-3)!(n-j-2)(n-j-1)(\kappa+j)!}$$

$$- \frac{(\kappa+n-3)!(\kappa+n-2)(n-2)(n-3)!n(2n-3+\kappa)}{(\kappa+j)!j!(n-j-2)(n-j-3)!}$$

$$= \frac{(\kappa+n-3)!(n-3)!}{(\kappa+j)!j!(n-j-3)!}n(n-2)(\kappa+n-2)\Bigg[\frac{(n-1)(\kappa+n-1)}{(n-j-2)(n-j-1)}$$

$$- \frac{j(\kappa+j)}{(n-j-2)(n-j-1)} - \frac{(2n-3+\kappa)(n-j-1)}{(n-j-2)(n-j-1)}\Bigg]$$

$$= n(n-2)(\kappa+n-2)\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!}\left[\frac{-n^2+nj+n+nj-j^2-j+2n-2j-2}{n^2-nj-n-nj+j^2+j-2n+2j+2}\right]$$

$$= -n(n-2)(\kappa+n-2)\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!},$$

and we are done.

To prove the induction hypotheses we divide $R(i)$ by $R(i+1)$ to obtain $R(i+2)$. Here it is assumed that i is odd. The proof with even i differs only in small details from this proof and is therefore omitted here. For simplicity we use the notations

$$\begin{cases} A = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)\dots(n-i)(\kappa+n-i), \\ B = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)\dots(n-i)(\kappa+n-i). \end{cases}$$

Dividing $R(i)$ by $R(i+1)$:

$$\frac{A}{B}z + \frac{A}{B}(2n-2i-3+\kappa)$$

$$B\sum_{j=0}^{n-i-2}\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}z^j \quad \Bigg| \quad A\sum_{j=0}^{n-i-1}\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}z^j$$

$$-\left[A\sum_{j=1}^{n-i-1}\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}z^j\right]$$

$$= A\sum_{j=0}^{n-i-2}\left[\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}\right]z^j$$

$$-\left[A\sum_{j=0}^{n-i-2}(2n{-}2i{-}3{+}\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}z^j\right]$$

$$= A\sum_{j=0}^{n-i-3}\left[\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!} - (2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}\right]z^j$$

and it remains to prove that the negative of this remainder equals the excpected (by hypothesis)

$$R(i+2) = A(n-i-2)(\kappa+n-i-2)\sum_{j=0}^{n-i-3}\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}z^j,$$

i.e. we have to prove the following equality:

$$\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!} - (2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$
$$= -(n-i-2)(\kappa+n-i-2)\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}.$$

But

$$\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!} - (2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$
$$= \frac{(n-i-1)!}{j!(n-i-j-1)!}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \frac{(n-i-2)!}{(j-1)!(n-i-j-1)!}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}$$
$$- (2n-2i-3+\kappa)\frac{(n-i-2)!}{j!(n-i-j-2)!}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$
$$= \frac{(n-i-3)!(n-i-2)(n-i-1)(\kappa+n-i-3)!(\kappa+n-i-2)(\kappa+n-i-1)}{j!(n-i-j-1)(n-i-j-2)(n-i-j-3)!(\kappa+j)!}$$
$$- \frac{(n-i-3)!(n-i-2)j(\kappa+n-i-3)!(\kappa+n-i-2)(c+j)}{j!(n-i-j-3)!(n-i-j-2)(n-i-j-1)(\kappa+j)!}$$
$$- (2n-2i-3+\kappa)\frac{(n-i-3)!(n-i-2)(\kappa+n-i-3)!(\kappa+n-i-2)}{j!(n-i-j-3)!(n-i-j-2)(\kappa+j)!}$$
$$= (n-i-2)(\kappa+n-i-2)\frac{(n-i-3)!}{j!(n-i-j-3)!}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}$$
$$\cdot\left[\frac{(n-i-1)(\kappa+n-i-1) - j(\kappa+j) - (2n-2i-3+\kappa)(n-i-j-1)}{(n-i-j-1)(n-i-j-2)}\right]$$
$$= (n-i-2)(\kappa+n-i-2)\frac{(n-i-3)!}{j!(n-i-j-3)!}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}(-1)$$
$$= -(n-i-2)(\kappa+n-i-2)\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}.$$

□

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