On Asymptotics of Polynomial Eigenfunctions for Exactly-Solvable Differential Operators

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Abstract
In this paper we study the class of differential operators \( T = \sum_{j=1}^{k} Q_j D^j \) with polynomial coefficients \( Q_j \) and \( \deg Q_j \leq j \) with equality for at least one \( j \). We show that if \( \deg Q_k < k \) then the root of the \( n \)th degree eigenpolynomial \( p_n \) of \( T \) with the largest absolute value tends to infinity when \( n \to \infty \), as opposed to the case when \( \deg Q_k = k \). Moreover we present an explicit conjecture and partial results on the growth of the largest root. Based on this conjecture we deduce the algebraic equation satisfied by the asymptotic Cauchy transform of the appropriately scaled eigenpolynomials.

1 Introduction
In this paper we study asymptotic properties of zeros in families of polynomials satisfying certain linear differential equations. Namely, consider a linear differential operator

\[ T = \sum_{j=1}^{k} Q_j D^j, \]

where \( D = \frac{d}{dz} \) and the \( Q_j \) are complex polynomials in a single variable \( z \). We are interested in the case when \( \deg Q_j \leq j \) for all \( j \), and in particular \( \deg Q_k < k \) for the leading term. Such operators are referred to as degenerate exactly-solvable operators, see Definition 1 below. In this paper we study the polynomial eigenfunctions of this operator, that is polynomials satisfying

\[ T(p_n) = \lambda_n p_n \quad (1) \]

for some value of the spectral parameter \( \lambda_n \), where \( n \) is a nonnegative integer and \( \deg p_n = n \).
The basic motivation for this study comes from two sources: 1) a classical question going back to S. Bochner, and 2) the generalized Bochner problem, which we describe below.

1) In 1929 Bochner asked about the classification of differential equations (1) having an infinite sequence of orthogonal polynomial solutions, see [11]. Such a system of polynomials \( \{p_n\}_{n=0}^\infty \) which are both eigenpolynomials of some finite order differential operator and orthogonal with respect to some suitable inner product, are referred to as Bochner-Krall orthogonal polynomial systems (BKS), and the corresponding operators are called Bochner-Krall operators. It is an open problem to classify all BKS - a complete classification is only known for Bochner-Krall operators of order \( k \leq 4 \), and the corresponding BKS are various classical systems such as the Jacobi type, the Laguerre type, the Legendre type and the Bessel and Hermite polynomials (see [5]).

Notice that for the operators considered below, the sequence of eigenpolynomials is in general not an orthogonal system of polynomials, and can therefore not be studied by means of the extensive theory known for such systems.

2) The problem of a general classification of linear differential operators for which the eigenvalue problem (1) has a certain number of eigenfunctions in the form of a finite-order polynomial in some variables, is referred to as the generalized Bochner problem, see [15] and [16]. In the former paper a classification of operators possessing infinitely many finite-dimensional subspaces with a basis in polynomials is presented, and in the latter paper a general method has been formulated for generating eigenvalue problems for linear differential operators in one and several variables possessing polynomial solutions.

Definition 1. We call a linear differential operator \( T \) of the \( k \)th order exactly-solvable if it preserves the infinite flag \( \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots \), where \( \mathcal{P}_n \) is the linear space of all polynomials of degree less than or equal to \( n \). Or, equivalently, the problem (1) has an infinite sequence of polynomial eigenfunctions if and only if the operator \( T \) is exactly-solvable (see [17]).

The exactly-solvable operators \( T = \sum_{j=1}^k Q_j D^j \) with \( \deg Q_j \leq j \) for all \( j \), split into two major classes: non-degenerate and degenerate, where in the former case \( \deg Q_k = k \), and in the latter case \( \deg Q_k < k \) for the leading term. The major difference between these two classes is that in the non-degenerate case the union of all roots of all eigenpolynomials of \( T \) is contained in a compact set, contrary to the degenerate case, which we will prove in this paper.

Let us briefly recall our previous results for eigenpolynomials of the non-degenerate exactly-solvable operators. In [1] we proved that asymptotically as

\[ \text{Correspondingly, a linear differential operator of the } k \text{th order is called quasi-exactly-solvable if it preserves the space } \mathcal{P}_n \text{ for some fixed } n. \]
n \to \infty$, the zeros of the $n$th degree eigenpolynomials $p_n$ of the non-degenerate exactly-solvable operators are distributed according to a certain probability measure which has compact support and which depends only on the leading polynomial $Q_k$. These are our main results from [1]:

**Theorem A.** Let $Q_k$ be a monic polynomial of degree $k$. Then there exists a unique probability measure $\mu_{Q_k}$ with compact support whose Cauchy transform $C(z) = \int \frac{d\mu_{Q_k}(\zeta)}{\zeta - z}$ satisfies $C(z)^k = 1/Q_k(z)$ for almost all $z \in \mathbb{C}$.

**Theorem B.** Let $Q_k$ and $\mu_{Q_k}$ be as in Theorem A. Then $\text{supp} \, \mu_{Q_k}$ is the union of finitely many smooth curve segments, and each of these curves is mapped to a straight line by the locally defined mapping $\Psi(z) = \int Q_k(z)^{-1/k}dz$. Moreover, $\text{supp} \, \mu_{Q_k}$ contains all the zeros of $Q_k$, is contained in the convex hull of the zeros of $Q_k$, is connected and has connected complement.

If $p_n$ is a polynomial of degree $n$ we construct the probability measure $\mu_n$ by placing the point mass of size $\frac{1}{n}$ at each zero of $p_n$, and we call $\mu_n$ the root measure of $p_n$. We then have the (main) result:

**Theorem C.** Let $p_n$ be the monic degree $n$ eigenpolynomial of a non-degenerate exactly solvable operator $T$ and let $\mu_n$ be the root measure of $p_n$. Then $\mu_n$ converges weakly to $\mu_{Q_k}$ when $n \to \infty$.

To illustrate, we show the zeros of the polynomial eigenfunctions $p_{50}, p_{75}$ and $p_{100}$ for the non-degenerate exactly-solvable operator $T = Q_5D^5$ where $Q_5 = (z - 2 + 2i)(z + 1 - 2i)(z + 3 + i)(z + 2i)(z + 2i)$. In the pictures below, the large dots represent the zeros of $Q_5$ and the small dots represent the zeros of the eigenpolynomials $p_{50}, p_{75}$ and $p_{100}$ respectively:

As a result of this study, we were then able to prove a special case of a general conjecture describing the leading terms of all Bochner-Krall operators, see [2].

In the present paper we are interested in the class of degenerate exactly-solvable operators, that is operators $T = \sum_{j=1}^{k} Q_j D^j$ where $\deg Q_j \leq j$ for all $j$ with equality for at least one $j$, and $\deg Q_k < k$. Without loss of generality we
assume that the \( n \)th degree eigenpolynomial \( p_n \) of \( T \) is monic. Some well-known classical polynomials, such as the Laguerre polynomials, appear as polynomial solutions to the eigenvalue problem (1) for certain choices on the polynomials coefficients \( Q_j \). Studies on the asymptotic zero behaviour for these polynomials can be found in [4], [7], [9], [13] and [14].

Computer experiments indicate the existence of a limiting measure for the asymptotic zero distribution of the \( n \)th degree polynomial eigenfunction \( p_n \) of any degenerate exactly-solvable operator after an appropriate scaling. Without such a scaling the roots of \( p_n \) tend to infinity when \( n \to \infty \), see Theorem 1. Based on calculations involving the Cauchy transform we conjecture how the largest modulus of all roots of \( p_n \) grows as \( n \to \infty \) for any given degenerate exactly-solvable operator, see Main Conjecture. All experiments performed by the author are consistent with this conjecture (see numerical evidence in Section 4), and we also prove it partially (lower bounds on the largest roots) for some classes of degenerate exactly-solvable operators, see Theorems 3 and 4.

The appropriately scaled eigenpolynomials will then (conjecturally) have nice compactly supported zero distribution in the limit as \( n \to \infty \). Under the same assumptions as in Main Conjecture, we then derive the algebraic equation satisfied by the asymptotic Cauchy transform of the scaled eigenpolynomials for any given degenerate exactly-solvable operator (see Main Corollary). From this equation it is possible to obtain detailed information on the asymptotic zero distribution of the scaled eigenpolynomials.

These are our main results:

**Theorem 1.**

Let \( T = \sum_{j=1}^k Q_j D^j \) be a degenerate exactly-solvable operator of order \( k \), and let \( r_n \) be the largest modulus of all roots of the unique and monic \( n \)th degree eigenpolynomial \( p_n \) of \( T \). Then \( r_n \to \infty \) as \( n \to \infty \).

**Main Conjecture.** Let \( T = \sum_{j=1}^k Q_j D^j \) be a degenerate exactly-solvable operator of order \( k \), and denote by \( j_0 \) the largest \( j \) for which \( \deg Q_j = j \). Denote by \( r_n \) be the largest modulus of all roots of the unique and monic \( n \)th degree eigenpolynomial \( p_n \) of \( T \). Then

\[
\lim_{n \to \infty} \frac{r_n}{n^d} = c_0,
\]

where \( c_0 > 0 \) is a positive constant and

\[
d := \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right).
\]

Based on this Main Conjecture, we now introduce the scaled eigenpolynomial \( q_n(z) = p_n(n^d z) \), for which the union of all roots are (conjecturally) contained

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\(^2\)This theorem is joint work with H. Rullgård.
in a compact set. We then make the following basic assumption: assume that $C_n(z) := C_{n,0}(z) = C_{n,1}(z) = \ldots = C_{n,k-1}(z)$ as $n \to \infty$ for the Cauchy transforms$^3$ of the scaled eigenpolynomial $q_n(z)$ and its derivatives. This means that we assume that the root measures $\mu_n^0, \mu_n^1, \mu_n^2, \ldots, \mu_{n,k-1}$ of $q_n, q_n^{(1)}, q_n^{(2)}, \ldots, q_n^{(k-1)}$ respectively, are all equal as $n \to \infty$, and let $C(z) := \lim_{n \to \infty} C_n(z)$ (computer experiments strongly indicate that this assumption is true, see Section 4.3).

Now let $T = \sum_{j=1}^{k} Q_j D^j = \sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} z^i \right) D^j$ be a degenerate exactly-solvable operator and denote by $\alpha_{j,0}$ the largest $j$ such that $\deg Q_j = j$. Moreover, with no loss of generality, we make a normalization by assuming that $Q_{j,0}$ is monic, i.e. $\alpha_{j,0,j_0} = 1$. Consider the scaled polynomial $q_n(z) = p_n(n^dz)$, where $p_n(z)$ is the unique and monic $n$th degree eigenpolynomial of $T$, and $d := \max_{j \in [j_0+1,k]} \left( \frac{\deg Q_j}{j} \right)$. We then have the following:

**Main Corollary.** Assume that $C_n(z) := C_{n,0}(z) = C_{n,1}(z) = \ldots = C_{n,k-1}(z)$ when $n \to \infty$ for the Cauchy transforms of the scaled eigenpolynomial $q_n(z)$ and its derivatives. Then, for almost all complex $z$ in the usual Lebesgue measure on $C$, the function $C(z) := \lim_{n \to \infty} C_n(z)$ satisfies the following equation:

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1.$$

Here $A$ is the set consisting of all $j$ for which the maximum $d := \max_{j \in [j_0+1,k]} \left( \frac{\deg Q_j}{j} \right)$ is attained, i.e. $A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$ where $d$ is as above.

In the following theorem we prove a lower bound for the largest modulus of all roots of $p_n$ when $n \to \infty$ for any degenerate exactly-solvable operator:

**Theorem 2.** Let $T = \sum_{j=1}^{k} Q_j D^j = \sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} z^i \right) D^j$ be a degenerate exactly-solvable operator of order $k$. Let $z_n$ be the root with the largest modulus, $|z_n| = r_n$, of the unique and monic $n$th degree eigenpolynomial $p_n$ of $T$. Then there exists a positive constant $c_0 > 0$ such that

$$\lim_{n \to \infty} \frac{r_n}{(n - k + 1)^\gamma} > c_0,$$

$^3$If $q_n$ is a polynomial of degree $n$ we construct the probability measure $\mu_n$ by placing a point mass of size $\frac{1}{n}$ at each zero of $q_n$. We call $\mu_n$ the root measure of $q_n$. By definition, for any polynomial $q_n$, the Cauchy transform $C_{n,j}$ of the root measure $\mu_{n,j}^{(j)}$ for the $j$th derivative $q_n^{(j)}$ is defined by

$$C_{n,j}(z) := \frac{q_n^{(j)}(z)}{(n - j)q_n^{(j)}(z)} = \int \frac{d\mu_n^{(j)}(\zeta)}{z - \zeta},$$

and it is well-known that the measure $\mu$ can be reconstructed from $C$ by the formula $\mu = \frac{1}{\pi} \frac{\partial C}{\partial z}$, where $\partial/\partial z = \frac{1}{2} (\partial/\partial x + i \partial/\partial y)$. 


for any $\gamma < b$ where

$$b := \min_{j \in [1, k - 1]} \left( \frac{k - j}{k - j + \deg Q_j - \deg Q_k} \right),$$

where the notation $\min^+$ means that the minimum is taken only over positive terms $(k - j + \deg Q_j - \deg Q_k)$.

The following two theorems are partial results supporting Main Conjecture:

**Theorem 3.** Let $T$ be a degenerate exactly solvable operator of order $k$ consisting of precisely two terms: $T = Q_j D_j + Q_k D_k$. Let $z_n$ be the root with the largest modulus of the unique and monic $n$th degree eigenpolynomial $p_n$ of $T$, and let $|z_n| = r_n$. Then there exists a positive constant $c > 0$ such that

$$\lim_{n \to \infty} \frac{r_n}{(n - k + 1)^d} \geq c,$$

where $d := \max_{j \in [k_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) = \frac{k - j_0}{k - \deg Q_k}$.

This result can be generalized, but with certain conditions on the polynomials $Q_j$ for $j > j_0$:

**Theorem 4.** Let $T$ be a degenerate exactly-solvable operator of order $k$. Denote by $j_0$ the largest $j$ such that $\deg Q_j = j$. Furthermore, let $(j - \deg Q_j) \geq (k - \deg Q_k)$ for every $j > j_0$. Let $z_n$ be the root with the largest modulus of the unique and monic $n$th degree eigenpolynomial $p_n$ of $T$, and let $|z_n| = r_n$. Then there exists a positive constant $c > 0$ such that

$$\lim_{n \to \infty} \frac{r_n}{(n - k + 1)^d} \geq c,$$

where $d := \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) = \frac{k - j_0}{k - \deg Q_k}$.

Conjecturally, for any degenerate exactly-solvable operator $T$, the support of the asymptotic zero distribution of the scaled eigenpolynomial $q_n$ is the union of a finite number of analytic curves in the complex plane, which we denote by $\Xi_T$. We then have the following conjecture:

**Conjecture 1.** [Interlacing property] For any family $\{q_n\}$ of appropriately**5** scaled polynomial eigenfunctions of any degenerate exactly-solvable operator $T$, the zeros of any two consecutive polynomials $q_{n+1}$ and $q_n$ interlace along $\Xi_T$ for all sufficiently large integers $n$.6

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**Footnotes:**

4 We believe the interlacing property also holds for the non-degenerate exactly-solvable operators, but without such a scaling of the eigenpolynomials.

5 According to the scaling in Main Conjecture.

6 The question concerning interlacing was raised by B. Shapiro. For details see Section 4.4.
We now present some typical pictures of the zero distribution of the scaled eigenpolynomials for some degenerate exactly-solvable operators. Below, \( p_n \) denotes the \( n \)-th degree monic polynomial eigenfunction of a given operator \( T \), and \( q_n \) denotes the corresponding (appropriately) scaled polynomial.

**Fig.1:** \( T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5 \).

![Roots of \( q_{50}(z) = p_{50}(50z) \)](image)

![Roots of \( q_{75}(z) = p_{75}(75z) \)](image)

![Roots of \( q_{100}(z) = p_{100}(100z) \)](image)

**Fig.2:** \( T_2 = z^2D^2 + D^7 \).

![Roots of \( q_{50}(z) = p_{50}(50^{5/7}z) \)](image)

![Roots of \( q_{75}(z) = p_{75}(75^{5/7}z) \)](image)

![Roots of \( q_{100}(z) = p_{100}(100^{5/7}z) \)](image)

**Fig.3:** \( T_3 = z^3D^3 + z^2D^4 + zD^5 \).

![Roots of \( q_{50}(z) = p_{50}(50^{1/2}z) \)](image)

![Roots of \( q_{75}(z) = p_{75}(75^{1/2}z) \)](image)

![Roots of \( q_{100}(z) = p_{100}(100^{1/2}z) \)](image)

The algebraic equation satisfied by the asymptotic Cauchy transform in Main Corollary indicates that the asymptotic zero distribution of the scaled eigenpolynomials depends only on the term \( z^nD^m \) and the term(s) \( \alpha_j \deg Q_j z^{\deg Q_j} D^j \) of
For which $d = \max_{j \in [n+1, k]} \left( \frac{1}{j-\deg q_j} \right)$ is attained. To illustrate this fact we present below some pictures of the zero distributions of the scaled eigenpolynomials for some distinct operators for which the asymptotic Cauchy transform $C(z)$ of the scaled eigenpolynomials satisfy the \textit{same} equation.

As a first example, consider the operator $T_4 = z^3 D^3 + z^2 D^5$. Clearly $d = \max_{j \in [n+1, k]} \left( \frac{1}{j-\deg q_j} \right) = (5 - 3)/(5 - 2) = 2/3$ and the asymptotic Cauchy transform of the scaled eigenpolynomial $q_n(z) = P_n(n^{2/3} z)$ satisfies the equation $z^3 C^3 + z^2 C^5 = 1$ in the limit when $n \to \infty$. Now consider the slightly modified operator $\tilde{T}_4 = z^2 D^2 + z D^4 + z^2 D^6 + D^8$ and note that $d = \max_{j \in [n+1, k]} \left( \frac{1}{j-\deg q_j} \right)$ is obtained again (only) for $j = 5$ (for $j = 4$ we have $(4 - 3)/(4 - 1) = 1/3 < 2/3$ and for $j = 6$ we have $(6 - 3)/(6 - 0) = 3/6 = 1/2 < 2/3$). We therefore obtain the same asymptotic equation in $C(z)$ for the scaled eigenpolynomials of $\tilde{T}_4$ as for the scaled eigenpolynomials of $T_4$; hence we can consider the added terms $z^2 D^2$, $z D^4$ and $D^6$ in $\tilde{T}_4$ as "irrelevant" for the asymptotic zero distribution. The pictures below clearly illustrate this:

\[ T_4 = z^3 D^3 + z^2 D^5, \quad \text{roots of } q_{100}(z) = P_{100}(100^{2/3} z) \]

\[ \tilde{T}_4 = z^2 D^2 + z^3 D^3 + z D^4 + z^2 D^5 + D^6, \quad \text{roots of } q_{100}(z) = P_{100}(100^{2/3} z) \]

As a second example, consider the operators $T_5 = z^5 D^5 + z^4 D^6 + z^2 D^8$ and $\tilde{T}_5 = z^2 D^2 + z^5 D^5 + z^4 D^6 + z D^7 + z^2 D^8$ whose scaled eigenpolynomials $q_n(z) = P_n(n^{1/2} z)$ both satisfy the Cauchy transform equation $z^5 C^5 + z^4 C^6 + z^2 C^8$ in the limit when $n \to \infty$. In the pictures below one can see that the "irrelevant" terms $z^2 D^2$ and $z D^7$ of $\tilde{T}_5$ seem to have no affect on the zero distribution of the scaled eigenpolynomials for sufficiently large $n$.

\[ T_5 = z^5 D^5 + z^4 D^6 + z^2 D^8, \quad \text{roots of } q_{100}(z) = P_{100}(100^{1/2} z) \]

\[ \tilde{T}_5 = z^2 D^2 + z^5 D^5 + z^4 D^6 + z D^7 + z^2 D^8, \quad \text{roots of } q_{100}(z) = P_{100}(100^{1/2} z) \]
In the sequel we will settle our Main Conjecture for some special classes of degenerate exactly-solvable operators, and then describe the asymptotic zero distribution of the scaled polynomial eigenfunctions for these operators in detail.

Let us finally mention some possible applications of our results and directions for further research. Operators of the type we consider occur, as was mentioned earlier, in the theory of Bochner-Krall orthogonal systems. A great deal is known about the asymptotic zero distribution of orthogonal polynomials, and by comparing such results with results on the asymptotic zero distribution of eigenpolynomials of degenerate exactly-solvable operators, we believe it will be possible to gain new insight into the nature of BKS.

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2 Proofs

We start with the following

Lemma 1. Let \( T = \sum_{j=1}^{k} Q_j D^j \) be a degenerate exactly-solvable operator of order \( k \). Then, for a sufficiently large integer \( n \), there exists a unique constant \( \lambda_n \) and a unique monic polynomial \( p_n \) of degree \( n \) which satisfy \( T(p_n) = \lambda_n p_n \).

If \( \deg Q_j = j \) for precisely one value \( j < k \), then there exists a unique constant \( \lambda_n \) and a unique monic polynomial \( p_n \) of degree \( n \) which satisfy \( T(p_n) = \lambda_n p_n \) for every \( n = 1, 2, \ldots \).

Proof of Lemma 1. In [1] we proved that for any exactly-solvable operator \( T \), the eigenvalue problem \( T(p_n) = \lambda_n p_n \) can be written as a linear system \( MX = Y \), where \( X \) is the coefficient vector of the monic \( n \)th degree eigenpolynomial \( p_n \) with components \( a_n,0, a_n,1, a_n,2, \ldots, a_n,n-1 \), \( Y \) is a vector and \( M \) is an upper triangular \( n \times n \) matrix, both with entries expressible in the coefficients of \( T \). With \( T = \sum_{j=1}^{k} Q_j D^j \), \( Q_j = \sum_{i=0}^{j} \alpha_{j,i} z^i \), and \( p_n(z) = \sum_{i=0}^{n} a_{n,i} z^i \), the eigenvalue \( \lambda_n \) is given by

\[
\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!},
\]

and the diagonal elements of \( M \) are given by

\[
M_{i+1,i+1} = \sum_{1 \leq j \leq \min(i,k)} \alpha_{j,j} \frac{i!}{(i-j)!} - \lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \left[ \frac{i!}{(i-j)!} - \frac{n!}{(n-j)!} \right].
\]
Then, for any complex number $j$.

**Lemma 2.** From the expression

\[ -M_{i+1,i+1} = \sum_{j=1}^{k} \alpha_{j,j} \left[ \frac{n!}{(n-j)!} - \frac{i!}{(i-j)!} \right] \]

it is clear that $M_{i+1,i+1} \neq 0$ for every $i \in [0, n-1]$ and every $n$ if $\alpha_{i,j} \neq 0$ for precisely one $j$, that is if $\deg Q_j = j$ for precisely one $j$ - thus we have proved the second part of Lemma 1.

Now assume that $\deg Q_j = j$ for more than one $j$ and denote by $j_0$ the largest such $j$ (clearly $\alpha_{j_0,j_0} \neq 0$). We then have

\[ -M_{i+1,i+1} = \sum_{j=1}^{j_0} \alpha_{j,j} \left[ \frac{n!}{(n-j)!} - \frac{i!}{(i-j)!} \right] \]

\[ = \frac{n!}{(n-j_0)!} \left[ \alpha_{j_0,j_0} \left( 1 - \frac{i!/(i-j_0)!}{n!/(n-j_0)!} \right) + \sum_{1 \leq j < j_0} \alpha_{j,j} \left( \frac{(n-j_0)!}{(n-j)!} - \frac{(n-j_0)!i!}{n!(i-j)!} \right) \right] \]

The last two sums on the right-hand side of the equality above tend to zero as $n \to \infty$, since $j_0 > j$ and $i \leq n-1$. Thus for sufficiently large $n$ we have

\[ -M_{i+1,i+1} = \frac{n!}{(n-j_0)!} \left[ \alpha_{j_0,j_0} \left( 1 - \frac{i!/(i-j_0)!}{n!/(n-j_0)!} \right) \right] \neq 0 \]

for every $i \in [0, n-1]$, and we have proved the first part of Lemma 1.

To prove Theorem 1 we need the following lemma. Recall that $\frac{p_n^{(j)}(z)}{(n-j)!} = \frac{1}{z^n} \frac{dz_n(z)}{dz_n(z)} =: C_{n,j}(z)$. Then we have:

**Lemma 2.** Let $z_n$ be the root of $p_n$ with the largest modulus, say $|z_n| = r_n$. Then, for any complex number $z_0$ such that $|z_0| = r_0 \geq r_n$, we have $|C_{n,j}(z_0)| \geq \frac{1}{z_0} \frac{1}{1-\zeta}$ for all $j \geq 0$.

**Proof.** With $\zeta$ being some root of $p_n^{(j)}$ we have $|\zeta| \leq |z_0|$ by Gauss Lucas’ Theorem. Thus $\frac{1}{z_0} \frac{1}{1-\zeta} = \frac{1}{z_0} \frac{1}{1-\zeta/z_0} = \frac{1}{z_0} \frac{1}{1-\theta}$ where $|\theta| = |\zeta/z_0| \leq 1$. With
$w = \frac{1}{1-\theta}$ we obtain

$$|w - 1| = \frac{|\theta|}{|1-\theta|} = |\theta||w| \leq |w| \Leftrightarrow |w - 1| \leq |w| \Rightarrow Re(w) \geq 1/2.$$ 

Thus

$$|C_{n,j}(z_0)| = \left| \int \frac{d\mu_n^{(j)}(\zeta)}{z_0 - \zeta} \right| = \frac{1}{r_0} \left\{ \int \frac{d\mu_n^{(j)}(\zeta)}{1-\theta} \right\} = \frac{1}{r_0} \left\{ \int w d\mu_n^{(j)}(\zeta) \right\} \geq \frac{1}{r_0} \left\{ \int Re(w) d\mu_n^{(j)}(\zeta) \right\} \geq \frac{1}{2r_0} \frac{1}{2}.$$

**Proof of Theorem 1.** Take $T = \sum_{j=1}^{k} Q_j D^j$ and denote by $j_0$ the largest $j$ such that $\deg Q_j = j$ (clearly $j_0 < k$). From the definition of $C_{n,j}$ we get

$$p_n^{(j)}(z) = C_{n,0}(z)C_{n,1}(z) \cdots C_{n,j-1}(z) \cdot n(n-1) \cdots (n-j+1)$$

$$= \frac{n!}{(n-j)!} \sum_{m=0}^{j-1} C_{n,m}(z).$$

With $Q_j(z) = \sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i$ we have $\lambda_n = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}$, and dividing the eigenvalue equation $T(p_n(z)) = \lambda_n p_n(z)$ by $p_n(z)$ we thus obtain

$$Q_k(z) \frac{p_n^{(k)}}{p_n(z)} + Q_{k-1}(z) \frac{p_n^{(k-1)}}{p_n(z)} + \cdots + Q_1(z) \frac{p_n'}{p_n(z)} = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}$$

$$\Leftrightarrow$$

$$Q_k(z) \frac{n!}{(n-k)!} \prod_{m=0}^{k-1} C_{n,m}(z) + Q_{k-1}(z) \frac{n!}{(n-k+1)!} \prod_{m=0}^{k-2} C_{n,m}(z) + \cdots$$

$$+ Q_1(z) \frac{n!}{(n-1)!} C_{n,0}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}. \quad (2)$$

Now assume that the largest modulus $r_n$ of all roots of $p_n$ (and hence, by Gauss Lucas’ Theorem, of any derivative $p_n^{(j)}$) is (strictly) less than some fixed constant $R < \infty$. We can always assume that $R$ is (strictly) larger than the largest absolute value of all roots of $Q_k$. Now let $\bar{z}$ be such that $|\bar{z}| = R$. Then

$$\frac{1}{2\pi} \leq |C_{n,j}(\bar{z})| \text{ by Lemma 2.}$$

Inserting $\bar{z}$ in equation (2) we obtain:

$$Q_k(\bar{z}) \frac{n!}{(n-k)!} \prod_{m=0}^{k-1} C_{n,m}(\bar{z}) + Q_{k-1}(\bar{z}) \frac{n!}{(n-k+1)!} \prod_{m=0}^{k-2} C_{n,m}(\bar{z}) + \cdots$$

$$+ Q_1(\bar{z}) \frac{n!}{(n-1)!} C_{n,0}(\bar{z}) = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}.$$
Note that by the choice of ~ \( \tilde{z} \) clearly \( Q_k(\tilde{z}) \neq 0 \) and \( p_n(\tilde{z}) \neq 0 \). Dividing both sides of this equation by \( \frac{n!}{(n-k)!} \) we get

\[
Q_k(\tilde{z}) \prod_{m=0}^{k-1} C_{n,m}(\tilde{z}) \left[ 1 + \frac{(n-k)!}{(n-k+1)!} \frac{1}{C_{n,k-1}(\tilde{z})} Q_{k-1}(\tilde{z}) + \frac{(n-k)!}{(n-k+2)!} \frac{1}{C_{n,k-1}(\tilde{z}) C_{n,k-2}(\tilde{z})} Q_{k-2}(\tilde{z}) + \cdots + \frac{(n-k)!}{(n-1)!} \frac{1}{\prod_{m=1}^{k-1} C_{n,m}(\tilde{z})} Q_1(\tilde{z}) \right] = \sum_{j=1}^{k} \alpha_{j} \frac{(n-k)!}{(n-j)!}.
\]

(3)

In this equation, the right-hand side tends to zero when \( n \to \infty \) since \( j_0 < k \). On the other hand, in the left-hand side of (3), the terms in the bracket (except for the constant term 1) all tend to zero when \( n \to \infty \), since \( \left| \frac{1}{C_{n,m}(\tilde{z})} \right| \leq 2R \) and \( R < \infty \) by assumption. Thus, for sufficiently large \( n \), we can find a positive constant \( K_n \), with \( \lim_{n \to \infty} K_n = 1 \), such that the modulus of the left-hand side of equation (3) equals

\[
|\text{LHS}| = K_n \cdot |Q_k(\tilde{z})| \prod_{m=0}^{k-1} |C_{n,m}(\tilde{z})| \geq K_n \cdot |Q_k(\tilde{z})| \frac{1}{2^k R^k} = K_0 > 0
\]

when \( n \to \infty \) for some positive constant \( K_0 > 0 \), since \( R < \infty \). Thus we obtain the contradiction \( K_0 \leq 0 \) when \( n \to \infty \), and therefore the largest modulus \( r_n \) of all roots of \( p_n \) must tend to infinity when \( n \to \infty \).

In order to prove Theorem 2 we need the following lemma:

**Lemma 3.** Let \( T = \sum_{j=1}^{k} Q_j D^j = \sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} z^i \right) D^j \) be a degenerate exactly-solvable operator of order \( k \). With no loss of generality we assume that \( Q_k \) is monic, i.e. \( \alpha_{k,\deg Q_k} = 1 \). Let \( z_n \) be the root with the largest modulus of all roots of the unique and monic \( n \)-th degree eigenpolynomial \( p_n \) of \( T \), and let \( |z_n| = r_n \). Then the following inequality holds:

\[
1 \leq \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_k + 1} |\alpha_{j,i}| \frac{r_n^{k-j} (n-k+1)^{k-j}}{(n-k+1)(n-k+2) \cdots (n-j) \prod_{m=1}^{n-1} C_{n,m}(z)} \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{i,j}|}{r_n^{\deg Q_k - i}}.
\]

**Proof of Lemma 3.** From the definition \( C_{n,j}(z) = \frac{p^{(j+1)}(z)}{(n-j)!p^{(i)}(z)} \) we easily derive

\[
p^{(j)}(z) = \frac{p^{(k)}(z)}{(n-k+1)(n-k+2) \cdots (n-j) \prod_{m=1}^{n-1} C_{n,m}(z)} \quad \forall \quad j < k.
\]

(4)

Inserting \( z_n \) in our eigenvalue equation \( T(p_n(z)) = \lambda_n p_n(z) \) we obtain

\[
\sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_k} \alpha_{j,i} z_n^i p^{(j)}(z_n) + \left( \sum_{i=0}^{\deg Q_k} \alpha_{k,i} z_n^i \right) p^{(k)}(z_n) = \lambda_n p_n(z_n) = 0.
\]

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Dividing this equation by $z_n^{\deg Q_k}p_n^{(k)}(z_n)$ we get
\[
\sum_{j=1}^{k-1} \left( \sum_{i=0}^{j} \alpha_{j,i} \frac{1}{z_n^{\deg Q_k-i}} \right) p_n^{(j)}(z_n) + \sum_{0 \leq i < \deg Q_k} \alpha_{k,i} \frac{1}{z_n^{\deg Q_k-i}} + 1 = 0,
\]
and from this, using (4) and Lemma 2, we obtain the following inequality:
\[
1 = \left| \sum_{j=1}^{k-1} \left( \sum_{i=0}^{j} \alpha_{j,i} \frac{1}{z_n^{\deg Q_k-i}} \right) p_n^{(j)}(z_n) + \sum_{0 \leq i < \deg Q_k} \alpha_{k,i} \frac{1}{z_n^{\deg Q_k-i}} \right| \\
\leq \sum_{j=1}^{k-1} \left| \sum_{i=0}^{j} \alpha_{j,i} \frac{1}{z_n^{\deg Q_k-i}} \right| + \sum_{0 \leq i < \deg Q_k} \frac{1}{z_n^{\deg Q_k-i}} \\
\leq \sum_{j=1}^{k-1} \sum_{i=0}^{j} |\alpha_{j,i}| \frac{1}{r_n^{\deg Q_k-i} (n-k+1)^{j-k+1}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k-i}} \\
= \sum_{j=1}^{k-1} \sum_{i=0}^{j} |\alpha_{j,i}| \frac{2^{j-k}}{(n-k+1)^{j-k}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k-i}}.
\]

\[\square\]

Now, using Theorem 1 and Lemma 3, we can prove Theorem 2:

**Proof of Theorem 2.** Consider the inequality in Lemma 3. Applying Theorem 1 we see that the last sum on the right-hand side of this inequality tends to zero as $n \to \infty$.

Now consider the double sum on the right-hand side of the inequality in Lemma 3. If the exponent $(k-j-\deg Q_k+i)$ of $r_n$ for given $i$ and $j$ is negative or zero, the corresponding term tends to zero when $n \to \infty$ by Theorem 1. Consider the remaining terms in the double sum, namely those for which the exponent $(k-j-\deg Q_k+i)$ of $r_n$ is positive. Assume that $r_n \leq c_0(n-k+1)^\gamma$ where $c_0 > 0$ is a positive constant and $\gamma < \frac{k-j+i-\deg Q_k}{k-j}$ for given $i \in [1, k-1]$ and given $i \in [0, j]$. Then for the corresponding term in the double sum we get
\[
\frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} = \left( \frac{r_n^{k-j}}{(n-k+1)^{k-j+i-\deg Q_k}} \right)^{k-j+i-\deg Q_k} \to 0
\]
when $n \to \infty$. Assume that $r_n \leq c_0(n-k+1)^\gamma$ where $c_0 > 0$ is a positive constant and $\gamma < b$, where
\[
b = \max_{j \in [0, k-1]} \frac{k-j}{k-j+i-\deg Q_k} = \max_{j \in [1, k-1]} \frac{k-j}{k-j+i-\deg Q_k} = \frac{k-j}{k-j+i-\deg Q_k}.
\]
The notation \( \min^+ \) means that we only take the minimum over positive terms 
\((k - j + \deg Q_j - \deg Q_k)\). (Above we have written the minimum over \(i \in [0,j]\),
but actually \(i \in [0,\deg Q_j]\) for any given \(j\), so since we look for the minimal value we can put \(i = \deg Q_j\) in this expression). Then \(\gamma < \frac{k-j}{k-j+i+\deg Q_k}\) for every \(j \in [1,k-1]\) and every \(i \in [0,j]\); thus every term with positive exponent \((k - j + i - \deg Q_k)\) will tend to zero when \(n \to \infty\). Therefore, assuming that \(r_n \leq c_0(n - k + 1)^\gamma\) and \(\gamma < b\) where \(b\) is as above, we get that every term on the right-hand side of the inequality in Lemma 3 tends to zero as \(n \to \infty\), and we arrive at the contradiction \(1 \leq 0\). From this we conclude that for sufficiently large choices on \(n\) there must exist a positive constant \(c_0 > 0\) such that \(r_n > c_0(n - k + 1)^\gamma\) for all \(\gamma < b\), where \(b\) is as above, and hence \(\lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} > c_0\) for any \(\gamma < b\). \(\square\)

We have conjectured that \(\lim_{n \to \infty} \frac{r_n}{n^p} = c_0 > 0\) for the largest modulus \(r_n\) of all roots of \(p_n\) for all degenerate exactly-solvable operators, where \(d := \max_{j \in [j_0+1,k]} \left( \frac{j-j_0}{j - \deg Q_j} \right)\) and \(j_0\) is the largest \(j\) such that \(\deg Q_j = j\). Thus, if the following condition is fulfilled:

\[
b := \min_{j \in [1,k-1]} \left( \frac{k-j}{k-j+\deg Q_j-\deg Q_k} \right) = \max_{j \in [1,k]} \left( \frac{j-j_0}{j - \deg Q_j} \right) := d,
\]

then there exists a positive constant \(c_0 > 0\) such that \(\lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} > c_0\) for any \(\gamma < d\).

Below we describe two classes of degenerate exactly-solvable operators for which the above condition is satisfied, namely:

**Corollary 1.** Let \(T = \sum_{j=1}^k Q_j D^j\) be a degenerate exactly-solvable operator of order \(k\) such that \(\deg Q_j \leq j_0\) for all \(j > j_0\), and in particular \(\deg Q_k = j_0\), where \(j_0\) is the largest \(j\) such that \(\deg Q_j = j\). If \(r_n\) is the largest modulus of all roots of the unique and monic \(n\)th degree eigenpolynomial \(p_n\) of \(T\), then there exists a positive constant \(c_0 > 0\) such that \(\lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} > c_0\) for any \(\gamma < 1\).

**Proof of Corollary 1.** For this class of operators it is conjectured that \(\lim_{n \to \infty} \frac{r_n}{n} = c_0 > 0\), since

\[
d := \max_{j \in [j_0+1,k]} \left( \frac{j-j_0}{j - \deg Q_j} \right) = \frac{k-j_0}{k-j_0} = 1.
\]

The maximum is attained by choosing any \(j > j_0\) with \(\deg Q_j = j_0\), e.g. \(j = k\). Also, for this class of operators we have

\[
b := \min_{j \in [1,k-1]} \left( \frac{k-j}{k-j+\deg Q_j-\deg Q_k} \right) = \max_{j \in [1,k-1]} \left( \frac{k-j}{k-j+\deg Q_j-j_0} \right) = \frac{k-j_0}{k-j_0} = 1,
\]

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and the proof is complete by applying Theorem 2.

Corollary 2. Let \( T = \sum_{j=1}^{k} Q_j D^j \) be a degenerate exactly-solvable operator of order \( k \) such that \( \deg Q_j = 0 \) for all \( j > j_0 \), where \( j_0 \) is the largest \( j \) such that \( \deg Q_j = j \). Let \( r_n \) be the largest modulus of all roots of the unique and monic \( n \)th degree eigenpolynomial \( p_n \) of \( T \). Then there exists a positive constant \( c_0 > 0 \) such that \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^{-\gamma}} > c_0 \) for any \( \gamma < \frac{k-j_0}{k} \).

Proof of Corollary 2. For this class of operators it is conjectured that \( \lim_{n \to \infty} \frac{r_n}{n^{k-j_0}/k} = c_0 > 0 \), since

\[
d := \max_{j \in [j_0+1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) = \max_{j \in [j_0+1, k]} \left( \frac{j - j_0}{j} \right) = \frac{k - j_0}{k}.
\]

Also, for this class of operators we have

\[
b := \min_{j \in [1, k-1]} \left( \frac{k - j}{k - j + \deg Q_j - \deg Q_k} \right) = \min_{j \in [1, j_0]} \frac{k - j}{k} = \frac{k - j_0}{k}
\]

where the third equality follows choosing any \( j \) such that \( \deg Q_j = j \), and the minimum is attained for \( j = j_0 \) (for \( j > j_0 \) we get \( \frac{k - j}{k - j + \deg Q_j} = 1 > (k - j_0)/k \)), and the proof is complete by applying Theorem 2.

Remark. For the classes of operators considered in Corollary 1 and Corollary 2 we can actually prove that \( \lim_{n \to \infty} \frac{r_n}{n^{k-j_0}/k} \geq c_0 \), where \( d \) is as in Main Conjecture, if we assume that we already have the upper bound \( \lim_{n \to \infty} \frac{r_n}{n^{k-j_0}/k} \leq c_1 \) for some positive constant \( c_1 \), see Section 5.

Proof of Theorem 3. First we note that \( \deg Q_{j_0} = j_0 \) since there must exist at least one such \( j < k \). Let

\[
T = Q_{j_0} D^{j_0} + Q_k D^k = \sum_{j=0}^{j_0} \alpha_{j_0, i} z^i D^{j_0} + \sum_{i=0}^{\deg Q_k} \alpha_{k, i} z^i D^k,
\]

where \( \alpha_{j_0, 0} \neq 0 \), and where we wlog assume that \( Q_k \) is monic. From Lemma 3 we get:

\[
1 \leq \sum_{i=0}^{j_0} |\alpha_{j_0, i}| 2^{k-j_0} \frac{r_n^{-\deg Q_k + k-j_0}}{(n-k+1)^{k-j_0}} + \sum_{0 \leq i < \deg Q_k} |\alpha_{k, i}| \frac{1}{r_n^{\deg Q_k - i}} \leq \sum_{i=0}^{j_0} |\alpha_{j_0, i}| 2^{k-j_0} \frac{r_n^{-\deg Q_k + k-j_0}}{(n-k+1)^{k-j_0}} + \epsilon,
\]

where we choose \( n \) large enough that \( \epsilon < 1 \) (this is possible since \( \epsilon \to 0 \) when
$n \to \infty$). Thus for sufficiently large $n$ we have the following inequality:

$$c_0 \leq \sum_{i=0}^{\tilde{j}_0} (\alpha_{j_0,i} |2^{k-j_0} r_n^{1-\deg Q_k+k-j_0} (n-k+1)^{k-j_0})$$

$$\leq \sum_{i=0}^{\tilde{j}_0} (\alpha_{j_0,i} |2^{k-j_0} r_n^{k-\deg Q_k} (n-k+1)^{k-j_0})$$

$$= K \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}},$$

where $1 - \epsilon = c_0 > 0$ and $K > 0$ since $\alpha_{j_0,j_0} \neq 0$ (the last inequality follows since $i \leq \tilde{j}_0$). Thus

$$r_n \geq \frac{c_0}{K} (n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}$$

for sufficiently large $n$, and hence

$$\lim_{n \to \infty} \frac{r_n}{(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}} \geq \frac{c_0}{K} = c > 0.$$  

Finally, it is clear that for this two-term operator we have

$$d := \max_{j \in [\tilde{j}_0+1,k]} \left( \frac{j - \tilde{j}_0}{k - \deg Q_k} \right) = \frac{k-j_0}{k-\deg Q_k},$$

and we are done. \(\square\)

**Remark.** If $Q_k$ is a monomial ($Q_k = z^{\deg Q_k}$), then there exists a positive constant $c$ such that $r_n \geq c(n-k+1)^d$ for every $n$, where $d := \max_{j \in [\tilde{j}_0+1,k]} (\frac{j - \tilde{j}_0}{k - \deg Q_k}) = \frac{k-j_0}{k-\deg Q_k}$. This is easily seen from the calculations in the proof of Theorem 3 above. Note that the sum $\sum_{0 \leq i < \deg Q_k} |\alpha_{k,i} r_n^{1-\deg Q_k}}$ on the right-hand side of the inequality in Lemma 3 vanishes, and therefore $1 \leq K \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}$ for every $n$. Also, from the second part of Lemma 1 we know that for this class of operators there exists a unique monic $n$th degree eigenpolynomial for every $n$, and the conclusion follows.

**Proof of Theorem 4.** First, since $j \leq k$ and $(j - \deg Q_j) \geq (k - \deg Q_k)$ for every $j > \tilde{j}_0$ for this class of operators, it is clear that

$$d := \max_{j \in [\tilde{j}_0+1,k]} \left( \frac{j - \tilde{j}_0}{j - \deg Q_j} \right) = \frac{k-j_0}{k-\deg Q_k}.$$  

We assume, with no loss of generality, that $Q_k$ is monic, i.e. $\alpha_{k,\deg Q_k} = 1$. From Lemma 3 we then have the inequality

$$1 \leq \sum_{j=1}^{k-1} \sum_{i=0}^{j} |\alpha_{j,i}| 2^{k-j} r_n^{k-j+i-\deg Q_k} (n-k+1)^{k-j} + \sum_{0 \leq i < \deg Q_k} |\alpha_{k,i} r_n^{1-\deg Q_k-i}}.$$  

(5)
Clearly the last sum here tends to zero as $n \to \infty$ by Theorem 1. Considering the double sum on the right-hand side of the inequality above it is clear that for every $j$ we have, using $i \leq \deg Q_j$, that

$$
\sum_{i=0}^{\deg Q_j} |\alpha_{j,i}||2^{k-j}r_n^{k-j+i-\deg Q_k}/(n-k+1)^{k-j}| = \sum_{i=0}^{\deg Q_j} |\alpha_{j,i}||2^{k-j}r_n^{k-j+\deg Q_j-\deg Q_k}/(n-k+1)^{k-j}|r_n^{-\deg Q_j}
$$

$$
= \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \left( 2^{k-j}|\alpha_{j,\deg Q_j}| + \sum_{i<\deg Q_j} 2^{k-j}|\alpha_{j,i}|r_n^{-\deg Q_j} \right)
$$

$$
= K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}},
$$

where

$$
K_j^n = 2^{k-j}|\alpha_{j,\deg Q_j}| + \sum_{i<\deg Q_j} 2^{k-j}|\alpha_{j,i}|r_n^{-\deg Q_j} > 0,
$$

since $\alpha_{j,\deg Q_j} \neq 0$. Also, $K_j^n < \infty$, since $i \leq 0$, $\deg Q_j$ and thus $(i-\deg Q_j) < 0$ for every $j$ (note that $K_j^n \to 2^{k-j}|\alpha_{j,\deg Q_j}|$ when $n \to \infty$ due to Theorem 1). Thus, with the decomposition

$\mathcal{A} = \{ j : \deg Q_j < j \}$ and $(k-j+\deg Q_j - \deg Q_k) > 0$,

$\mathcal{B} = \{ j : \deg Q_j < j \}$ and $(k-j + \deg Q_j - \deg Q_k) \leq 0$,

and using (6), inequality (5) is equivalent to:

$$
1 \leq \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_k} |2^{k-j}r_n^{k-j+i-\deg Q_k}/(n-k+1)^{k-j}| + \sum_{0 \leq i < \deg Q_k} |\alpha_{k,i}|r_n^{-\deg Q_k-i}
$$

$$
= \sum_{j \in \mathcal{A}} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in \mathcal{B}} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}}
$$

$$
+ \sum_{j \in \mathcal{C}} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} |\alpha_{k,i}|r_n^{-\deg Q_k-i}.
$$

Consider the last two sums on the right hand side of this inequality. They both tend to zero as $n \to \infty$, the last one due to Theorem 1, and the sum over $\mathcal{C}$ since we have $(j-\deg Q_j) \geq (k-j - \deg Q_k) \Rightarrow (k-j + \deg Q_j - \deg Q_k) \leq 0$ for every $j \in \mathcal{C}$ by assumption, and then applying Theorem 1.

Therefore, in the limit when $n \to \infty$, we get the inequality

$$
c_0 \leq \sum_{j \in \mathcal{A}} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in \mathcal{B}} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}}
$$

$$
(7)
$$

where

$$
0 < c_0 = 1 - \sum_{j \in \mathcal{C}} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} - \sum_{0 \leq i < \deg Q_k} |\alpha_{k,i}|r_n^{-\deg Q_k-i}.
$$
for sufficiently large $n$.

Now assume that the set $B$ is empty. This corresponds to an operator with
$(j - \deg Q_j) \geq (k - \deg Q_k)$ for every $j$ for which $\deg Q_j < j$. Then the inequality (7) above becomes
\[
c_0 \leq \sum_{j \in A} K_j^n \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j}}
= \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j}} \left( K_{j_0}^n + \sum_{j \in A \setminus \{j_0\}} K_j^n \frac{1}{(n-k+1)^{j-j_0}} \right)
\leq K_A \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j_0}}
\]
where $K_A > 0$ is a positive constant (since $A$ is nonempty) which is finite when $n \to \infty$, since $j_0 - j > 0$ for every $j \in A \setminus \{j_0\}$ (recall that $j_0$ is the largest element in $A$ by definition). Thus for sufficiently large $n$ there exists a positive constant $c = c_0/K_A > 0$ such that
\[
r_n \geq c(n-k+1)^{k-j_0}
\]
and thus
\[
\lim_{n \to \infty} \frac{r_n}{c(n-k+1)^{k-j_0}} \geq c > 0.
\]

Now assume that $B$ is nonempty. Then for sufficiently large $n$ there exists a positive constant $c_0 > 0$ such that (as in the case of empty $B$) inequality (7) above holds:
\[
c_0 \leq \sum_{j \in A} K_j^n \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in B} K_j^n \frac{r_n^{k - j + \deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}}.
\]
For the sum over $A$ we previously concluded in (8) that there exists a positive and finite constant $K_A$ such that
\[
\sum_{j \in A} K_j^n \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j}} \leq K_A \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j_0}}
\]
for sufficiently large $n$, whence we get the following inequality from (7):
\[
c_0 \leq \sum_{j \in A} K_j^n \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in B} K_j^n \frac{r_n^{k - j + \deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}}\]
\[
\leq K_A \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j_0}} + \sum_{j \in B} K_j^n \frac{r_n^{k - j + \deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}}\]
\[
= \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j_0}} \left( K_A + \sum_{j \in B} K_j^n \frac{r_n^{\deg Q_j - j}}{(n-k+1)^{j-j_0}} \right)\]
\[
\leq K_{AB} \frac{r_n^{k - \deg Q_k}}{(n-k+1)^{k-j_0}}.
\]
Here $K_{AB}$ is a positive and finite constant for sufficiently large $n$ (note that $K_{AB} \to K_A$ when $n \to \infty$, since $(\deg Q_j - j) < 0$ and $j_0 - j > 0$ for every $j \in B$). Thus there exists a positive constant $c = c_0/K_{AB} > 0$ such that

$$r_n \geq c(n - k + 1)^{-\frac{k-j_0}{k-j_0+2}},$$

for sufficiently large choices on $n$, and thus

$$\lim_{n \to \infty} \frac{r_n}{(n - k + 1)^{-\frac{k-j_0}{k-j_0+2}}} \geq c > 0.$$

\[\square\]

3 How did we arrive at Main Conjecture?

Our Main Conjecture is based on calculations involving the Cauchy transform and is consistent with all experiments we have performed concerning the zero distribution of eigenpolynomials of degenerate exactly-solvable operators.

With $p_n$ being the unique monic $n$th degree eigenpolynomial of $T$, we define the corresponding scaled polynomial $q_n(z) = p_n(n^d z)$, where we need to find the positive number $d$ specific for each operator. For a polynomial $q_n$ of degree $n$, the Cauchy transform $C_n,j$ of the root measure $\mu_n$ for the $j$th derivative $q_n^{(j)}$ is given by

$$C_n,j(z) := \frac{q_n^{(j+1)}(z)}{(n-j)q_n^{(j)}(z)} = \int \frac{d\mu_n^{(j)}(\zeta)}{z - \zeta}.$$

From this definition we obtain

$$\prod_{i=0}^{j-1} C_{n,i}(z) = \frac{q_n^{(j+1)}(z)}{(n-j)q_n^{(j)}(z)} \cdot \frac{q_n^{(j)}(z)}{(n-j)q_n^{(j-1)}(z)} \cdot \frac{q_n^{(j-1)}(z)}{(n-j)q_n^{(j-2)}(z)} \cdot \ldots \cdot \frac{q_n^{(1)}(z)}{nq_n(z)}.$$

Now the basic assumption (see also section 4.3) we make to get our conjecture is the following. Assume that the Cauchy transforms of the scaled polynomial $q_n(z)$ and its derivatives are all equal when $n \to \infty$, i.e. $C_n(z) := C_n,0(z) = C_n,1(z) = \ldots = C_n,k-1(z)$ when $n \to \infty$. This means that we assume that the
root measures $\mu_n^0, \mu_n^1, \mu_n^2, \ldots, \mu_n^{k-1}$ of $q_n, q_n^{(1)}, q_n^{(2)}, \ldots, q_n^{(k-1)}$ respectively, are all equal as $n \to \infty$. Then

$$C_j^i(z) = \prod_{i=0}^{j-1} C_{n,i}(z) = \frac{q_n^{(j)}(z)}{n(n-1) \cdots (n-j+1) q_n(z)}.$$  \hspace{1cm} (9)

and with $C(z) := \lim_{n \to \infty} C_n(z)$ (we call this function the asymptotic Cauchy transform of $q_n$), we get

$$C_j^i(z) = \lim_{n \to \infty} C_j^i(z) = \lim_{n \to \infty} \prod_{i=0}^{j-1} C_{n,i}(z) = \frac{q_n^{(j)}(z)}{n(n-1) \cdots (n-j+1) q_n(z)}.$$  \hspace{1cm} (10)

With $q_n(z) = p_n(n^dz)$ the scaling factor $n^d$ is now appropriately chosen in the sense that we obtain a "nice" equation in the asymptotic Cauchy transform $C(z)$ for the scaled polynomials. Then the asymptotic zero distribution of the scaled polynomials will (conjecturally) be compactly supported.

Let $T = \sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} z^i \right) D_j$ be a degenerate exactly-solvable operator and denote by $j_0$ the largest $j$ such that $\deg Q_j = j$. Consider the equation

$$T(p_n(z)) = \lambda_n p_n(z)$$

where

$$\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1) \cdots (n-j+1).$$

Clearly this sum ends at $j_0$ since $\alpha_{j,j} = 0$ for all $j > j_0$ by definition of $j_0$. We then have

$$T(p_n(z)) = \lambda_n p_n(z)$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} z^i \right) p_n^{(j)}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1) \cdots (n-j+1) p_n(z)$$

Now letting $z \to n^dz$ in this equation we obtain

$$\sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} n^{di} z^i \right) p_n^{(j)}(n^dz) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1) \cdots (n-j+1) p_n(n^dz),$$

and making the substitution $q_n(z) = p_n(n^dz)$ the equation above will be equivalent to the following:

$$\sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} n^{di} \frac{z^i}{n^{d(j-i)}} \right) q_n^{(j)}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1) \cdots (n-j+1) q_n(z).$$
Dividing this equation by \( \frac{n!}{(n-j_0)!} q_n(z) = n(n-1) \cdots (n-j_0+1)q_n(z) \) we get

\[
\text{LHS} = \sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \right) \frac{q_n^{(j)}(z)}{n(n-1) \cdots (n-j_0+1)q_n(z)}
\]

= \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n(n-1) \cdots (n-j+1)}{n(n-1) \cdots (n-j_0+1)} = \text{RHS} \quad \text{(11)}

where \( \alpha_{j_0,j_0} \neq 0 \). Consider the right-hand side (RHS) of equation (11). Since \( j \leq j_0 \), all terms for which \( j < j_0 \) (if not already zero, which is the case if \( \alpha_{j,j} = 0 \), i.e. if \( \deg Q_j < j \)) tend to zero when \( n \to \infty \), and therefore

\[
\text{RHS} = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n(n-1) \cdots (n-j+1)}{n(n-1) \cdots (n-j_0+1)} \to \alpha_{j_0,j_0} = 1 \quad \text{as} \quad n \to \infty.
\]

Here we wlog have made a normalization by assuming that \( Q_{j_0} \) is monic, i.e. \( \alpha_{j_0,j_0} = 1 \).

Now consider the \( j \)th term in the sum on the left-hand side (LHS) of equation (11). Using (9) and (10) we get, for any given \( j \):

\[
\sum_{i=0}^{j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1) \cdots (n-j_0+1)q_n(z)} = \sum_{i=0}^{j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1) \cdots (n-j+1)} \cdot \frac{n(n-1) \cdots (n-j_0+1)}{n(n-1) \cdots (n-j_0+1)}
\]

\[
\to \sum_{i=0}^{j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)+j_0-j}} C_j^j(z) \quad \text{when} \quad n \to \infty.
\]

Thus, for the left-hand side of (11) we have

\[
\text{LHS} = \sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)+j_0-j}} \right) C_j^j(z) \quad \text{when} \quad n \to \infty.
\]
Adding up we have the following equation satisfied by the asymptotic Cauchy transform $C$ for the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$:

$$
\sum_{j=1}^{k} \left( \sum_{i=0}^{j} \alpha_{j,i} \frac{z_i}{n^{d(j-i)+j_0-j}} \right) C^j(z) = 1. 
$$

(12)

In order to make (12) a "nice" equation we need to (in order to avoid infinities in the denominator) impose the following condition on the exponent $d$ of $n$:

$$
d(j - i) + j_0 - j \geq 0 \iff d \geq \frac{j - j_0}{j - i}
$$

for all $j \in [1, k]$ and all $i \in [0, j]$. Therefore we take $d = \max_{j \in [1, k]} \left( \frac{j - j_0}{j - i} \right)$, but this maximum is clearly obtained for the maximum value of $i$ for a given $j$. Since $i \in [0, \deg Q_j]$ for any given $j$, we may as well put $i = \deg Q_j$. Our condition then becomes\(^7\) $d = \max_{j \in [1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right)$. But clearly we need only take this maximum over $j \in [j_0 + 1, k]$, since $j_0 < k$ and therefore there always exists a positive value on $d$ for any operator of the type we consider; thus our condition becomes:

$$
d = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right).
$$

This is how we arrived at the scaling factor $n^d$. If we put this $d$ into equation (12) and let $n \to \infty$ we obtain an equation satisfied by the asymptotic Cauchy transform of the scaled polynomial $q_n(z) = p_n(n^d z)$ - namely the algebraic equation in Main Corollary.

**Arriving at Main Corollary.** We insert $d$ in (12), where $d$ is as above (i.e. as in Main Conjecture). We then get the following equation:

$$
\sum_{j=1}^{k} \left( \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z_i}{n^{\max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j-i)+j_0-j}} \right) C^j(z) = 1.
$$

(13)

Denote by $N_{j,i}$ the exponent of $n$ in (13) for given $j$ and $i$. Thus

$$
N_{j,i} = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j-i)+j_0-j.
$$

The terms for which this exponent is positive tend to zero as $n \to \infty$.

First we consider $j$ for which $\deg Q_j = j$, and denote, as usual, by $j_0$ the largest such $j$. If $j = j_0$, then $i \leq \deg Q_{j_0} = j_0$; thus for $j = j_0$ and $i = j_0$ we have

\[\begin{align*}
\frac{\alpha_{j, \deg Q_j}}{\alpha_{j, \deg Q_j}} \max_{j \in [1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) = 1,
\end{align*}\]

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corresponding terms in (13) tend to zero: that is attained. Note that there may be several distinct

Thus \( N_{j_0,i} = \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - i) + j_0 - j \)

and for \( j < j_0 \) we have

Thus \( N_{j_0,j} = 0 \) and \( N_{j_0,i} > 0 \) for \( i < j_0 \), and for the term corresponding to \( j = j_0 \) in (13) we get

in the limit when \( n \to \infty \), assuming that \( Q_{j_0} \) is monic (\( \alpha_{j_0,j_0} = 1 \)).

Now let \( j \) be such that \( \deg Q_j = j \) and \( j < j_0 \). Then \( i \leq \deg Q_j = j \) and

that is \( N_{j,i} > 0 \) for all \( j < j_0 \) such that \( \deg Q_j = j \) and for all \( i \leq j \). Thus the corresponding terms in (13) tend to zero:

when \( n \to \infty \) for every \( j < j_0 \) such that \( \deg Q_j = j \).

Now denote by \( j_m \) the \( j \) for which the maximum \( d = \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) \) is attained. Note that there may be several distinct \( j \) for which this maximum
is attained! Then

\[
N_{j_m, \deg Q_{j_m}} = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - i) + j_0 - j
\]

\[
= \left( \frac{j_m - j_0}{j_m - \deg Q_{j_m}} \right) (j_m - \deg Q_{j_m}) + j_0 - j_m
\]

\[
= j_m - j_0 + j_0 - j_m = 0,
\]

and for \( i < \deg Q_{j_m} \) we get

\[
N_{j_m, i} = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - i) + j_0 - j
\]

\[
> \left( \frac{j_m - j_0}{j_m - \deg Q_{j_m}} \right) (j_m - \deg Q_{j_m}) + j_0 - j_m
\]

\[
= j_m - j_0 + j_0 - j_m = 0,
\]

i.e. \( N_{j_m, \deg Q_{j_m}} = 0 \) and \( N_{j_m, i} > 0 \) for \( i < \deg Q_{j_m} \), and for the term corresponding to \( j = j_m \) in (13) we get

\[
\sum_{i=0}^{\deg Q_{j_m}} \frac{z^i}{\max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j_m - 1) + j_0 - j_m} C^{j_m}(z) \rightarrow \alpha_{j_m, \deg Q_{j_m}} z^{\deg Q_{j_m}} C^{j_m}(z)
\]

when \( n \to \infty \). In case of several \( j \) for which \( d \) is attained, we put \( A = \{ j : (j-j_0)/(j - \deg Q_j) = d := \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) \} \), and for the corresponding terms in (13) we get

\[
\sum_{j \in A} \sum_{i=0}^{\deg Q_j} \frac{z^i}{\max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - i) + j_0 - j} C^j(z) \rightarrow \sum_{j \in A} \alpha_{j, \deg Q_j} z^{\deg Q_j} C^j(z)
\]

when \( n \to \infty \). Now consider the remaining terms in (13), namely terms for which \( j < j_0 \) such that \( \deg Q_j < j \), terms for which \( j_0 < j < j_m \), and terms for which \( j_m < j \leq k \), (clearly this last case does not exist if \( j_m = k \)). We start with \( j < j_0 \) for which \( \deg Q_j < j \). Then \( i \leq \deg Q_j < j \) and

\[
N_{j, i} = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - i) + j_0 - j
\]

\[
> \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - j) + j_0 - j = j_0 - j > 0,
\]

\[\text{Consider, for example, the Laplace type operator (that is with all polynomial coefficients}} \]
\( Q_j \) \text{linear} \( T = zD + zD^2 + \ldots zD^k \). Here \( j_0 = 1 \) and the equation satisfied by the asymptotic Cauhy transform of the scaled eigenpolynomial \( q_n(z) = p_n(nz) \) is given by \( zC(z) + zC^2(z) + \ldots zC^k(z) = 1 \), since the maximum \( d = \max_{j \in [2, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) = 1 \) is attained for every \( j = 2, 3, \ldots k \).
and the corresponding terms in (13) for which \( j < j_0 \) such that \( \deg Q_j < j \) tend to zero when \( n \to \infty \):

\[
\sum_{j \in \{ j < j_0 : \deg Q_j < j \}} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n \max_{j \in [j_0+1,k]} \left( \frac{j - \deg Q_j}{j - \deg Q_j} \right)^{(j-i)+j_0-j}} C^j(z) \to 0
\]

when \( n \to \infty \).

Now assume that \( j_m < k \) and consider \( j_m < j \leq k \). Clearly \( j_m > j_0 \) since the maximum is taken over \( j \in [j_0+1,k] \), and therefore \( i \leq \deg Q_j < j \) for \( j_m < j \leq k \). Also,

\[
\max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) = \left( \frac{j_m - j_0}{j_m - \deg Q_j} \right) > \left( \frac{j - j_0}{j - \deg Q_j} \right),
\]

since the maximum is attained for \( j_m \) by assumption. Thus we get

\[
N_{j,i} = \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j-i) + j_0 - j = \left( \frac{j_m - j_0}{j_m - \deg Q_j} \right) (j-i) + j_0 - j
\]

\[
> \left( \frac{j - j_0}{j - \deg Q_j} \right) (j-i) + j_0 - j \geq \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - \deg Q_j) + j_0 - j
\]

\[= j - j_0 + j_0 - j = 0, \]

i.e. \( N_{j,i} > 0 \) for every \( j_m < j \leq k \) and every \( i \leq \deg Q_j \). The corresponding terms in (13) therefore tend to zero when \( n \to \infty \):

\[
\sum_{j_m < j \leq k} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n \max_{j \in [j_0+1,k]} \left( \frac{j - \deg Q_j}{j - \deg Q_j} \right)^{(j-i)+j_0-j}} C^j(z) \to 0
\]

as \( n \to \infty \).

Finally we consider \( j_0 < j < j_m \). Note that this also covers the case \( j_{m_1} < j < j_{m_2} \) where the maximum \( d = \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) \) is attained for \( j_{m_1} \) and \( j_{m_2} \). Since \( i \leq \deg Q_j < j \) we get

\[
N_{j,i} = \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right) (j-i) + j_0 - j = \left( \frac{j_m - j_0}{j_m - \deg Q_j} \right) (j-i) + j_0 - j
\]

\[
> \left( \frac{j - j_0}{j - \deg Q_j} \right) (j-i) + j_0 - j \geq \left( \frac{j - j_0}{j - \deg Q_j} \right) (j - \deg Q_j) + j_0 - j
\]

\[= j - j_0 + j_0 - j = 0, \]

i.e. \( N_{j,i} > 0 \) for every \( j_0 < j < j_m \) and every \( i \leq \deg Q_j \). Thus the corresponding terms in (13) tend to zero when \( n \to \infty \):

\[
\sum_{j_0 < j < j_m} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n \max_{j \in [j_0+1,k]} \left( \frac{j - \deg Q_j}{j - \deg Q_j} \right)^{(j-i)+j_0-j}} C^j(z) \to 0
\]
when $n \to \infty$.

Adding up these results we get the following equation from (13) for the asymptotic Cauchy transform $C(z)$ of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ where $d$ is as in Main Conjecture:

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j, \deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where $j_0$ is the largest $j$ such that $\deg Q_j = j$, and $A$ is the set consisting of all $j$ for which the maximum $d = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{\deg Q_j} \right)$ is attained, i.e.

$A = \{ j : (j - j_0)/(j - \deg Q_j) = d \}$ where $d$ is as above.

\[ \square \]

\section{Numerical evidence}

\subsection{Evidence for Main Conjecture}

On page 30 we present numerical evidence for Main Conjecture on the asymptotic root growth. We have performed similar computer experiments for a large number of other degenerate exactly-solvable operators, and the results are in all cases consistent with this conjecture.

\subsection{Comments on Main Corollary}

We here present and comment some pictures of the zero distribution for some properly (according to Main Conjecture) scaled eigenpolynomials of some degenerate exactly-solvable operators. In Section 1 we presented pictures of the zero distributions of the scaled eigenpolynomials for some distinct operators for which the Cauchy transform $C$ of the scaled eigenpolynomials satisfy the same equation in the limit as $n \to \infty$. We considered the operator $T_4 = z^3 D^3 + z^2 D^5$ for which $d = 2/3$, and for which the asymptotic Cauchy transform of the scaled eigenpolynomial $q_n(z) = p_n(n^{d_2} z)$ satisfies the equation $z^3 C^3 + z^2 C^5 = 1$. For the slightly modified operator $\tilde{T}_4 = z^2 D^2 + z^3 D^3 + z D^4 + z^2 D^5 + D^6$ we noted that $d$ is obtained again (only) for $j = 5$ and we therefore obtain the same equation in $C$ for the scaled eigenpolynomials of $\tilde{T}_4$ as for the scaled eigenpolynomials of $T_4$, whence we can consider the added terms $z^2 D^2$, $z D^4$ and $D^6$ as irrelevant for the zero distribution.

However, instead of $D^6$, we may add the “more disturbing” term $z D^6$ to $T_4$.

Consider the operator $\tilde{T}_4 = z^2 D^2 + z^3 D^3 + z D^4 + z^2 D^5 + z D^6$ and note that for $j = 6$ we have $(6 - 3)/(6 - 1) = 3/5 = 0.6 < 2/3$ - it is clear that the closer the value $(j - j_0)(j - \deg Q_j)$ of the added term $Q_j D^j$ is to $d = 2/3$, the more disturbing is this term, since, besides the term for which $j = j_0$, it is precisely the terms for which $(j - j_0)/(j - \deg Q_j) = d = 2/3$ that are involved in the asymptotic Cauchy transform equation. Se pictures below.
The term \( zD^6 \) should however be irrelevant in the limit when \( n \to \infty \) (according to the asymptotic Cauchy transform equation), and experiments indicate that for sufficiently large \( n \) the zero distributions for the scaled eigenpolynomials of \( T_4 \) and \( \tilde{T}_4 \) coincide, as they (conjecturally) should.

Note also that it is only the term of highest degree in a given (relevant) \( Q_j \), i.e. \( \alpha_j \deg Q_j \), that is relevant for the zero distribution of the scaled polynomials in the limit when \( n \to \infty \). This is illustrated by the following example, where adding lower degree terms in the (relevant) \( Q_j \) clearly does not affect the zero distribution of the scaled eigenpolynomials for large \( n \). Below, \( T_6 = z^3D^3 + z^2D^6 \), and \( \tilde{T}_6 = [(1 + 13i) + (24i - 3)z + 11iz^2 + z^3]D^3 + [(22i - 13) + (-9 - 14i)z + z^2]D^6 \) (note the difference in scaling between the pictures).

4.3 On the basic assumption

Finally, we show some pictures which support the basic assumption upon which Main Conjecture and Main Corollary are built, namely that the Cauchy transforms for the scaled eigenpolynomial and its derivatives are all equal when \( n \to \infty \), i.e. \( C_{n,0} = C_{n,1} = \ldots = C_{n,k-1} \) in the limit when \( n \to \infty \). This means that we assume that the zero distributions \( \mu_n, \mu_n^{(1)}, \ldots, \mu_n^{(k)} \) of the scaled eigenpolynomial and its derivatives \( q_n, q_n', \ldots, q_n^{(k)} \), respectively, are all equal when \( n \to \infty \). Below, \( p_n \) denotes the \( n \)th degree monic eigenpolynomial of the given operator, and \( q_n = p_n(n^d) \) denotes the corresponding appropriately scaled polynomial.
Fig. 1: $T_7 = zD + D^3$ and $q_n(z) = p_n(n^{2/3}z)$.

Fig. 2: $T_8 = z^2D^2 + D^5$ and $q_n(z) = p_n(n^{3/5}z)$.

Fig. 3: $T_9 = zD + zD^4 + z^3D^7$ and $q_n(z) = p_n(n^{3/2}z)$. 
4.4 Interlacing property

We now state the exact meaning of interlacing on curves in the complex plane. Conjecturally the support of the asymptotic zero distribution of the scaled eigen-polynomial \( q_n \) of \( T \) is the union of a finite number of analytic curves in the complex plane, which we denote by \( \Xi_T \). When defining the interlacing property some caution is required since the zeros of \( q_n \) do not lie exactly on \( \Xi_T \). Thus, identify some sufficiently small neighbourhood \( N(\Xi_T) \) of \( \Xi_T \) with the normal bundle to \( \Xi_T \) by equipping \( N(\Xi_T) \) with the projection onto \( \Xi_T \) along the fibres which are small curvilinear segments orthogonal to \( \Xi_T \). Then we say that two sets of points in \( N(\Xi_T) \) interlace if their (orthogonal) projections on \( \Xi_T \) interlace in the usual sense. If \( \Xi_T \) has singularities one should first remove some sufficiently small neighbourhoods of these singularities and the proceed in the above way on the remaining part of \( \Xi_T \). Conjecture 1 then states that for any sufficiently small neighbourhood \( N(\Xi_T) \) of \( \Xi_T \) there exists \( N \) such that the interlacing property holds for the roots of \( q_n \) and \( q_{n+1} \) for all \( n \geq N \). Below, small dots are roots of \( q_{n+1} \) and large dots are roots of \( q_n \) for some fixed \( n \).

\[
T = zD^2 + zD^3 + zD^5, \quad \text{roots of } q_{25} \text{ and } q_{24}.
\]

\[
T = zD + z^2D^2 + D^3, \quad \text{roots of } q_{20} \text{ and } q_{19}.
\]

\[
T = zD + zD^2 + zD^3 + zD^4 + zD^5, \quad \text{roots of } q_{23} \text{ and } q_{22}.
\]

\[
T = z^3D^3 + z^2D^5 + zD^9, \quad \text{roots of } q_{20} \text{ and } q_{19}.
\]
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<th>$r_n$ experimental</th>
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5 Appendix

For the classes of degenerate exactly-solvable operators considered in Corollary 1 and Corollary 2, what we really want is the lower bound \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} \geq c_0 > 0 \), since we have conjectured \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} = c_0 > 0 \), where \( d := \max_{j \in [j_0+1, k]} \left( \frac{j-j_0}{j-\deg Q_j} \right) \) and \( j_0 \) is the largest \( j \) such that \( \deg Q_j = j \). Recall that in Corollary 1 and 2 we obtained the result \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} > c_0 > 0 \) for any \( \gamma < d \).

Here we prove that for a class of operators containing the operators considered in Corollary 1 and 2, the lower bound \( r_n \geq c_0(n-k+1)^d \) follows automatically from the inequality in Lemma 3, if we assume that the upper bound \( r_n \leq c_1(n-k+1)^d \) holds for large \( n \), where \( c_1 > 0 \) is a positive constant and \( c_0 \leq c_1 \).

Theorem 5. Let \( T \) be a degenerate exactly-solvable operator which satisfies the following condition:

\[
b := \min_{j \in [1, k-1]} \left( \frac{k-j}{k-j + \deg Q_j - \deg Q_k} \right) = \max_{j \in [j_0+1, k]} \left( \frac{j-j_0}{j-\deg Q_j} \right) =: d,
\]

where the notation \( \min^+ \) means that the minimum is taken only over positive values of \((k-j + \deg Q_j - \deg Q_k)\). Assume that \( r_n \leq c_1(n-k+1)^d \) holds for large \( n \), where \( c_1 > 0 \) is a positive constant. Then there exists a positive constant \( c_0 \) such that \( c_0 \leq c_1 \) and \( r_n \geq c_0(n-k+1)^d \) for sufficiently large \( n \), and thus \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^\gamma} = \tilde{c} \), where \( c_0 \leq \tilde{c} \leq c_1 \) and \( d := \max_{j \in [j_0+1, k]} \left( \frac{j-j_0}{j-\deg Q_j} \right) \).

Proof. From Lemma 3 and using \( i \leq \deg Q_j \) for every given \( j \), we have

\[
1 \leq \sum_{j=1}^{k-1} \sum_{i=0}^{j} |\alpha_{j,i}|^2 \frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k-i}} \leq \sum_{j=1}^{k-1} K_j \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < c_k} \frac{|\alpha_{k,i}|}{r_n^{k-i}} \tag{14}
\]

where \( K_j > 0 \) is a positive constant. The second sum on the right-hand side of this inequality tends to zero as \( n \to \infty \) due to Theorem 1. To prove our theorem we decompose the first sum on the right-hand side of the inequality above into three parts:

- \( j \) for which \( \frac{k-j}{k-j + \deg Q_j - \deg Q_k} = d \),
- (note that \((k-j + \deg Q_j - \deg Q_k) > 0 \) here since \( d > 0 \)),

\[
(14) \implies \frac{r_n}{(n-k+1)^\gamma} \geq \tilde{c}.
\]
• $j$ for which $\frac{k-j}{k-j+\deg Q_j - \deg Q_k} > d$,
  (note that $(k-j+\deg Q_j - \deg Q_k) > 0$ here since $d > 0$),

• $j$ for which $(k-j+\deg Q_j - \deg Q_k) \leq 0$,

where $d := \max_{j \in [b_n+1,k]} \left( \frac{j-b_n}{j-\deg Q_j} \right)$. Clearly there are no terms for which
$\frac{k-j}{k-j+\deg Q_j - \deg Q_k} < d$ and $(k-j+\deg Q_j - \deg Q_k) > 0$, due to the condition $b = d$.

In the first case, for any term for which $(k-j)/(k-j+\deg Q_j - \deg Q_k) = d$, we have
$$\frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} = \left( \frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k}.$$ The second part we consider consists of terms for which $(k-j)/(k-j+\deg Q_j - \deg Q_k) > d$, i.e. $d(k-j+\deg Q_j - \deg Q_k) < (k-j)$, and this inequality together with the upper bound $r_n \leq c_1(n-k+1)^d$ gives the following estimation of the corresponding terms in (14):
$$\frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} \leq c_1(n-k+1)^d k-j+\deg Q_j - \deg Q_k \leq 0 \text{ when } n \to \infty.$$ The third part we consider consists of the remaining terms, namely terms for which $(k-j+\deg Q_j - \deg Q_k) \leq 0$, since $(k-j) > 0$. But clearly the corresponding terms $r_n^{k-j+\deg Q_j - \deg Q_k} / (n-k+1)^{k-j}$ in (14) tend to zero when $n \to \infty$, using Theorem 1.

Thus, decomposing the first sum on the right-hand side of the last inequality in (14) in this way, we obtain the following inequality:
$$1 \leq \sum_{j=1}^{k-1} K_j \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i \leq t_k} \frac{|\alpha_{k,i}|}{r_n^{t_k-i}}$$
$$\leq \sum_{j \in A} K_j \left( \frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k} + \sum_{j \in B} K_j c_1(n-k+1)^d(k-j+\deg Q_j - \deg Q_k)$$
$$+ \sum_{j \in C} K_j \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i \leq t_k} \frac{|\alpha_{k,i}|}{r_n^{t_k-i}}$$
where $A = \{ j : k-j+\deg Q_j - \deg Q_k = d \}$, $B = \{ j : k-j+\deg Q_j - \deg Q_k > d \}$ and $C = \{ j : (k-j+\deg Q_j - \deg Q_k) \leq 0 \}$. The last three sums in on the right-hand
side of this inequality tend to zero when \( n \to \infty \), the last one due to Theorem 1, the sum over \( B \) since \( d(k-j + \deg Q_j - \deg Q_k) < (k-j) \), and the sum over \( C \) due to Theorem 1.

Thus, when \( n \to \infty \), there exists a positive constant \( c' > 0 \) such that

\[
c' \leq \sum_{j \in A} K_j \left( \frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k}
\]

(15)

where \( A = \{ j : \frac{k-j}{k-j+\deg Q_j - \deg Q_k} = d \} \) and \( A \) is nonempty. If \( A \) contains precisely one element, then the sum in the inequality (15) consists of one single term, and we are done; there exists a positive constant \( c_0 \) such that \( r_n \geq c_0(n-k+1)^d \) for sufficiently large \( n \). But clearly for some operators \( A \) will contain more elements.\(^9\) If this is the case, let \( m = \min_{j \in A}(k-j + \deg Q_j - \deg Q_k) \) and denote by \( j_m \) the corresponding \( j \). Using the upper bound \( r_n \leq c_1(n-k+1)^d \) we then get the following inequality from (15):

\[
c' \leq \sum_{j \in A} K_j \left( \frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k} \leq K_{j_m} \left( \frac{r_n}{(n-k+1)^d} \right)^{m} + \sum_{j \in A \setminus \{j_m\}} K_j \left( \frac{r_n}{(n-k+1)^d} \right)^{m} \left( \frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k - m}
\]

\(^9\)Consider for example the operator \( T = zD + D^2 + zD^3 + zD^4 \). Then, from Lemma 3, we get the following inequality (here \( k = 4 \) and \( \deg Q_k = 1 \)):

\[
1 \leq \sum_{j=1}^{3} 2^{4-j} \frac{r_n}{(n-3)^{4-j}} = 8 \frac{r_n}{(n-3)^3} + 4 \frac{r_n}{(n-3)^2} + 2 \frac{r_n}{(n-3)}
\]

where \( r_n \) is the largest modulus of all roots of the unique and monic eigenpolynomial of \( T \). For this operator \( d = 1 \) and we see that \( \frac{4-j}{k-j+\deg Q_j} = d \) for the first \((j = 1)\) and the last \((j = 3)\) term. Now assuming that \( r_n \leq c_1(n-3) \) our inequality becomes

\[
1 \leq 8 \frac{r_n}{(n-3)^3} + 4 \frac{r_n}{(n-3)^2} + 2 \frac{r_n}{(n-3)} \leq 8 \frac{r_n}{(n-3)} + \frac{c_1(n-3)^2}{(n-3)^2} + 2 \frac{r_n}{(n-3)} = (8c_1^2 + 2) \frac{r_n}{(n-3)} + \frac{4c_1}{(n-3)}
\]

where the last term tends to zero as \( n \to \infty \). Thus \( r_n \geq c_0(n-3) \) for some positive constant \( c_0 \) for sufficiently large choices on \( n \).
\[
\begin{align*}
K_j \left( \frac{r_n}{n-k+1} \right)^d m & + \sum_{j \in A \setminus \{j_m\}} K_j \left( \frac{r_n}{n-k+1} \right)^d \cdot \frac{c_1}{c_1} \cdot k_j - \deg Q_j - \deg Q_k - m \\
= \left( \frac{r_n}{n-k+1} \right)^d \left( K_{j_m} + \sum_{j \in A \setminus \{j_m\}} K_j \cdot \frac{c_1}{c_1} \cdot k_j - \deg Q_j - \deg Q_k - m \right) \\
= \left( \frac{r_n}{n-k+1} \right)^d K
\end{align*}
\]

where \( K > 0 \). Thus there exists a positive constant \( c_0 = (c'/K)^{1/m} > 0 \) such that \( r_n \geq c_0(n-k+1)^d \) for sufficiently large choices on \( n \), i.e. \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^d} \geq c_0 \), and thus \( \lim_{n \to \infty} \frac{r_n}{(n-k+1)^d} = \tilde{c} \) for some positive constant \( \tilde{c} \) such that \( c_0 \leq \tilde{c} \leq c_1 \). \( \square \)

References


