On Roots of Eigenpolynomials for Degenerate Exactly-Solvable Differential Operators

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Abstract

In this paper we partially settle our conjecture from [1] on the roots of eigenpolynomials for degenerate exactly-solvable operators. Namely, for any such operator we establish a lower bound (which supports our conjecture) for the largest modulus of all roots of its unique and monic eigenpolynomial p_n as the degree n tends to infinity. The main theorem below thus extends earlier results obtained in [1] for a restrictive class of operators.

1 Introduction

We are interested in roots of eigenpolynomials satisfying certain linear differential equations. Namely, consider an operator

$$T = \sum_{j=1}^{k} Q_j D^j$$

where $D = d/dz$ and the Q_j are complex polynomials in one variable satisfying the condition $\deg Q_j \leq j$, with equality for at least one j, and in particular $\deg Q_k < k$ for the leading term. Such operators are referred to as *degenerate exactly-solvable operators*[1], see [1]. We are interested in eigenpolynomials of T, that is polynomials satisfying

$$T(p_n) = \lambda_n p_n \tag{1}$$

for some value of the spectral parameter λ_n, where n is a positive integer and $\deg p_n = n$. The importance of studying eigenpolynomials for these operators is among other things motivated by numerous examples coming from classical orthogonal polynomials, such as the Laguerre and Hermite polynomials, which

[1]Correspondingly, operators for which $\deg Q_k = k$ are called *non-degenerate exactly-solvable operators*. We have treated roots of eigenpolynomials for these operators in [2].

appear as solutions to (1) for certain choices on the polynomials Q_j when $k = 2$. Note however that for the operators considered here the sequence of eigenpolynomials $\{p_n\}$ is in general *not* an orthogonal system.
Let us briefly recall our previous results:

A. In [2] we considered eigenpolynomials of *non-degenerate exactly-solvable operators*, that is operators of the above type but with the condition $\deg Q_k = k$ for the leading term. We proved that when the degree n of the unique and monic eigenpolynomial p_n tends to infinity, the roots of p_n stay in a compact set in $\mathbb{C}$ and are distributed according to a certain probability measure which is supported by a tree and which depends only on the leading polynomial Q_k.

B. In [1] we studied eigenpolynomials of *degenerate exactly-solvable operators* ($\deg Q_k < k$). We proved that there exists a unique and monic eigenpolynomial p_n for all sufficiently large values on the degree n, and that the largest modulus of the roots of p_n tends to infinity when $n \to \infty$. We also presented an explicit conjecture and partial results on the growth of the largest root. Namely,

Conjecture (from [1]). *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k and denote by j_0 the largest j for which $\deg Q_j = j$. Let $r_n = \max\{|\alpha| : p_n(\alpha) = 0\}$, where p_n is the unique and monic nth degree eigenpolynomial of T. Then*

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_0,$$

where $c_0 > 0$ is a positive constant and

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right).$$

Extensive computer experiments listed in [1] confirm the existence of such a constant c_0. Now consider the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$. We construct the probability measure μ_n by placing a point mass of size $1/n$ at each zero of q_n. Numerical evidence indicates that for each degenerate exactly-solvable operator T, the sequence $\{\mu_n\}$ converges weakly to a probability measure μ_T which is (compactly) supported by a tree. In [1] we deduced the algebraic equation satisfied by the Cauchy transform of μ_T.[2] Namely, let $T = \sum_{j=1}^{k} Q_j(z) D^j = \sum_{j=1}^{k} \big(\sum_{i=0}^{\deg Q_j} q_{j,i} z^i \big) D^j$ and denote by j_0 the largest j for which $\deg Q_j = j$. Assuming wlog that Q_{j_0} is monic, i.e. $q_{j_0,j_0} = 1$, we have

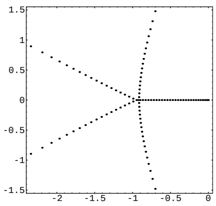
$$z^{j_0} C^{j_0}(z) + \sum_{j\in A} q_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where $C(z) = \int \frac{d\mu_T(\zeta)}{z-\zeta}$ is the Cauchy transform of μ_T and $A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$, where d is defined in the conjecture. Below we present some

[2] It remains to prove the existence of μ_T and to describe its support explicitly.

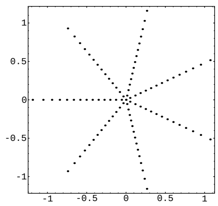
typical pictures of the roots of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$.

Fig.1:



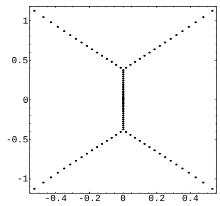
roots of
$q_{100}(z) = p_{100}(100z)$

Fig.2:



roots of
$q_{100}(z) = p_{100}(100z)$

Fig.3:



roots of
$q_{100}(z) = p_{100}(100z)$

Fig.1: $T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$.
Fig.2: $T_2 = z^2D^2 + D^7$.
Fig.3: $T_3 = z^3D^3 + z^2D^4 + zD^5$.

In this paper we extend the results from [1] by establishing a lower bound for r_n for *all* degenerate exactly-solvable operators and which supports the above conjecture.[3] This is our main result:

Main Theorem. *Let* $T = \sum_{j=1}^{k} Q_j D^j$ *be a degenerate exactly-solvable operator and denote by* j_0 *the largest* j *for which* $\deg Q_j = j$. *Let* p_n *be the unique and monic nth degree eigenpolynomial of* T *and* $r_n = \max\{|\alpha| : p_n(\alpha) = 0\}$. *Then there exists a positive constant* $c > 0$ *such that*

$$\lim_{n\to\infty} \frac{r_n}{n^d} \geq c,$$

where

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j}\right).$$

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2 Proofs

Lemma 1. *For any monic polynomial* $p(z)$ *of degree* $n \geq 2$ *for which all the zeros are contained in a disc of radius* $A \geq 1$, *there exists an integer* $n(j)$ *and an absolute constant* C_j *depending only on* j, *such that for every* $j \geq 1$ *and*

[3] It is still an open problem to prove the upper bound.

every $n \geq n(j)$ we have

$$\frac{1}{C_j} \cdot \frac{n^j}{A^j} \leq \left|\left|\frac{p^{(j)}(z)}{p(z)}\right|\right|_{2A} \leq C_j \cdot \frac{n^j}{A^j} \tag{2}$$

where $p^{(j)}(z)$ denotes the jth derivative of $p(z)$, and where we have used the maximum norm $||p(z)||_{2A} = \max_{|z|=2A} |p(z)|$.

Remark. The right-hand side of the above inequality actually holds for all $n \geq 2$, whereas the left-hand side holds for all $n \geq n(j)$.

Proof. To obtain the inequality on the *right-hand side* we use the notation $p(z) = \prod_{i=1}^{n}(z - \alpha_i)$ where by assumption $|\alpha_i| \leq A$ for every complex root of $p(z)$. Then $p^{(j)}(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms, each being the product of $(n-j)$ factors $(z-\alpha_i)$.[4] Thus $p^{(j)}(z)/p(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms, each equal to 1 divided by a product consisting of $n-(n-j) = j$ factors $(z-\alpha_i)$. If $|z| = 2A$ we get $|z-\alpha_i| \geq A \Rightarrow \frac{1}{|z-\alpha_i|} \leq \frac{1}{A}$, and thus

$$||\frac{p^{(j)}}{p}||_{2A} \leq \frac{n(n-1)\cdots(n-j+1)}{A^j} \leq C_j \cdot \frac{n^j}{A^j}.$$

Here we can choose $C_j = 1$ for all j, but we refrain from doing this since we will need C_j large enough to obtain the constant $1/C_j$ in the left-hand side inequality. To prove the *left-hand side* inequality we will need inequalities (i)-(iv) below, where we need (i) to prove (ii), and we need (ii) and (iii) to prove (iv), from which the left-hand side inequality of this lemma follows.

For every $j \geq 1$ we have

(i) $\quad ||\frac{d}{dz}\left(\frac{p^{(j)}(z)}{p(z)}\right)||_{2A} \leq j \cdot \frac{n^j}{A^{j+1}}$.

For every $j \geq 1$ there exists a positive constant C'_j depending only on j, such that

(ii) $\quad \left|\left|\frac{p^{(j)}}{p} - \frac{(p')^j}{p^j}\right|\right|_{2A} \leq C'_j \cdot \frac{n^{j-1}}{A^j}$.

(iii) $\quad ||\frac{p'}{p}||_{2A} \geq \frac{n}{3A}$.

For every $j \geq 1$ there exists a positive constant C''_j and some integer $n(j)$ such that for all $n \geq n(j)$ we have

(iv) $\quad ||\frac{p^{(j)}}{p}||_{2A} \geq C''_j \cdot \frac{n^j}{A^j}$.

To prove (i), let $p(z) = \prod_{i=1}^{n}(z-\alpha_i)$, where $|\alpha_i| \leq A$ for each complex root

[4] Differentiating $p(z) = \prod_{i=1}^{n}(z-\alpha_i)$ once yields $\binom{n}{1} = n$ terms each term being a product of $(n-1)$ factors $(z-\alpha_i)$, differentiating once again we obtain $n\binom{n-1}{1} = n(n-1)$ terms, each being the product of $(n-2)$ factors $(z-\alpha_i)$, etc.

α_i of $p(z)$. Then again $p^{(j)}(z)/p(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms and each term equals 1 divided by a product consisting of j factors $(z-\alpha_i)$. Differentiating *each* such term we obtain a sum of j terms each being on the form (-1) divided by a product consisting of $(j+1)$ factors $(z-\alpha_i)$.[5] Thus $\frac{d}{dz}\big(\frac{p^{(j)}(z)}{p(z)}\big)$ is a sum consisting of $j\cdot n(n-1)\cdots(n-j+1)$ terms, each on the form (-1) divided by $(j+1)$ factors $(z-\alpha_i)$. Using $\frac{1}{|z-\alpha_i|}\le\frac{1}{A}$ for $|z|=2A$ since $|\alpha_i|\le A$ for all $i\in[1,n]$, we thus get

$$||\frac{d}{dz}(\frac{p^{(j)}(z)}{p(z)})||_{2A}\le\frac{j\cdot n(n-1)\cdots(n-j+1)}{A^{j+1}}\le j\cdot\frac{n^j}{A^{j+1}}.$$

To prove (ii) we use (i) and induction over j. The case $j=1$ is trivial since $\frac{p'}{p}-\frac{(p')^1}{p^1}=0$. If we put $j=1$ in (i) we get $||\frac{d}{dz}\big(\frac{p'}{p}\big)||_{2A}\le\frac{n}{A^2}$. But $\frac{d}{dz}\big(\frac{p'}{p}\big)=\frac{p^{(2)}}{p}-\frac{(p')^2}{p^2}$, and thus $\big|\big|\frac{p^{(2)}}{p}-\frac{(p')^2}{p^2}\big|\big|\le\frac{n}{A^2}$, so (ii) holds for $j=2$. We now proceed by induction. Assume that (ii) holds for some $j=p\ge 2$, i.e. $||\frac{p^{(p)}}{p}-\frac{(p')^p}{p^p}||_{2A}\le C'_p\cdot\frac{n^{p-1}}{A^p}$. Also note that with $j=p$ in (i) we have

$$||\frac{p^{(p+1)}}{p}-\frac{p^{(p)}\cdot p'}{p^2}||_{2A}=||\frac{d}{dz}(\frac{p^{(p)}}{p})||_{2A}\le p\cdot\frac{n^p}{A^{p+1}},$$

and also $||\frac{p'}{p}||_{2A}\le\frac{n}{A}$ (from the right-hand side inequality of this lemma). Thus we have

$$\begin{aligned}\left|\left|\frac{p^{(p+1)}}{p}-\frac{(p')^{p+1}}{p^{p+1}}\right|\right|_{2A} &= \left|\left|\frac{p^{(p+1)}}{p}-\frac{p^{(p)}\cdot p'}{p^2}+\frac{p^{(p)}\cdot p'}{p^2}-\frac{(p')^{p+1}}{p^{p+1}}\right|\right|_{2A}\\ &\le \left|\left|\frac{p^{(p+1)}}{p}-\frac{p^{(p)}\cdot p'}{p^2}\right|\right|_{2A}+\left|\left|\frac{p'}{p}\left(\frac{p^{(p)}}{p}-\frac{(p')^p}{p^p}\right)\right|\right|_{2A}\\ &\le p\cdot\frac{n^p}{A^{p+1}}+\frac{n}{A}\cdot C'_p\cdot\frac{n^{p-1}}{A^p}\\ &= (p+C'_p)\cdot\frac{n^p}{A^{p+1}}=C'_{p+1}\cdot\frac{n^p}{A^{p+1}}.\end{aligned}$$

To prove (iii) observe that $\frac{p'(z)}{p(z)}=\sum_{i=1}^n\frac{1}{(z-\alpha_i)}=\sum_{i=1}^n\frac{1}{z}\cdot\frac{1}{1-\frac{\alpha_i}{z}}$. By assumption $|\alpha_i|\le A$ for all complex roots α_i of $p(z)$, so for $|z|=2A$ we have $|\frac{\alpha_i}{z}|\le\frac{A}{2A}=\frac{1}{2}$ for all $i\in[1,n]$. Writing $w_i=\frac{1}{1-\frac{\alpha_i}{z}}$ we obtain

$$|w_i-1|=\left|\frac{1}{1-\frac{\alpha_i}{z}}-\frac{1-\frac{\alpha_i}{z}}{1-\frac{\alpha_i}{z}}\right|=\frac{\left|\frac{\alpha_i}{z}\right|}{\left|1-\frac{\alpha_i}{z}\right|}\le\frac{1}{2}|w_i|,$$

[5] With $D=d/dz$ consider for example $D\frac{1}{\prod_{i=1}^j(z-\alpha_i)}=\frac{-1\cdot D\prod_{i=1}^j(z-\alpha_i)}{\prod_{i=1}^j(z-\alpha_i)^2}$, which is a sum of j terms, each being on the form (-1) divided by a product consisting of $2j-(j-1)=(j+1)$ factors $(z-\alpha_i)$.

which implies

$$Re\left(\frac{1}{1-\frac{\alpha_i}{z}}\right) = Re(w_i) \geq \frac{2}{3} \quad \forall i \in [1,n] \Rightarrow Re\left(\sum_{i=1}^{n} \frac{1}{1-\frac{\alpha_i}{z}}\right) \geq \frac{2n}{3}.$$

Thus

$$\begin{aligned}
\left|\left|\frac{p'(z)}{p(z)}\right|\right|_{2A} &= \max_{|z|=2A}\left|\frac{p'(z)}{p(z)}\right| = \max_{|z|=2A}\frac{1}{|z|}\cdot\left|\sum_{i=1}^{n}\frac{1}{1-\frac{\alpha_i}{z}}\right| \\
&\geq \frac{1}{2A}\cdot\left|\sum_{i=1}^{n}\frac{1}{1-\frac{\alpha_i}{z}}\right|_{2A} \geq \frac{1}{2A}\cdot Re\left(\sum_{i=1}^{n}\frac{1}{1-\frac{\alpha_i}{z}}\right) \\
&\geq \frac{n}{3A}.
\end{aligned}$$

To prove (iv) we note that from (iii) we obtain $||\big(\frac{p'}{p}\big)^j||_{2A} \geq \frac{n^j}{3^j A^j}$, and this together with (ii) yields

$$\begin{aligned}
\left|\left|\frac{p^{(j)}}{p}\right|\right|_{2A} &= \left|\left|\left(\frac{p'}{p}\right)^j + \frac{p^{(j)}}{p} - \left(\frac{p'}{p}\right)^j\right|\right|_{2A} \geq \left|\left|\left(\frac{p'}{p}\right)^j\right|\right|_{2A} - \left|\left|\frac{p^{(j)}}{p} - \left(\frac{p'}{p}\right)^j\right|\right|_{2A} \\
&\geq \frac{n^j}{3^j A^j} - C'_j\cdot\frac{n^{j-1}}{A^j} = \frac{n^j}{A^j}\left(\frac{1}{3^j} - \frac{C'_j}{n}\right) \geq C''_j\cdot\frac{n^j}{A^j},
\end{aligned}$$

where C''_j is a positive constant such that $C''_j \leq \big(\frac{1}{3^j} - \frac{C'_j}{n}\big)$ for all $n \geq n(j)$.

The left-hand side inequality in this lemma now follows from (iv) if we choose the constant C_j on right-hand side inequality so large that $\frac{1}{C_j} \leq C''_j$. □

To prove Main Theorem we will need the following lemma, which follows from Lemma 1:

Lemma 2. Let $0 < s < 1$ and $d > 0$ be real numbers. Let $p(z)$ be any monic polynomial of degree $n \geq 2$ such that all its zeros are contained in a disc of radius $A = s\cdot n^d$, and let $Q_j(z)$ be arbitrary polynomials. Then there exists some positive integer n_0 and positive constants K_j such that

$$\frac{1}{K_j}\cdot n^{d(\deg Q_j - j)+j}\cdot\frac{s^{\deg Q_j}}{s^j} \leq \left|\left|Q_j(z)\cdot\frac{p^{(j)}}{p}\right|\right|_{2sn^d} \leq K_j\cdot n^{d(\deg Q_j - j)+j}\cdot\frac{s^{\deg Q_j}}{s^j}$$

for every $j \geq 1$ and all $n \geq \max(n_0, n(j))$, where $n(j)$ is as in Lemma 1.

Proof. Let $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i}z^i$. Then for $|z| = 2A >> 1$ we have

$$|Q(z)|_{2A} = |q_{j,\deg Q_j}|2^{\deg Q_j}A^{\deg Q_j}\left(1 + O(\frac{1}{A})\right).$$

Since $A = s \cdot n^d$ there exists some integer n_0 such that $n \geq n_0 \Rightarrow A \geq A_0 >> 1$, and thus by Lemma 1 there exists a positive constant K_j such that the following inequality holds for all $n \geq \max(n(j), n_0)$ and all $j \geq 1$:

$$\frac{1}{K_j} \cdot \frac{n^j}{A^j} \cdot A^{\deg Q_j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2A} \leq K_j \cdot \frac{n^j}{A^j} \cdot A^{\deg Q_j}.$$

Inserting $A = s \cdot n^d$ in this inequality we obtain

$$\frac{1}{K_j} \cdot \frac{n^j}{s^j n^{dj}} \cdot s^{\deg Q_j} n^{d \cdot \deg Q_j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2sn^d} \leq K_j \cdot \frac{n^j}{s^j n^{dj}} \cdot s^{\deg Q_j} n^{d \cdot \deg Q_j}$$

$$\Leftrightarrow$$

$$\frac{1}{K_j} \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2sn^d} \leq K_j \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j}$$

for every $j \geq 1$ and all $n \geq \max(n_0, n(j))$. □

Proof of Main Theorem. Let $d = \max_{j \in [j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right)$ where j_0 is the largest j for which $\deg Q_j = j$ in the degenerate exactly-solvable operator $T = \sum_{j=1}^{k} Q_j D^j$, where $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i} z^i$. Let $p_n(z)$ be the nth degree unique and monic eigenpolynomial of T and denote by λ_n the corresponding eigenvalue. Then the eigenvalue equation can be written

$$\sum_{j=1}^{k} Q_j(z) \cdot \frac{p_n^{(j)}(z)}{p_n(z)} = \lambda_n \tag{3}$$

where $\lambda_n = \sum_{j=1}^{j_0} q_{j,j} \cdot \frac{n!}{(n-j)!}$. We will now use the result in Lemma 2 to estimate each term in (3).

* Denote by j_m the largest j for which d is attained. Then $d = (j_m - j_0)/(j_m - \deg Q_{j_m}) \Rightarrow d(\deg Q_{j_m} - j_m) + j_m = j_0$, and $j_m - \deg Q_{j_m} = (j_m - j_0)/d$. By Lemma 2 we have:

$$\frac{1}{K_{j_m}} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq \left|\left| Q_{j_m}(z) \cdot \frac{p^{(j_m)}}{p} \right|\right|_{2sn^d} \leq K_{j_m} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}}. \tag{4}$$

Note that the exponent of s is positive since $j_m > j_0$ and $d > 0$. In what follows we will only need the left-hand side of the above inequality.

* Consider the remaining (if there are any) $j_0 < j < j_m$ for which d is attained. For such j we have (using the right-hand side inequality of Lemma 2):

$$\begin{aligned} \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2sn^d} &\leq K_j n^{j_0} \cdot \frac{1}{s^{\frac{j - j_0}{d}}} = K_j n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \cdot s^{\frac{j_m - j}{d}} \\ &\leq K_j n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \cdot s^{1/d} \end{aligned} \tag{5}$$

where we have used that $(j_m - j) \geq 1$ and $s < 1 \Rightarrow s^{(j_m-j)/d} \leq s^{1/d}$.

* Consider all $j_0 < j \leq k$ for which d is *not* attained. Then $(j - \deg Q_j) > 0$ and $(j - j_0)/(j - \deg Q_j) < d \Rightarrow d(\deg Q_j - j) + j < j_0$ and we can write $d(\deg Q_j - j) + j \leq j_0 - \delta$ where $\delta > 0$. Then we have:

$$\begin{aligned} \left\| Q_j(z) \cdot \frac{p^{(j)}}{p} \right\|_{2sn^d} &\leq K_j \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j} \leq K_j \cdot n^{j_0-\delta} \cdot \frac{s^{\deg Q_j}}{s^j} \\ &\leq K_j \cdot n^{j_0-\delta} \cdot \frac{1}{s^k}, \end{aligned} \tag{6}$$

where the last inequality follows since $\deg Q_j \geq 0 \Rightarrow s^{\deg Q_j} \leq s^0 = 1$ and $j \leq k \Rightarrow s^j \geq s^k$ since $0 < s < 1$.

* For $j = j_0$ by definition $\deg Q_{j_0} = j_0$ and thus:

$$\left\| Q_{j_0}(z) \cdot \frac{p^{(j_0)}}{p} \right\|_{2sn^d} \leq K_{j_0} \cdot n^{d(\deg Q_{j_0} - j_0)+j_0} \cdot \frac{s^{\deg Q_{j_0}}}{s^{j_0}} = K_{j_0} \cdot n^{j_0}. \tag{7}$$

* Now consider all $1 \leq j \leq j_0 - 1$. Since $n \geq n_0 \Rightarrow A = sn^d >> 1$ we get $(sn^d)^{j-\deg Q_j} \geq 1$ and thus:

$$\begin{aligned} \left\| Q_j(z) \cdot \frac{p^{(j)}}{p} \right\|_{2sn^d} &\leq K_j \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j} = K_j \cdot n^j \cdot (sn^d)^{(\deg Q_j - j)} \\ &= K_j \cdot n^j \cdot \frac{1}{(sn^d)^{j-\deg Q_j}} \leq K_j \cdot n^j \leq K_j \cdot n^{j_0-1}. \end{aligned} \tag{8}$$

* Finally we estimate the eigenvalue $\lambda_n = \sum_{i=1}^{j_0} q_{j,j} \cdot \frac{n!}{(n-j)!}$, which grows as n^{j_0} for large n, since there exists an integer n_{j_0} and some positive constant K'_{j_0} such that for all $n \geq n_{j_0}$ we obtain:

$$\begin{aligned} |\lambda_n| &\leq \sum_{j=1}^{j_0} |q_{j,j}| \cdot \frac{n!}{(n-j)!} = |q_{j_0,j_0}| \cdot \frac{n!}{(n-j_0)!} \left[1 + \sum_{1 \leq j < j_0} \left| \frac{q_{j,j}}{q_{j_0,j_0}} \right| \cdot \frac{(n-j_0)!}{(n-j)!} \right] \\ &\leq K'_{j_0} \cdot n^{j_0}. \end{aligned} \tag{9}$$

Finally we rewrite the eigenvalue equation (3) as follows:

$$Q_{j_m}(z) \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)} = \lambda_n + \sum_{j \neq j_m} Q_j(z) \frac{p_n^{(j)}(z)}{p_n(z)}.$$

Applying inequalities (5)-(9) to this we obtain

$$
\begin{aligned}
\left\| Q_{j_m} \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)} \right\|_{2sn^d} &\leq |\lambda_n| + \sum_{j \neq j_m} \left\| Q_j \frac{p_n^{(j)}(z)}{p_n(z)} \right\|_{2sn^d} \\
&\leq K'_{j_0} n^{j_0} + K_{j_0} n^{j_0} + \sum_{1 \leq j < j_0} K_j n^{j_0 - 1} \\
&+ \sum_{\substack{j_0 < j \leq k: \\ (\frac{j - j_0}{j - \deg Q_j} < d}}) K_j \frac{n^{j_0 - \delta}}{s^k} + \sum_{\substack{j_0 < j < j_m \\ (\frac{j - j_0}{j - \deg Q_j} = d}}) K_j n^{j_0} \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}} \\
&\leq K \cdot n^{j_0} + K \cdot \frac{n^{j_0 - \delta}}{s^k} + K \cdot n^{j_0} \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}} \qquad (10)
\end{aligned}
$$

for all $n \geq \max(n_0, n(j), n_{j_0})$, where K is some positive constant and $0 < s < 1$. For the term on the left-hand side of the rewritten eigenvalue equation above we obtain using (4) the following estimation:

$$
\frac{1}{K} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq \frac{1}{K_{j_m}} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq \left\| Q_{j_m} \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)} \right\|_{2sn^d} \qquad (11)
$$

for some constant $K \geq K_{j_m}$ which also satisfies (10). Now combining (10) and (11) we get

$$
\frac{1}{K} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq K \cdot n^{j_0} + K \cdot \frac{n^{j_0 - \delta}}{s^k} + K \cdot n^{j_0} \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}}.
$$

Dividing this inequality by n^{j_0} and multiplying by K we have

$$
\frac{1}{s^{\frac{j_m - j_0}{d}}} \leq K^2 + K^2 \cdot \frac{1}{n^\delta} \cdot \frac{1}{s^k} + K^2 \cdot \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}}.
$$

$$
\Leftrightarrow
$$

$$
\frac{1}{s^w} \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta} + K^2 \cdot \frac{s^{1/d}}{s^w}
$$

$$
\Leftrightarrow
$$

$$
\frac{1}{s^w}[1 - K^2 \cdot s^{1/d}] \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}. \qquad (12)
$$

where $w = (j_m - j_0)/d > 0$.

In what follows we will obtain a contradiction to this inequality for some small properly chosen $0 < s < 1$ and all sufficiently large n. Since $j_m \in [j_0 + 1, k]$ we have $w = (j_m - j_0)/d \geq 1/d$, and since $s < 1$ we get $s^w \leq s^{1/d} \Rightarrow 1/s^w \geq 1/s^{1/d}$. **Now choose $s^{1/d} = \frac{1}{4K^2}$, where K is the constant in (12)**. Then estimating the left-hand side of (12) we get

$$
\frac{1}{s^w}[1 - K^2 \cdot s^{1/d}] \geq \frac{1}{s^{1/d}}[1 - K^2 \cdot s^{1/d}] = 4K^2 - K^2 = 3K^2
$$

and thus from (12) we have

$$3K^2 \leq \frac{1}{s^w}[1 - K^2 \cdot s^{1/d}] \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}$$
$$\Leftrightarrow$$
$$2K^2 \leq \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}$$
$$\Leftrightarrow$$
$$n^\delta \leq \frac{1}{2} \cdot \frac{1}{s^k} = \frac{1}{2}(2K)^{2dk}.$$

We therefore obtain a contradiction to this inequality, and hence to inequality (12) and thus to the eigenvalue equation (3), if $n^\delta > \frac{1}{2}(2K)^{2dk}$ and $s = 1/(2K)^{2d}$, and consequently all roots of p_n cannot be contained in a disc of radius $s \cdot n^d$ for such choices on s and n, whence $r_n > s \cdot n^d$ where r_n denotes the largest modulus of all roots of p_n, so clearly there exists some positive constant c such that $\lim_{n\to\infty} \frac{r_n}{n^d} \geq c$. □

3 Open Problems and Conjectures

3.1 The upper bound

Based upon numerical evidence from computer experiments (some of which is presented in [1]) we are led to assert that there exists a constant C_0, which depends on T only, such that

$$r_n \leq C_0 \cdot n^d \tag{13}$$

holds for all sufficiently large integers n. We refer to this as the **upper-bound conjecture**. Computer experiments confirm that the zeros of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ are contained in a compact set when $n \to \infty$.

3.2 The measures $\{\mu_n\}$

Consider the sequence of discrete probability measures

$$\mu_n = \frac{1}{n} \sum_{\nu=1}^{\nu=n} \delta(\frac{\alpha_\nu}{n^d})$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of the eigenpolynomial p_n and d is as in Definition 1. Assuming (13) the supports of $\{\mu_n\}$ stay in a compact set in $\mathbb{C}$. Next, by a **tree** we mean a connected compact subset Γ of $\mathbb{C}$ which consists of a finite union of real-analytic curves and where $\hat{\mathbb{C}} \setminus \Gamma$ is simply connected (here $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is the extended complex plane). Computer experiments from [1] lead us to the following

Conjecture 1. For each operator T the sequence $\{\mu_n\}$ converges weakly to a probability measure μ_T which is supported on a certain tree Γ_T.

3.3 The determination of μ_T

Given $T = \sum_{j=1}^{k} Q_j(z)D^j$ and $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i}z^i$ we obtain an algebraic function $y_T(z)$ which satisfies the following algebraic equation (also see [1]):

$$q_{j_0,j_0} \cdot z^{j_0} \cdot y_T^{j_0}(z) + \sum_{j\in J} q_{j,\deg Q_j} \cdot z^{\deg Q_j} \cdot y_T^{j}(z) = q_{j_0,j_0},$$

where $J = \{j : (j - j_0)/(j - \deg Q_j) = d\}$, i.e. the sum is taken over all j for which d is attained. In addition y_T is chosen to be the unique single-valued branch which has an expansion

$$y_T(z) = \frac{1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \dots$$

at $\infty \in \hat{\mathbb{C}}$. Let $\mathbb{D}_T$ be the discriminant locus of y_T, i.e. this is a finite set in $\mathbb{C}$ such that the single-valued branch of y_T in an exterior disc $|z| > R$ can be continued to an (in general multi-valued) analytic function in $\hat{\mathbb{C}} \setminus \mathbb{D}_T$. If Γ_T is a tree which contains $\mathbb{D}_T$, we obtain a single-valued branch of y_T in the simply connected set $\Omega_{\Gamma_T} = \hat{\mathbb{C}} \setminus \Gamma_T$. It is easily seen that this holomorphic function in Ω_{Γ_T} defines a locally integrable function in the sense of Lebesgue outside the nullset Γ_T. A tree Γ_T which contains $\mathbb{D}_T$ is called T-positive if the distribution defined by

$$\nu_{\Gamma_T} = \frac{1}{\pi} \cdot \bar{\partial} y_T / \bar{\partial} \bar{z}$$

is a probability measure.

3.4 Main conjecture

Now we announce the following which is experimentally confirmed in [1]:

For each operator T, the limiting measure μ_T in Conjecture 1 exists. Moreover, its support is a T-positive tree Γ_T and one has the equality $\mu_T = \nu_{\Gamma_T}$ which means that when $z \in \hat{\mathbf{C}} \setminus \Gamma_T$ the following holds:

$$y_T(z) = \int_{\Gamma_T} \frac{d\mu_T(\zeta)}{z - \zeta}.$$

Remark. For *non-degenerate exactly-solvable operators* (i.e. when $\deg Q_k = k$) it was proved in [2] that the roots of all eigenpolynomials stay in a compact set of $\mathbb{C}$, and the unscaled sequence of probability measures $\{\mu_n\}$ converge to a measure supported on a tree, i.e. the analogue of the main conjecture above holds in the non-degenerate case.

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