TWO DIAGRAMMATIC DESCRIPTIONS OF THE EXTENDED SOERGEL CATEGORY OF AFFINE TYPE A

Abstract. We define a reduced version of the diagrammatic category of extended Soergel bimodules of affine type A introduced in [MT13] and we prove that these two diagrammatic categories are equivalent.

Introduction

To understand some algebraic properties of a given object like a group or an algebra, one can use the process called categorification. It consists in constructing a categorical analogue of the object that enables to derive informations about the structure of the object. Defining such categories and proving that their Grothendieck group is indeed isomorphic to the algebraic object often relies on deep algebra and geometry. A diagrammatic approach to categorification has started with [Lau10, KL09, KL11, KL10], [EK10], with the benefit of working with more handable categories but also providing diagrammatic descriptions of already existing categories.

Here we are interested in Hecke algebras $\mathcal{H}_{\mathcal{W}}$. Soergel [Soe07] obtained a categorification of those by considering some categories of bimodules over a polynomial ring. More precisely the category of Soergel bimodules is the Karoubi envelope of the $\mathbb{Q}$–linear monoidal graded category generated by the Bott-Samelson bimodules B_i, for all simple reflections σ_i of $\mathcal{W}$. In the finite type A case, Elias and Khovanov [EK10] gave a diagrammatic description of Soergel category. They associate an object i to any Bott-Samelson bimodule B_i, planar diagrams colored by these integers to certain generating bimodule maps and give a complete set of relations on them. Their diagrammatic category is equivalent to the category of Soergel bimodules and thus provides a presentation by generators and relations of Soergel category. Elias and Williamson [EW] generalized this diagrammatic approach to Soergel bimodules of other types, including the affine type A. In [MT13], the extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is categorified using an extension of the category of Soergel bimodules of affine type A, where some twisted bimodules $B_{\rho^{\pm 1}}$ are added to the generating Bott-Samelson bimodules. An Elias–Khovanov–like diagrammatic category for extended affine type A is also constructed, its definition is recalled in Section 2 of the present note. It is a straightforward generalization of Elias and Khovanov category, where oriented strands are introduced on top of their colored strands. The Karoubi envelope of the diagrammatic category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ obtained in this way is proved to be equivalent to the category of extended Soergel bimodules $\mathrm{Kar}\mathcal{EB}im_{\widehat{A}_{r-1}}$ of affine type A, and therefore provides a diagrammatic categorification of the extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$.

In the same way that the generator corresponding to the reflection σ_r is not necessary to present the extended affine Hecke algebra (see the two presentations $(P1)$ and $(P2)$ of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ in Section 1), removing B_r from the set of generators of the extended Soergel category $\mathrm{Kar}\mathcal{EB}im_{\widehat{A}_{r-1}}$ leads to a full subcategory which is in fact equivalent to $\mathrm{Kar}\mathcal{EB}im_{\widehat{A}_{r-1}}$ since there exists a unique isomorphism (up to scalar) between B_r and $B_\rho \otimes_R B_{r-1} \otimes_R B_{\rho^{-1}}$. That somehow reflects at a categorical level the two presentations of the extended affine Hecke algebra given in Section 1. To play the same game for the diagrammatic category, the naive idea would be to forget the integer r in the set of generating objects of $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ and any morphism in which color r appears. But removing color r this brutally in the diagrammatic setting affects the morphisms of the category in such a way that the objects $+(r-1)-$ and $-1+$ are not isomorphic anymore. Finding a diagrammatic category that reflects the presentation $(P2)$ of the extended affine Hecke algebra is a bit more subtle. Such a reduced version $\mathcal{DEB}im^{red}_{\widehat{A}_{r-1}}$ of the diagrammatic extended Soergel category is defined in Section 3 and it is proved to be equivalent to $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ in Section 4.

1. The extended affine Hecke algebra

Let q be a formal parameter. The extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is the q–deformation of the group algebra of the extended affine Weyl group $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ where the relations of involutivity of the generating simple reflections σ_i are replaced by the quadratic relations

$$T_{\sigma_i}^2 = (q^2-1)T_{\sigma_i} + q^2 \quad \text{for all } i = 1, \dots, r.$$

The extended Hecke algebra can be described using the Kazhdan-Lusztig generators $b_i := C'_{\sigma_i} = q^{-1}(1+T_{\sigma_i})$ instead of the standard generators T_{σ_i} providing a presentation whose relations possess only positive coefficients which is a desirable property for categorification purposes. The extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ admits the following presentation $(P1)$ with generators

$$\{T_\rho, T_\rho^{-1}, b_i, i = 1, \dots r\} \tag{1.1}$$

subject to the relations

$$b_i^2 = (q+q^{-1})b_i \qquad \text{for } i = 1, \dots, r \tag{1.2}$$

$$b_i b_j = b_j b_i \qquad \text{for distant } i, j = 1, \dots, r \tag{1.3}$$

$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i \qquad \text{for } i = 1, \dots, r \tag{1.4}$$

$$T_\rho b_i T_\rho^{-1} = b_{i+1} \qquad \text{for } i = 1, \dots, r \tag{1.5}$$

where the indices have to be understood modulo r, e.g. $b_{r+1} = b_1$ by definition. We say that i and j are distant (resp. adjacent) if $j \not\equiv i \pm 1 \mod r$ (resp. $i \equiv j \pm 1 \mod r$).

One can do without the generator b_r at the cost of adding the relations $T_\rho^r b_i T_\rho^{-r} = b_i$ for all $i = 1, \dots, r-1$ or equivalently the relation $T_\rho b_{r-1} T_\rho^{-1} = T_\rho^{-1} b_1 T_\rho$. Thus another presentation of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$, denoted $(P2)$, is given by the following generators

$$\{T_\rho, T_\rho^{-1}, b_i, i = 1, \dots r-1\} \tag{1.6}$$

satisfying the relations

$$\begin{aligned}
b_i^2 &= (q+q^{-1})b_i && \text{for } i=1,\dots,r-1 && (1.7)\\
b_ib_j &= b_jb_i && \text{for distant } i,j=1,\dots,r-1 && (1.8)\\
b_ib_{i+1}b_i + b_{i+1} &= b_{i+1}b_ib_{i+1} + b_i && \text{for } i=1,\dots,r-2 && (1.9)\\
T_\rho b_i T_\rho^{-1} &= b_{i+1} && \text{for } i=1,\dots,r-2 && (1.10)\\
T_\rho b_{r-1} T_\rho^{-1} &= T_\rho^{-1} b_1 T_\rho && && (1.11)
\end{aligned}$$

Another important presentation of this algebra is the so-called Bernstein presentation, it corresponds to the q–deformation of the presentation of the extended affine Weyl group with generators the $r-1$ simple reflections of non-affine type A and the r translations along the weight lattice $\mathbb{Z}^r$ of $\mathfrak{gl}_r$. Categorifying this latter presentation is a much more difficult problem and is beyond the scope of this note. For more details about the extended affine Hecke algebra and the relations between its different presentations, see [DG07, CP96, DDF12, MT13].

2. The diagrammatic extended Soergel category

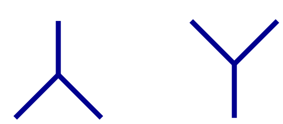
Consider the category whose objects are graded finite sequences of integers belonging to $\{1,\dots,r\}$ and the symbols $+$ and $-$; graphically represented by sequences of colored points (read from left to right) of the x–axis of the real plane $\mathbb{R}^2$. The morphisms are equivalence classes of $\mathbb{Q}$–linear combinations of graded planar diagrams in $\mathbb{R}\times[0,1]$ (read from bottom to top) and composition is defined by vertically glueing the diagrams and rescaling the vertical coordinate. These morphisms are defined by generators and relations listed below. This category possesses a monoidal structure given by stacking sequences and diagrams next to each other.

Let $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$ be the category containing all direct sums and grading shifts of these objects and diagrams and let $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ be the subcategory whose morphisms are only the degree-preserving diagrams. The diagrammatic extended Soergel category is by definition its Karoubi envelope $\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}}$.

In the diagrams, the strands whose endpoints are $+$ or $-$–signs are oriented and the other strands are non-oriented. The non-oriented strands are colored with integers belonging to $\{1,\dots,r\}$. By convention, no label means that the equation holds for any color $i\in\{1,\dots,r\}$.

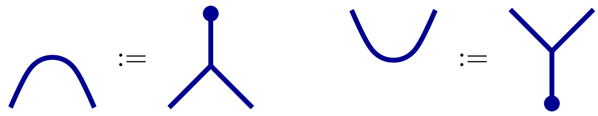
The morphisms of $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ are built out of the following generating diagrams.

- Generators involving only one color:

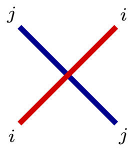


Degree	1	1	-1	-1
	enddot_i	$\mathrm{startdot}_i$	merge_i	split_i

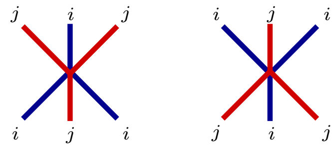
Let us define the cap and cup as the followinf composites



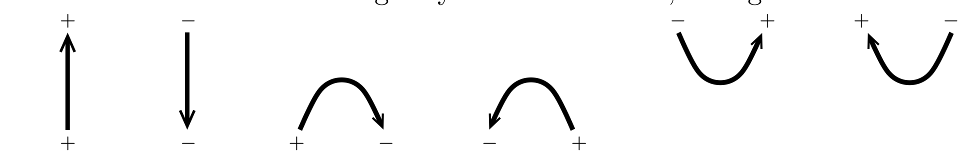
- Generators involving two colors:
  - the 4–valent vertex with distant colors, of degree 0, denoted $4\text{vert}_{i,j}$



  - and the 6–valent vertices with adjacent colors i and j, of degree 0, denoted $6\text{vert}_{i,j}$ and $6\text{vert}_{j,i}$

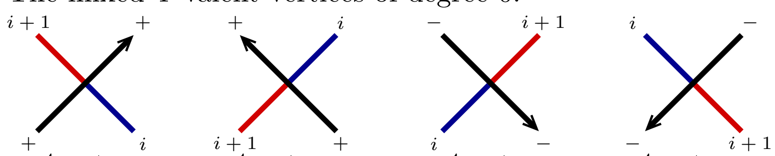


- Generators involving only oriented strands, of degree 0:



$+$strand $-$strand $+$cap $-$cap $-$cup $+$cup

- Generators involving oriented strands and adjacent colored strands. The mixed 4–valent vertices of degree 0:



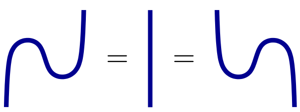
$4\text{vert}_{+,i}$ $4\text{vert}_{i+1,+}$ $4\text{vert}_{i,-}$ $4\text{vert}_{-,i+1}$

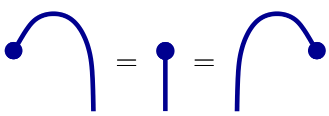
- Generators involving boxes, of degree 2, denoted box_i

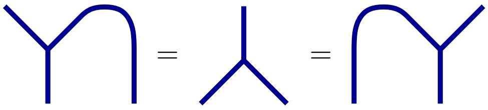
  for all $i = 1, \ldots, r$, and denoted box_y

The generating diagrams are subject to the following relations.

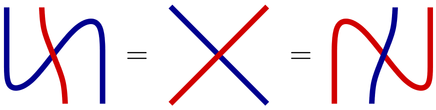
- Isotopy relations:

 (2.1)

 (2.2)



(2.3)



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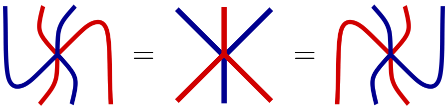

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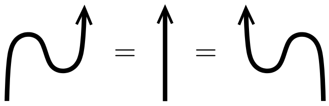

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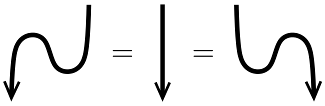

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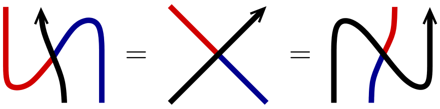

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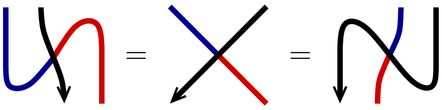
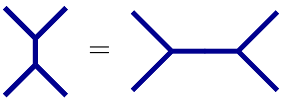


(2.9)

- Relations involving one color:



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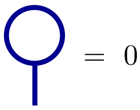

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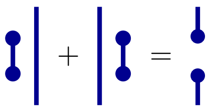
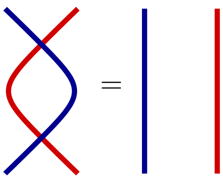
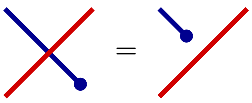


(2.12)

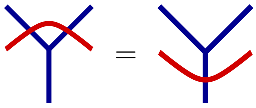
- Relations involving two distant colors:



(2.13)

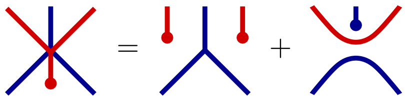


(2.14)

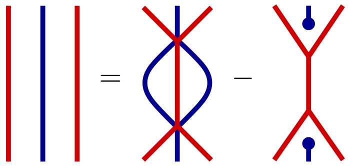


(2.15)

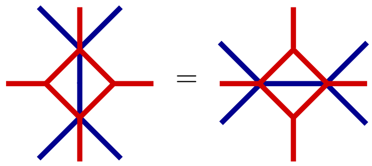
- Relations involving two adjacent colors:



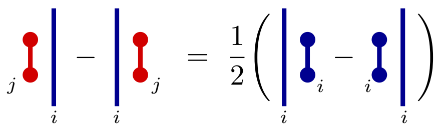
(2.16)



(2.17)

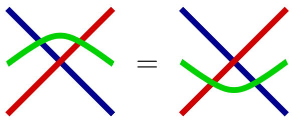


(2.18)



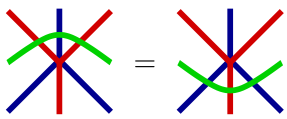
(2.19)

- Relation involving three distant colors (i.e. three colors forming a parabolic subgroup of type $A_1 \times A_1 \times A_1$):



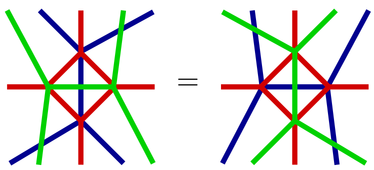
(2.20)

- Relation involving two adjacent colors and one distant from the other two (i.e. three colors forming a parabolic subgroup of type $A_2 \times A_1$):



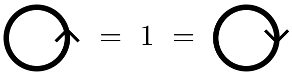
(2.21)

- Relation involving three adjacent colors forming a parabolic subgroup of type A_3 (i.e., the case $\widehat{A}_2$ is excluded):



(2.22)

- Relations involving only oriented strands:



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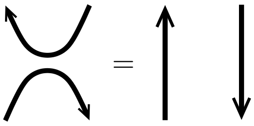
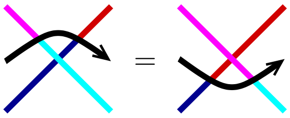


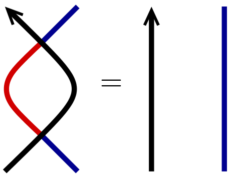
(2.24)

- Relations involving oriented strands and distant colored strands:



(2.25)

- Relations involving oriented strands and two adjacent colored strands:



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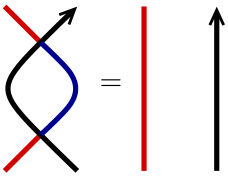

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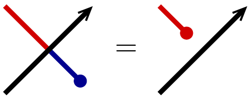

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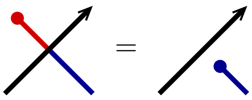

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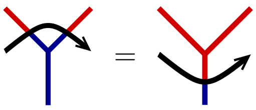
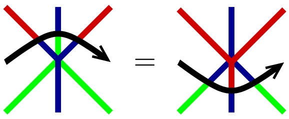
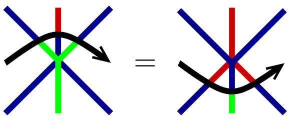


(2.30)

- Relations involving oriented strands and three adjacent colored strands:

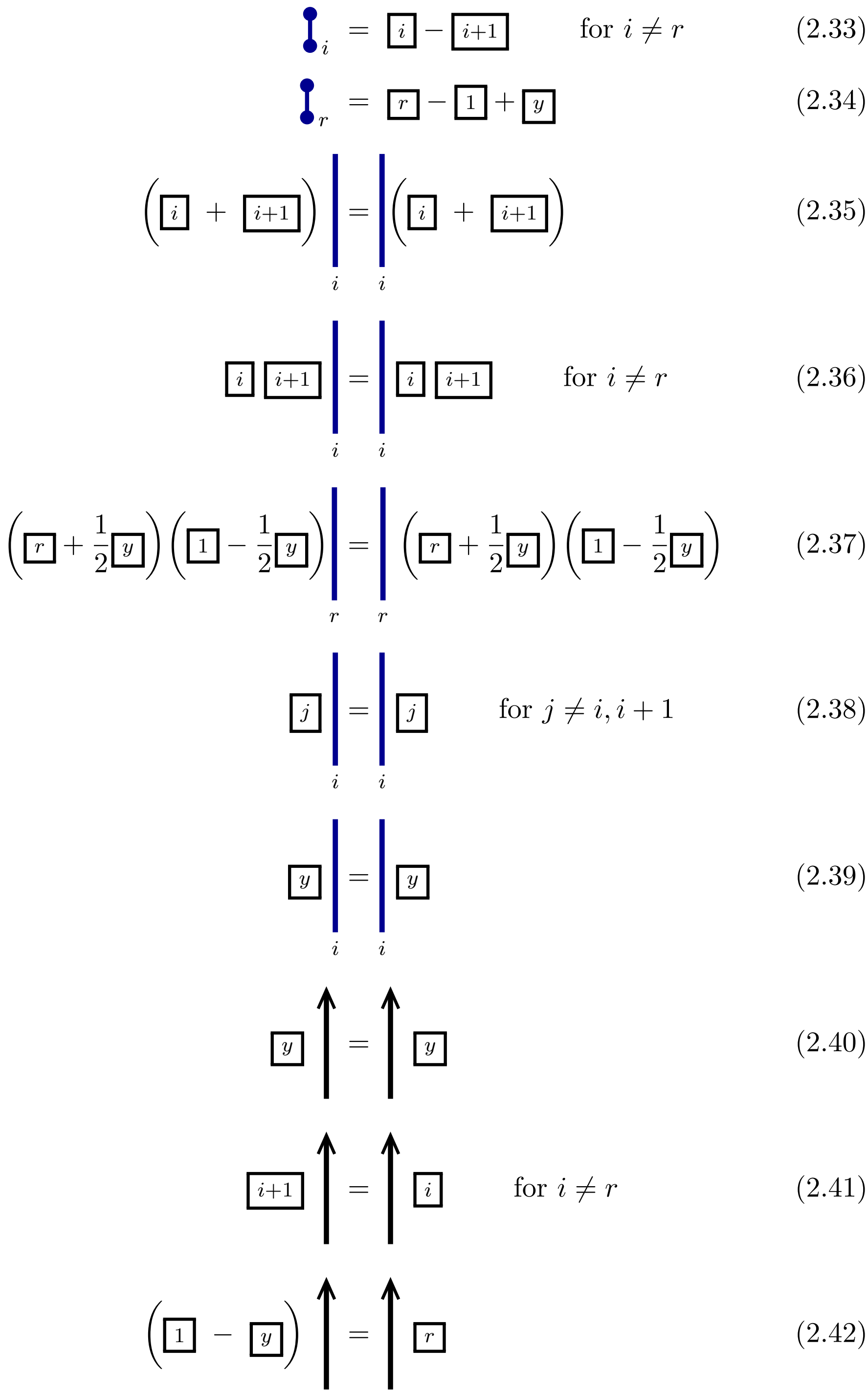


(2.31)



(2.32)

- Relations involving boxes:

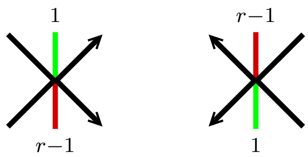


Remark 2.1. Similar relations also hold for opposite orientation of the oriented strands and can be deduced from the one listed using isotopy invariance.

3. The reduced diagrammatic extended Sorgel category

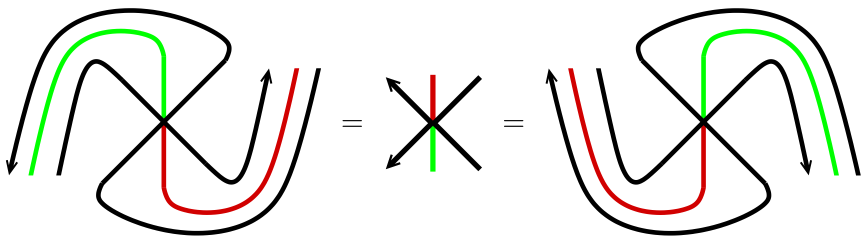
The definition of the reduced diagrammatic category $\mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}$ is very similar to the definition of the non-reduced one presented in the former section. The idea here is to forget about the color r. Consider as objects sums of graded finite sequences of integers belonging to $\{1, \ldots, r-1\}$ and the symbols $+$ and $-$. The morphisms are again $\mathbb{Q}$–linear combinations of graded planar diagrams given by generators and relations. More precisely the morphisms of the $\mathbb{Q}$–linear monoidal category $\mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}$ are the diagrams built out of:

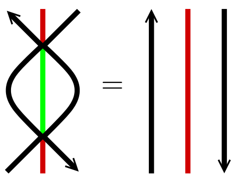
- all generators of $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ that do not contain a strand colored by r
- the mixed 6–valent vertices, of degree 0, denoted by $6\mathrm{vert}_{+,r-1}$ and $6\mathrm{vert}_{-,1}$

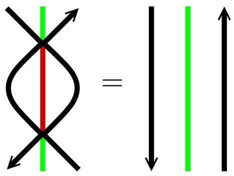


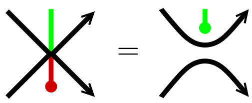
subject to the following relations:

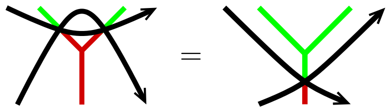
- all relations of $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ in which strands colored by r do not appear
- the following relations involving the new generators $6\mathrm{vert}_{+,r-1}$ and $6\mathrm{vert}_{-,1}$


(3.1)

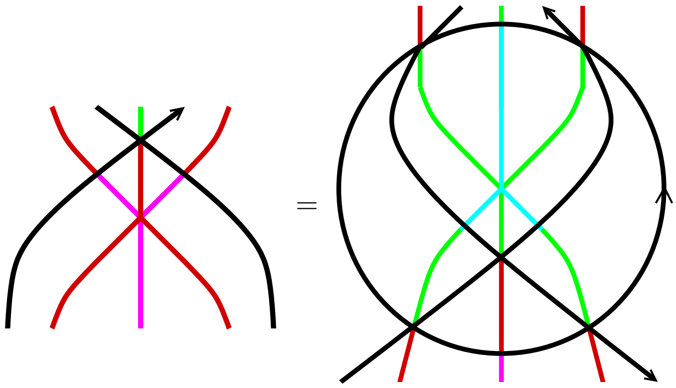

(3.2)


(3.3)

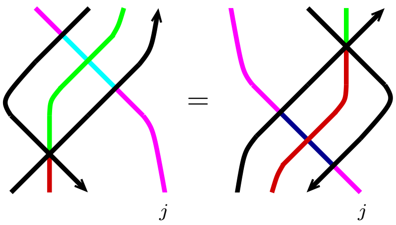

(3.4)



(3.5)



(3.6)



for all $j \neq 1, r-1$ (3.7)

Consider the subcategory $\mathcal{DEB}im^{red}_{\widehat{A}_{r-1}}$ with same objects but only degree-preserving morphisms. The reduced diagrammatic extended Soergel category is by definition its Karoubi envelope $\mathrm{Kar}\mathcal{DEB}im^{red}_{\widehat{A}_{r-1}}$.

Remark 3.1. We keep box_r in the set of generating morphisms.

4. Equivalence of the diagrammatic descriptions

Let us construct an equivalence between these two categories consisting in two functors $\mathcal{F}$ an $\mathcal{G}$ which are $\mathbb{Q}$–linear monoidal and preserve isotopy invariance and grading. Let us first define the functor $\mathcal{F} : \mathcal{DEB}im^{*}_{\widehat{A}_{r-1}} \to \mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}$ as follows:

• on objects:

$$\mathcal{F}(i) = i \qquad \text{for all } i = 1, \ldots, r-1 \tag{4.1}$$

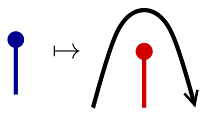
$$\mathcal{F}(r) = +(r-1)- \tag{4.2}$$

$$\mathcal{F}(\pm) = \pm \tag{4.3}$$

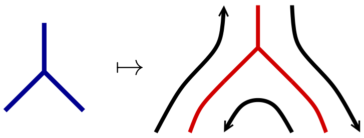
• on morphisms:

The functor $\mathcal{F}$ is the identity on any generator for which the color r does not appear. Otherwise the images of the generators enddot_r, merge_r,

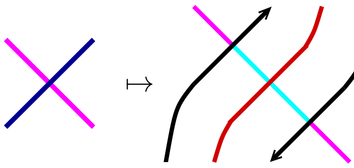
$4\mathrm{vert}_{r,i}$, $6\mathrm{vert}_{1,r}$, $6\mathrm{vert}_{r,r-1}$, $4\mathrm{vert}_{+,r}$ and $4\mathrm{vert}_{r,+}$ under $\mathcal{F}$ are given by



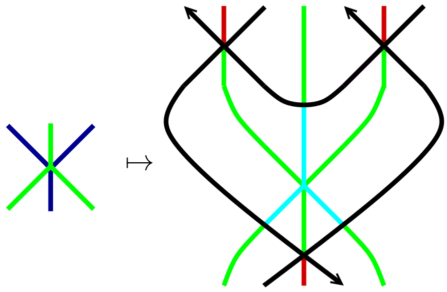
(4.4)



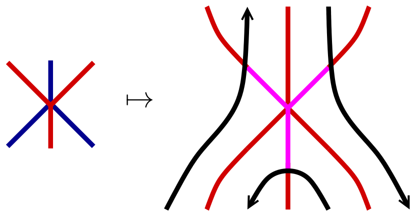
(4.5)



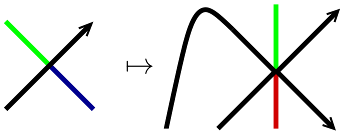
(4.6)



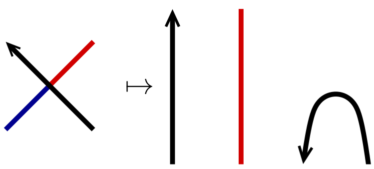
(4.7)



(4.8)



(4.9)



(4.10)

Finally we can deduce the images of the remaining generators from (4.4)–(4.10) since we require that $\mathcal{F}$ preserves isotopy invariance.

Now let us define the functor $\mathcal{G} : \mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}} \to \mathcal{DEB}im^{*}_{\widehat{A}_{r-1}}$ as follows:

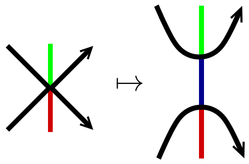
- on objects:

$$\mathcal{G}(i) = i \qquad \text{for all } i = 1, \ldots, r-1 \tag{4.11}$$

$$\mathcal{G}(\pm) = \pm \tag{4.12}$$

- on morphisms:

The functor $\mathcal{G}$ is the identity on all generators except on $6\text{vert}_{+,r-1}$ and $6\text{vert}_{-,1}$. The image of $6\text{vert}_{+,r-1}$ under $\mathcal{G}$ is given by



(4.13)

The image of $6\text{vert}_{-,1}$ under $\mathcal{G}$ follows from (4.13) and preservation of isotopy invariance.

One can verify that the relations between morphisms in these categories are preserved by the functors $\mathcal{F}$ and $\mathcal{G}$ ensuring that these two functors are well-defined $\mathbb{Q}$–linear monoidal grading preserving functors.

Theorem 4.1. *The categories $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ and $\mathcal{DEB}im^{red}_{\widehat{A}_{r-1}}$ are equivalent. The functors $\mathcal{F}$ and $\mathcal{G}$ realize this equivalence.*

Proof. Let us exhibit two natural isomorphisms

$$\eta : \mathbf{1}_{\mathcal{DEB}im^*_{\widehat{A}_{r-1}}} \to \mathcal{G} \circ \mathcal{F}$$

where $\mathbf{1}_{\mathcal{DEB}im^*_{\widehat{A}_{r-1}}}$ is the endofunctor identity of the category $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$, and

$$\eta' : \mathcal{F} \circ \mathcal{G} \to \mathbf{1}_{\mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}}$$

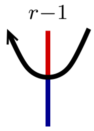
where $\mathbf{1}_{\mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}}$ is the endofunctor identity of the category $\mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}$.

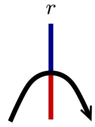
Since both functors $\mathcal{F}$ and $\mathcal{G}$ and both categories are monoidal and that their morphims are defined by generators and relations, we only need to define η and η' on generating objects and check the commutativity between the natural transformations and the morphisms of the corresponding category only for generating morphisms.

Let us start with the definition of η. The functor $\mathcal{G} \circ \mathcal{F}$ is equal to the identity on all generating objects of $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$ except on r. So

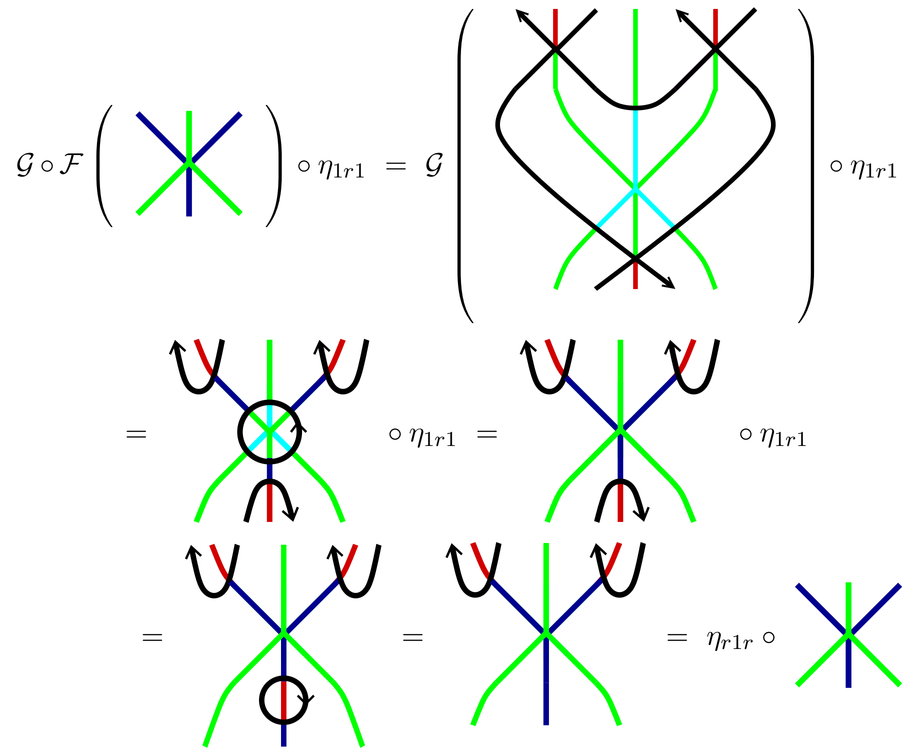
$\eta_- =$  , $\eta_+ =$  and $\eta_i =$ for all $i \neq r$

which are obviously all isomorphisms. While $\mathcal{G} \circ \mathcal{F}(r) = +(r-1)-$, so one defines η_r as follows

$\eta_r =$ 

whose inverse is given by 

So it only remains to check that η commutes with any generating morphism *i.e.* for any $f : X \to Y$ one has $\eta_Y \circ f = (\mathcal{G} \circ \mathcal{F})(f) \circ \eta_X$. This is straightforward when the color r does not appear in the generating morphism because then $\eta_X = \text{id}_X$, $\eta_Y = \text{id}_Y$ and $\mathcal{G} \circ \mathcal{F}(f) = f$. When color r appears, the reader can check that this follows immediately from the relations that hold in $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$. Let us just explicit the case of $f = 6\text{vert}_{1,r}$ as an example.



Now let us turn to the definition of the natural transformation η'. For any object X in $\mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}$, observe that $\mathcal{F} \circ \mathcal{G}(X) = X$ which implies

$$\eta'_X = \text{id}_X \quad \text{for all } X \in \mathcal{DEB}im^{red*}_{\widehat{A}_{r-1}}$$

So to prove that η' is a well-defined natural isomorphism it only remains to check that the functor $\mathcal{F} \circ \mathcal{G}$ is equal to the identity functor on all generating morphisms. This simply follows from the fact that $\mathcal{F}$ and $\mathcal{G}$ are themselves the identity functor when the generator is not $6\text{vert}_{+,r-1}$ or $6\text{vert}_{-,1}$. For these latter two, it is a simple verification that is left to the reader.

The degree-preserving functors $\mathcal{F}$ and $\mathcal{G}$ restrict to the categories $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ and $\mathcal{DEB}im^{red}_{\widehat{A}_{r-1}}$ and make an equivalence between these two categories totally explicit. □

Corollary 4.2. *The category $\mathcal{DEB}im^{red}_{\widehat{A}_{r-1}}$ provides another diagrammatic presentation by generators and relations of the category of extended Soergel bimodules of affine type A.*

Remark 4.3. We have made an arbitrary choice for the image $+(r-1)-$ of r under $\mathcal{F}$, we could have chosen any other isomorphic object *e.g.* $\mathcal{F}(r) = -1+$ or more generally $\mathcal{F}(r) = (-)^i i (+)^i$ and the images of the morphisms involving color r accordingly.

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