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DIFFERENTIAL GEOMETRY

PROBLEM 1-5

1. Show that the curvature of a plane curve is in general given by the formula

$$\kappa(t) = \frac{\det(\dot{c}(t),\, \ddot{c}(t))}{|\dot{c}(t)|^3}.$$

2. Show that the curvature and torsion of a space curve are in general given by the formulae

$$\kappa(t) = \frac{|\dot{c}(t) \times \ddot{c}(t)|}{|\dot{c}(t)|^3},$$

$$\tau(t) = \frac{\det(\dot{c}(t),\, \ddot{c}(t),\, c\,(t))}{|\dot{c}(t) \times \ddot{c}(t)|^2},$$

where $x \times y$ is the vector product in R^3.

3. Let $c(t)$ be a curve in R^n parametrized by arc length with the property that $|c(t)|^2$ has a local maximum at t_0. Let $p_0 = c(t_0)$ and $\rho^2 = |p_0|^2$. Show that

$$\kappa(t_0) \geq \frac{1}{\rho}$$

where $\kappa(t_0) = |\ddot{c}(t_0|$ (which is equal to the first curvature of $c(t)$ at t_0 if it is defined).

4. Let A, B, C, D, E, G be constants such that

$$AD + BE + CG \neq 0.$$

Consider the curves $c(t) = (x(t), y(t), z(t)) \in R^3$ whose tangent vectors at each point $P = (x, y, z)$ in space are in the plane L_P through P whose normal is

$$(Bz - Cy + D, Cx - Az + E, Ay - Bx + G).$$

It is clear that such curves satisfy

$$(Bz(t) - Cy(t) + D)\dot{x}(t) + (Cx(t) - Az(t) + E)\dot{y}(t) + (Ay(t) - Bx(t) + G)\dot{z}(t) = 0.$$

Let $c(t)$ be a curve which satisfies such a condition and assume that $\dot{c}(t)$, $\ddot{c}(t)$ are linearly independent at P. Let τ be the torsion of the curve at $P = (x, y, z)$.

a) Show that L_P is the osculating plane of the curve at P.

b) Show that τ at P satisfies the formula

$$\tau = \frac{-(AD + BE + CG)}{(Bz - Cy + D)^2 + (Cx - Az + E)^2 + (Ay - Bx + G)^2}.$$

5. Let $c(t)$ be a curve in R^3 parametrized by arc length and let $t_0 = 0 \in I$. Let the Frenet-frame at $c(0)$ be $e_i(0) = e_i$ and let $\kappa(0) = \kappa_0$, $\tau(0) = \tau_0$. We then have the following well known series expansion for t close to 0

$$c(t) - c(0) = te_1 + \frac{1}{2}\kappa_0 t^2 e_2 + \frac{1}{6}\kappa_0\tau_0 t^3 e_3 + o(t^3).$$

The projections of the curve in a small neighbourhood of $c(0)$ in the planes of the Frenet-frame at that point are therefore approximated by the following curves:

a) the projection onto the $(e_1, e_3)-$plane, the rectifying plane, is described by the cubical parabola

$$x = t, \quad y = 0, \quad z = \frac{1}{6}\kappa_0\tau_0 t^3;$$

b) the projection onto the $(e_2, e_3)-$plane, the normal plane, is described by the semi-cubical parabola with a cusp at origo

$$x = 0, \quad y = \frac{1}{2}\kappa_0 t^2, \quad z = \frac{1}{6}\kappa_0\tau_0 t^3;$$

c) the projection onto the $(e_1, e_2)-$plane, the osculating plane, is decribed by the parabola

$$x = t, \quad y = \frac{1}{2}\kappa_0 t^2, \quad z = 0.$$

i) Draw these projektions and their orientations in the case $\tau_0 > 0$ and $\tau_0 < 0$.

ii) We shall now study what the curve looks like along the negative e_1-axis if we raise or lower our eyes somewhat above or under the osculating plane. This means that we shall find the projection of the curve in a $(f_2, f_3)-$plane where the $ON-$system (f_1, f_2, f_3) is created from the system (e_1, e_2, e_3) by letting the e_1, e_3-plane rotate a small angle α $(-\varepsilon < \alpha < \varepsilon)$ with the e_2-axis as axis of rotation. Derive the analytical expression of the projection and draw pictures of how it looks for different values of α in the cases $\tau_0 > 0$ and $\tau_0 < 0$.