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DIFFERENTIAL GEOMETRY

PROBLEM 6

6. $f : U \to R^3$ is a regular surface with the external unit normal n, Gauss curvature $K = \kappa_1 \kappa_2 \neq 0$ and mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2)$. We define for every $r \in R$ a map $\bar{f} : U \to R^3$ by

$$\bar{f} = f + r\,n.$$

$\bar{f}$ is called a *parallel surface* of f.

a) Show that

$$\bar{f}_{u^1} \times \bar{f}_{u^2} = (1 + 2rH + r^2 K)(f_{u^1} \times f_{u^2}).$$

Remark. $n_{u^i} = \sum a_i^k\, f_{u^k}$ where (a_i^k) is the matrix of Weingarten's map.

b) For what values of r is $\bar{f}$ regular?

The result in a) shows that $\bar{n} = \pm n$ at all points where $\bar{f}$ is regular. We can therefore identify the tangent plane $T_u\bar{f}$ to the parallel surface $\bar{f}$ with the tangent plane $T_u f$ to the surface f at the point u.

c) Determine the matrix for $d\bar{f}$ with respect to the basis f_{u^1}, f_{u^2} i $T_u\bar{f}$.

d) Determine the matrix of Weingarten's map $\bar{L} = +d\bar{n} \circ (d\bar{f})^{-1} : T_u\bar{f} \to T_u\bar{f}$ with respect to the basis f_{u^1}, f_{u^2} ($\bar{n} = n$ is the external normal). Show that the Gauss curvature $\bar{K}$ and the mean curvature $\bar{H}$ on $\bar{f}$ is given by

$$\bar{K} = \frac{K}{1 + 2rH + r^2 K}, \qquad \bar{H} = \frac{H + rK}{1 + 2rH + r^2 K}.$$

e) Let M be a surface with constant positive Gauss curvature K and without umbilics. (Examples of such surfaces can be found in Figures 3.12(b) och (c) in Klingenberg).
Let $r_1 = \frac{1}{\sqrt{K}}$ och $r_2 = -\frac{1}{\sqrt{K}}$ define the parallel surfaces M_1 and M_2 respectively. Show that M_1 and M_2 are regular surfaces with constant mean curvature $\frac{\sqrt{K}}{2}$ resp. $-\frac{\sqrt{K}}{2}$.

f) Let M be a surface with constant mean curvature $H \neq 0$ and Gauss curvature $K \neq 0$. (K is not necessarily constant). Then $r = -\frac{1}{2H}$ gives a regular parallel surface with constant Gauss curvature $4H^2$.