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DIFFERENTIAL GEOMETRY

PROBLEM 7 - 8

7. Assume that $f : U \to R^3$ is a surface whose principal curvatures κ_1 and κ_2 satisfy $\kappa_1\kappa_2 \neq 0$ and $\kappa_1 \neq \kappa_2$ at all points in U. Let n be the unit normal and let (u^1, u^2) be *principal curvature coordinates.* Show that the functions

$$b_i(u) = f(u) + n(u)/\kappa_i(u), \ i = 1, 2$$

are regular surfaces if and only if $\kappa_{1,1} \neq 0$ and $\kappa_{2,2} \neq 0$. These surfaces are called the *caustic surfaces* of f.

8. Let $f : (a, b) \times (c, d) \to R^3$ be a surface in R^3 with constant Gauss curvature $K < 0$, defined in asymptotic coordinates, and parametrized by arc length so that

$$g_{11}(u) = g_{22}(u) = 1.$$

Let $\omega(u_0^1, u_0^2)$ be the unique number $0 < \omega(u_0^1, u_0^2) < \pi$ such that $\omega(u_0^1, u_0^2)$ is the angle between $f_{u^1}(u^1, u_0^2)|_{u^1=u_0^1}$ and $f_{u^2}(u_0^1, u^2)|_{u^2=u_0^2}$, i.e.

$$g_{12}(u) = \cos\omega(u).$$

a) Show that ω satisfies the differential equation

$$\frac{\partial^2\omega}{\partial u^1 \partial u^2} = (-K)\sin\omega$$

Hint: Use Gauss' equation which in the coordinates of the problem is

$$K = \frac{1}{2\sqrt{1-(g_{12})^2}}\left[\frac{\partial}{\partial u^1}\left(\frac{g_{12,2}}{\sqrt{1-(g_{12})^2}}\right) + \frac{\partial}{\partial u^2}\left(\frac{g_{12,1}}{\sqrt{1-(g_{12})^2}}\right)\right]$$

b) Show that every polygon Q with four sides and which is bounded by parameter curves has the area

$$\frac{1}{-K}\left(\sum_{i=1}^{4}\alpha_i - 2\pi\right) \leq \frac{2\pi}{-K},$$

where α_i are the inner angles in Q.

Hint: The area element is $\sqrt{g_{11}g_{22} - (g_{12})^2} = \sin\omega \, du^1 du^2$.