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DIFFERENTIAL GEOMETRY

PROBLEM 9 - 10

9. Let S^2 be the sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. Let Δ be the angle a vector has turned in relation to its original position after parallel translation of the vector around the latitude circle with latitude angle λ. Show that

$$\Delta = 2\pi(1 - \sin\lambda).$$

In which direction has the vector turned when it returns to its original position? What is the area of the spherical surface bounded by the latitude circle on the upper half sphere?

10. When $g_{12} = 0$, i.e. in orthogonal coordinates, the equations for the geodesic lines are

$$\begin{aligned} g_{11}\,\ddot{u}^1 + \frac{1}{2}g_{11,1}\,(\dot{u}^1)^2 + g_{11,2}\,\dot{u}^1\dot{u}^2 - \frac{1}{2}g_{22,1}\,(\dot{u}^2)^2 &= 0 \quad (1) \\ g_{22}\,\ddot{u}^2 - \frac{1}{2}g_{11,2}\,(\dot{u}^1)^2 + g_{22,1}\,\dot{u}^1\dot{u}^2 + \frac{1}{2}g_{22,2}\,(\dot{u}^2)^2 &= 0 \quad (2) \end{aligned}$$

In the case of a surface of rotation

$$f = (u^1 \cos u^2, u^1 \sin u^2, z(u^1))$$

the differential equation (2) of the geodesic lines is

$$\frac{d}{ds}\left[(u^1)^2\,\dot{u}^2\right] = 0.$$

One solution is $du^2 = 0$, which shows that *the meridians are geodesic lines.* In all other cases the constant value of $(u^1)^2\,\dot{u}^2$ is denoted $1/h$, so that

$$ds = h\,(u^1)^2\,du^2.$$

Show that

$$u^2 = C \pm \int \left[\frac{1 + {z_1}^2}{h^2(u^1)^2 - 1}\right]^{\frac{1}{2}} \frac{du^1}{u^1}.$$

Hint: $ds^2 = \sum\sum g_{ij}\,du^i\,du^j$.