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DIFFERENTIAL GEOMETRY

PROBLEM 12

The Poincaré Upper Half Plane Model of H^2

The surface H^2 is in the Poincaré upper half plane model the set

$$U = \{(u, v) \in R^2 |\, v > 0\}$$

with the Riemann metric

$$ds^2 = \frac{du^2 + dv^2}{v^2}.$$

Using Gauss' equation we find immediately that this surface has constant Gauss curvature $K = -1$. The line element ds^2 in H^2 is equal to the euclidean line element $du^2 + dv^2$ multiplied by a strictly positive function. Therefore an angle measured with respect to the Riemann metric coincides with the euclidean angle.

The geodesics in the upper half plane model of H^2 are the euclidean circles and straight lines which meet the boundary $v = 0$ orthogonally. This can be shown in the following way:

In H^2 we get $g_{11} = 1/v^2, \quad g_{12} = 0, \quad g_{22} = 1/v^2$ and it follows that

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = 0, \quad \Gamma^2_{11} = -\Gamma^2_{22} = -\Gamma^1_{21} = 1/v.$$

The differential equations of the geodesics can therefore be written

$$\ddot{u} - \frac{2\dot{u}\dot{v}}{v} = 0, \quad \ddot{v} + \frac{\dot{u}^2 - \dot{v}^2}{v} = 0.$$

If $\dot{u} = 0$ then $u =$ constant. In this case it is clear that the geodesic is a straight euclidean line orthogonal to $v = 0$.

If $\dot{u} \neq 0$ we get from the first equation that $\ln(\dot{u}/v^2)$ = constant so $\dot{u} = cv^2 \neq 0$ for some constant c. In the same way we get from the second equation that $\dot{u}^2 + \dot{v}^2 = bv^2 > 0$ for some constant b. By combining these equations we get $(dv/du)^2 = \dot{v}^2/\dot{u}^2 = b/c^2v^2 - 1$. Therefore $(u - a)^2 + v^2 = b/c^2$ for some constant a. This is a circle with centre on $v = 0$ and so meets $v = 0$ orthogonally.

The isometries of H^2 are well-known maps in the upper half plane model. Let $SL(2, R)$ be the special linear group in dimension 2, i.e. the group of all real (2×2)-matrices with determinant $= 1$. $SL(2, R)$ acts on H^2 in the following way. Let $z = u + iv$. The points (u, v) in the upper half plane correspond to $z = u + iv, \;\; v > 0$. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R),$$

let

$$gz = \frac{az + b}{cz + d}.$$

Proposition

The group $SL(2, R)$ acts as a group of isometries on H^2.

Proof: Let $u + iv = z$ och

$$\frac{az + b}{cz + d} = \tilde{z}.$$

If we write $dz\, d\bar{z}$ for $du^2 + dv^2$ the line element of H^2 can be written

$$ds^2(z) = \frac{-4dz\, d\bar{z}}{(z - \bar{z})^2}, \qquad \bar{z} = u - iv.$$

As $d\tilde{z} = d((az + b)(cz + d)) = dz/((cz + d)^2$ it follows that $ds^2(z) = ds^2(\tilde{z})$, which means that $z \mapsto \tilde{z}$ is an isometry.

a) Calculate the arc length of the geodesic $c(t) = (r\cos t, r\sin t), \; 0 < t < \pi$ starting from the top of the half circle , $t = \pi/2$. (**Result:** $|\ln\tan\frac{t}{2}|$)

Calculate also the arc length of the geodesic $u = u_0$ from $v = a$ till $v = b$.
(**Result:** $(|\ln\frac{a}{b}|)$

b) Calculate the geodesic curvature k_g of the curve $v = 1$. (**Result:** $k_g = 1$)

c) A vector is parallel translated the hyperbolic distance d along the curve $v = 1$.

Calculate the angle the vector has turned during this translation.

The Poincaré Disc Model

The surface H^2 in the Poincaré disc model is the set

$$U = \{(u,v) \in R^2 | u^2 + v^2 < 4\},$$

with the Riemann metric

$$ds^2 = \left(1 - \frac{u^2+v^2}{4}\right)^{-2} (du^2 + dv^2).$$

Using Gauss' equation we find that the surface has constant Gauss curvature $K = -1$. The geodesics in the disc model correspond to the circles orthogonal to the boundary of the disc and the diameters. This is most easily seen by showing that the map

$$w = \frac{z+2i}{iz+2}$$

is an isometry of the disc model onto the half plane model.

d) Calculate the arc length r of the geodesic $c(t) = (t\cos\vartheta, t\sin\vartheta)$, $0 \le t < 2$ beginning at origo.

(**Result:** $r = \ln\frac{2+t}{2-t} = 2\frac{1}{2}\ln\frac{1+t/2}{1-t/2} = 2\tanh^{-1}(t/2)$)

e) Show that in geodesic polar coordinates

$$ds^2 = dr^2 + \sinh^2(r)\, d\theta^2.$$

Hint: From the problem above follows that $t = 2\tanh(r/2)$.
Use $u = t\cos\theta, v = t\sin\theta$ and the given metric.