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DIFFERENTIAL GEOMETRY

EXAMPLES 1-4

1. Let $f : U \to R^3$ be

$$(u,v) \to (\cos u \cos v, \cos u \sin v, \sin u), \quad (u,v) \in]-\frac{\pi}{2}, \frac{\pi}{2}[\times R.$$

The image $f(U)$ of U is then the two-punctured sphere $S^2 \setminus \{0,0,\pm 1\}$. Then
a) $E = 1, \quad F = 0, \quad G = \cos^2 u$

b) $n(u,v) = \dfrac{f_u \times f_v}{|f_u \times f_v|} = -f(u,v)$

c) $II = -dn \cdot df = df \cdot df = I$

d) $\kappa_1 = \kappa_2 = 1, \quad H = K = 1.$
2. Let $f : U \to R^3$ be the torus

$$(u,v) \to ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u), \quad 0 < b < a, \quad (u,v) \in R \times R.$$

Then

a) f is regular at each point

b)$E = b^2, \quad F = 0, \quad G = (a + b\cos u)^2$

c)$L = b, \quad M = 0, \quad N = (a + b\cos u)\cos u$

d) $a_1^1 = \dfrac{1}{b}, \quad a_1^2 = a_2^1 = 0, \quad a_2^2 = \dfrac{\cos u}{a + b\cos u}, \quad \kappa_1 = a_2^2 < \kappa_2 = a_1^1 = \dfrac{1}{b}$

e) $K = \dfrac{\cos u}{b(a + b\cos u)}, \quad H = \dfrac{a + 2b\cos u}{2b(a + b\cos u)}.$

Remark $\quad a_i^k = \sum_j h_{ij} g^{jk}.$

3. A surface $f : U \to R^3$ defined by $f(u,v) = (u, v, F(u,v))$ can be described as $z = F(x,y)$. Then

a)

$$(g_{ij}) = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix}, \quad (h_{ij}) = \frac{1}{(1+F_u^2+F_v^2)^{\frac{1}{2}}} \begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix}$$

b)

$$2H = \frac{(1+F_u^2)F_{vv} + (1+F_v^2)F_{uu} - 2F_uF_vF_{uv}}{(1+F_u^2+F_v^2)^{\frac{3}{2}}}.$$

4. $f(u,v) = (u, v, u(u^2 - 3v^2))$ is a so called "monkey saddle". It is a saddle used by a monkey riding a bicycle. There are two depressions for its legs, and an extra one for its tail. Show that the surface has a planar, umbilic point at origo, i.e. that $K = 0$ at origo and that the normal curvatures there are equal in all directions.