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2-DIMENSIONAL INTRINSIC GEOMETRY

The axiomatic method in 2-dimensional geometry makes it possible to study a plane or a surface as an object in itself without an embedding in a surrounding space. The method has its limitations though, for example distances and areas are hard or even impossible to determine. In his famous paper 1827 on curved surfaces Gauss showed how to do geometric calculations in an efficient way in a surface without having to consider the surface as embedded in a surrounding space. This work became the foundation in 1854 for Riemann's theory of curved spaces, in which the geometry of the planes in a space are determined by the space itself. Locally (and sometimes even globally) intrinsic 2-dimensional geometry, i.e. geometry without any exterior space, can be described in an open set U in E^2, where we have the standard scalar product

$$x \cdot y = x^1 y^1 + x^2 y^2,$$

and where the vectors $x = (x^1, x^2)$, $y = (y^1, y^2)$ are expressed in the basis $e_1 = (1,0)$, $e_2 = (0,1)$.
To each point $P \in U$ we associate a tangent plane $T_P E^2$ which is a vector space isomorphic to R^2. To obtain a basis in each such tangent plane the basis e_1, e_2, in R^2 is translated parallel to itself to P. In every tangent plane $T_P E^2$, where the point $P = (u,v) \in U$ and

$$X_P = (x_P^1, x_P^2) \in T_P E^2 \quad \textit{and} \quad Y_P = (y_P^1, y_P^2) \in T_P E^2,$$

we define a scalar product

$$g_P(X_P, Y_P) = E(u,v) x_P^1 y_P^1 + F(u,v) x_P^1 y_P^2 + F(u,v) x_P^2 y_P^1 + G(u,v) x_P^2 y_P^2 \quad (1)$$

Such a set of scalar products is called a **metric** or **Riemannian metric** on U.
The form g_P is a scalar product if $E > 0$, $G > 0$, $EG - F^2 > 0$ for all $P = (u,v) \in U$. The functions E, F och G are assumed to be four times continuously differentiable with respect to u och v. So the metric defines a positively definite bilinear form g_P on each tangent plane $T_P E^2$ and is an example of a tensor. In the literature a metric is often written in the form

$$ds^2 = E(u,v)\, du^2 + 2F(u,v)\, dudv + G(u,v)\, dv^2 \quad (2)$$

The expression (2) is to be interpreted as the bilinear form in (1).

The metric determines uniquely a 2-dimensional geometry described in U. This geometry corresponds to the geometry in a plane or surface whose Gaussian curvature is K and which is determined by the formula

$$K(EG-F^2)^2 = (F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu})(EG-F^2) +$$
$$+\det\begin{pmatrix} 0 & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}.$$

This expression, the Gauss equation, is complicated but if $F=0$ the expression simplifies to

$$K = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right].$$

(The Gauss equation in the case $F=0$)
When the Gauss curvature is constant it is possible to describe the connection between the curvature and the geometry of the plane or surface by

$$\int_{\triangle} K\,dS = \alpha_1 + \alpha_2 + \alpha_3 - \pi,$$

where α_i are the inner angles of a triangle in a plane or a surface. So the angle sum in a triangle is greater than, equal to or less than π depending on whether the Gauss curvature K is positive, zero or negative.
By using the metric we can in principle solve all geometric problems in in a plane or a surface of Gaussian curvature K. To make calculations easier we introduce the so called Christofel symbols Γ^i_{jk}. They are calculated from the metric but are not tensors.

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)} \quad \Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG-F^2)} \quad \Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v + FE_u}{2(EG-F^2)} \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG-F^2)} \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}$$

The straight lines, also called geodesics, are determined from a system of ordinary differential equations

$$\frac{d^2u}{ds^2} + \Gamma^1_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^1_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^1_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

$$\frac{d^2v}{ds^2} + \Gamma^2_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^2_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^2_{22}\left(\frac{dv}{ds}\right)^2 = 0.$$

So the solution curves of this system, which correspond to the straight lines in the plane or the surface whose Gaussian curvature is K, are curves in the open set U in the euclidean plane in which we represent the geometry of the plane or the surface.

The distance $d(P, Q)$ in the surface between two points P and Q in U corresponds to the arc length of the straight line, i.e. the geodesic, between P and Q in U. This distance is determined by the metric and is obtained from the line integral

$$d(P,Q) = \int_P^Q \left[E\left(\frac{du}{ds}\right)^2 + 2F\frac{du}{ds}\frac{dv}{ds} + G\left(\frac{dv}{ds}\right)^2\right]^{\frac{1}{2}}.$$

Of course it is not the euclidean distance between P and Q, measured with the standard scalarproduct in U, which is the distance between P and Q in the surface. The area of a geometric figure in a plane or a surface is obtained from the integral

$$Area = \int_S \sqrt{EG - F^2}\, dudv.$$

If

$$U = \{(u,v) \mid -\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi\} \quad and \quad ds^2 = du^2 + \cos u^2\, dv^2$$

it follows from the Gauss equation that $K = 1$, i.e. the geometry is spherical. The area of the sphere is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos u\, dudv = 4\pi.$$

The curvature k_g of a curve $u = u(s), v = v(s)$ in a plane or a surface, the so called geodesic curvature, is given by

$$k_g = \left[\Gamma^2_{11} \left(\frac{du}{ds}\right)^3 + (2\Gamma^2_{12} - \Gamma^1_{11}) \left(\frac{du}{ds}\right)^2 \frac{dv}{ds} + (\Gamma^2_{22} - 2\Gamma^1_{12}) \frac{du}{ds} \left(\frac{dv}{ds}\right)^2 - \right.$$

$$\left. -\Gamma^1_{22} \left(\frac{dv}{ds}\right)^3 + \frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds} \right] \sqrt{EG - F^2}.$$

Along the curves $v = constant$ and $u = constant$ we get the simpler expressions

$$(k_g)_{v=constant} = \Gamma^2_{11} \frac{\sqrt{EG - F^2}}{E\sqrt{E}}, \qquad (k_g)_{u=constant} = -\Gamma^1_{22} \frac{\sqrt{EG - F^2}}{G\sqrt{G}}.$$

In case $F = 0$ we get the even more simpler expressions

$$(k_g)_{v=constant} = -\frac{E_v}{2E\sqrt{G}}, \qquad (k_g)_{u=constant} = \frac{G_u}{2G\sqrt{E}}.$$

The angle θ between two vectors X_P and Y_P at P is calculated as usual by the scalar product (1), i.e˙

$$\cos\theta = \frac{g_P(X_P, Y_P)}{\sqrt{g_P(X_P, X_P)}\sqrt{g_P(Y_P, Y_P)}}.$$

An important class of metrics are those which are proportional to the standard scalar product in each $T_P E^2$, i.e.

$$g_P(X_P, Y_P) = \lambda(u, v)(x^1_P y^1_P + x^2_P y^2_P), \quad \lambda(u, v) > 0 \quad in \quad U \quad (3)$$

They can also be written on the form

$$ds^2 = \lambda(u, v)(du^2 + dv^2), \quad \lambda(u, v) > 0 \quad (3')$$

Metrics of the type (3), (3') are called *conformal.* The euclidean angle measured between two vectors at P with the standard scalar product in $U \subset E^2$ is then the same as the angle between the vectors in the plane or surface whose geometry we study if we use a metric like (3), (3').

Some metrics which give 2-dimensional hyperbolic geometry

An open subset $U \subset R^2$ together with a metric ds^2 is often called a *model* of the corresponding geometry. The italian mathematician Beltrami found in 1868 a number of suitable models for hyperbolic geometry by using the theories of Gauss and Riemann. These models were later used by Poincaré and Klein in important works and therefore carry their names instead.

1. *The Poincaré Upper Half Plane Model* H^2.

$$U = \{(u,v) | v > 0\}$$

with the metric

$$ds^2 = \frac{du^2 + dv^2}{v^2}.$$

The Gauss equation gives $K = -1$. The model is conformal so angles are determined directly in U. The straight lines are all euclidean circles and euclidean straight lines in U which are orthogonal to $v = 0$.

2. *The Poincaré Disc Model* B^2.

$$U = \{(u,v) | u^2 + v^2 < 4\}$$

with the metric

$$ds^2 = (du^2 + dv^2)\left(1 - \frac{u^2 + v^2}{4}\right)^{-2}.$$

In this model $K = -1$ and it is also conformal. The straight lines are all diameters and all circles orthogonal to $u^2 + v^2 = 4$.

There are simple formulas for distances in the conformal models. Define *the double ratio*

$$[A, B; P, Q] = \frac{|AP|}{|AQ|} : \frac{|BP|}{|BQ|}$$

where $|AP|$ denotes the euclidean distance between A and P etc.

PROP 1 The distance between two points A and B is in the conformal models H^2 and B^2 given by the formula

$$d(A, B) = |\log[A, B; P, Q]|,$$

where P and Q are the end points of the straight line in the model. In H^2 one of the points P or Q can be ∞ (say Q) and then we get

$$d(A, B) = \left|\log \frac{|AP|}{|BP|}\right|.$$

3. *The Klein Model* K^2.

$$U = \{(u,v) | u^2 + v^2 < 1\}$$

with the metric

$$ds^2 = \frac{(1-v^2)du^2 + 2uvdudv + (1-u^2)dv^2}{(1-u^2-v^2)^2}.$$

In this model $K < 0$ and constant but the model is not conformal. The straight lines correspond to all euclidean chords in $u^2 + v^2 < 1$ including all diameters. Even if the model is not conformal the euclidean angle between two diameters is in fact also the angle in the hyperbolic plane. A chord which can be extended through the pole of another chord cuts this chord under right angles in the hyperbolic plane.

PROP 2 The distance between two points A and B is in the Klein model K^2 obtained from the formula

$$d(A,B) = \frac{1}{2} |\log[A,B;P,Q]|,$$

where P and Q are the end points of the chord in the model.