FROM GEOMETRY TO NUMBER

The Arithmetic Field implicit in Geometry

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Contents

Preface

This is a short monograph on geometry, written by the author as a supplement to David C. Kay's "College Geometry" while taking a course based on that book. The presentation in this monograph is very indebted to Robin Hartshorne's book "Geometry: Euclid and Beyond."

To be more precise we might say this is a text on classical, synthetic, or axiomatic geometry, but we consider this the heart of geometry and prefer to refer to the subject simply as "geometry." Most texts on geometry focus on the parallel axiom. Little fuzz is made about the other axioms, and one easily gets the idea that they serve only to provide a sensible foundation. The parallel axiom, or some version of it, then provides the essential structure of the geometry, deciding whether it is Euclidean or hyperbolic. While doubtlessly having its value, we shall not follow this approach, but instead focus mainly on the circle-circle intersection axiom and Archimedes' axiom. These two axioms, as we shall see, put interesting constraints on what number field we may use to model our geometry.

High school or college level geometry books tend to base the presentation of geometry heavily on the field of real numbers. Verily, most such texts include a "ruler axiom," which states that the points of a geometrical line can be put into one-to-one correspondence with the real numbers in such a way that the distance between two points is the difference of the corresponding real numbers. As a consequence, the axioms of geometry come to depend on the axioms of the real numbers. This is very unfortunate, being both mathematically incorrect and historically completely backwards. Analytical methods are exeedingly powerful, sure, so it would be pure idiocy if we insisted on not using them in favour of purely synthetic methods. But the use of the real numbers obscures one of the most interesting aspects of the development of geometry: namely, how the concept of continuity, which belonged originally to geometry only, came gradually by analogy to be applied to numbers, leading eventually to Dedekind's construction of the field of real numbers.

We regard the algebraic and analytic representations of geometry as just that: representations, or models rather. Ideally, the model should faithfully represent the axiomatic geometry we have imagined. For this to be the case, we do not want statements that cannot be proven from the axioms to be provable by analytic or algebraic means. The way to avoid this is to find the appropriate number field for our geometry. We should allow geometry to discover the appropriate numbers for us. Axiomatic geometry is utterly free. Only the imagination and the demand for self-consistency limit the axioms. If we are to quick to dress our geometry in real numbers much of the freedom evaporates.

We follow the kind of thinking we find embodied in Euclid's "Elements," but in order to conform to modern rigour we replace Euclid's axioms with those given by Hilbert in his "Foundations of Geometry." For a list of Hilbert's axioms, we refer the reader to the appendix ("Hilbert's axioms"). The philosophy we follow is ancient, though we claim not antiquated. In the first chapter, which is historical and philosophical, we try to convey some of the awe and respect we feel for the ancients that developed our subject. The first chapter also serves the important role of providing a context for the later mathematical developments in the monograph. In the third chapter we combine the historical roots and the modern theory in a brief study of the three classical problems of geometry.

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
> - Joseph Louis Lagrange (1736 to 1813).

CHAPTER 1

Kinds of Number in Early Geometry

1. Pythagorean natural philosophy

Pythagoras is known primarily as a contributor to science, philosophy and mathematics. But Pythagoras was more than a mathematician and physicist. He was universally recognized in the ancient world as a teacher of a whole way of life, and as a distinguished initiate and magician in the Orphic mysteries. Based on Orphic cult doctrines and his own philosophy, Pythagoras founded a school in southern Italy, around 518 BC. The inner circle of this school formed a society, subsequently called the Pythagorean society, and the members of the school acquired the name Pythagoreans. The Pythagorean school quickly rose to become the chief Greek scientific school of its time, flourishing in the latter half of the sixth century BC. The Pythagoreans developed number theory, geometry (both plane and solid geometry), and the theory of irrational or incommensurable numbers. Pythagoras himself is among other things credited with discovering the relationship between the length of a string and the emitted pitch, and from that the Pythagoreans developed the structure of western musical theory. That Pythagoras is credited for coining both the word mathematics ("mathemata") and the word philosophy reveals his immense impact. Yet it is uncertain precisely what his teaching constituted, as Pythagoras himself wrote nothing down.

Scholars agree that we can safely reckon Pherecydes, Thales and Anaximander among the philosophers, and the common mythological background of eastern Greece (the region in which Orphism thrived), as Pythagoras' principal sources. The common ground for how the Pythagoreans projected the Orphic doctrines and the cosmology of Hesiod, onto the world as a context for physical understanding and study, can be seen in their concept of "apeiron." The early Greeks noted that all things in this world are bounded, or defined ("pera" in Greek, meaning "boarder"). The cosmos, they felt, can be described as the set of all defined or bounded things. Hand in hand with those trails of thought the concept of that which is not bounded was formed, hence the a-peiron. The "discovery" of the apeiron is attributed to Anaximander, one of Pythagoras' principal influences. The surviving lines of Anaximander's discovery read:

> The beginning and the origin of all being things (of the all-there-is) is the apeiron And therefrom is the emergence (waxing) of all the being things Thereinto is also their waning (destruction, annihilation) according to their fate And they pay each other their justified debt and penance for their injustice according to the law of the time.

What Anaximander did was to put forth a rationalized version and understanding of the mythological cosmology of Hesiod and the Orphics. This was further developed by Pythagoras and the Pythagoreans, casting the limited and unlimited into a mathematical and geometrical scheme.

Who we count as Pythagorean and who we count as something else is largely based on what Aristotle and his pupil Aristoxenus had to say on the matter, and when Aristotle spoke about "the so-called Pythagoreans" he spoke about the fifth-century BC Pythagorean Philolaus and the circle around him. Unlike the Pythagoreans under Pythagoras, Philolaus wrote at least one book. Diogenes Laertius quotes the beginning of the book as: "Nature in the universe was harmonized from both unlimiteds and limiters - both the universe as a whole and everything in it." The theme of unlimiteds and limiters is very reminiscent of Anaximander. Scholars believe Philolaus' book to have been Aristotle's principal source for what he said about Pythagoreanism. Aristotle explains how the Pythagoreans developed Anaximander's ideas about the apeiron and the peiron, the unlimited and limited, by writing that:

> ...for they [the Pythagoreans] plainly say that when the one had been constructed, whether out of planes or of surface or of seed or of elements which they cannot express, immediately the nearest part of the unlimited began to be drawn in and limited by the limit.

And Stobaues in his turn, quoting Aristotle's now lost book wholly devoted to the Pythagoreans:

> In the first book of his work "On the Philosophy of Pythagoras" he writes that the universe is one, and that from the unlimited there are drawn into it time, breath and the void, which always distinguishes the places of each thing.

This inhalation of the unlimited was what, in Pythagorean thought, made it possible to describe the world in mathematical terms. Quoting Aristotle again, commenting on Philolaus' teachings:

> The Pythagoreans, too, held that void exists, and that it enters the heaven from the unlimited breath it, so to speak, breathes in void. The void distinguishes the natures of things, since it is the thing that separates and distinguishes the successive terms in a series. This happens in the first case of numbers; for the void distinguishes their nature.

When the apeiron is inhaled by the peiron it causes separation, which also apparently means that it "separates and distinguishes the successive terms in a series." Instead of an undifferentiated whole we have a living whole of inter-connected parts separated by "void" between them. This inhalation of the apeiron is also what makes the world mathematical, not just possible to describe using math, but truly mathematical since it shows numbers and reality to be upheld by the same principle: both the field of numbers (a series of successive terms, separated by void) and the field of reality, the cosmos - both are a play of emptiness and form, apeiron and peiron. What really sets this apart from Anaximander's original ideas is that this play of apeiron and peiron must take place according to harmonia (harmony), about which Stobaeus commentated:

> About nature and harmony this is the position. The being of the objects, being eternal, and nature itself admit of divine, not human, knowledge except that it was not possible for any of the things that exist and are known by us to have come into being, without there existing the being of those things from which the universe was composed, the limited and the unlimited. And since these principles existed being neither alike nor of the same kind, it would have been impossible for them to be ordered into a universe if harmony had not supervened in whatever manner this came into being. Things that were alike and of the same kind had no need of harmony, but those that were unlike and not of the same kind and of unequal order it was necessary for such things to have been locked together by harmony, if they are to be held together in an ordered universe.

A musical scale presupposes an unlimited continuum of pitches, which must be limited in some way in order for a scale to arise. The crucial point is that not just any set of limiters will do. We cannot just pick pitches at random along the continuum and produce a scale that will be musically pleasing. The diatonic scale, also known as "Pythagorean," is such that the ratio of the highest to the lowest pitch is $2:1$, which produces the interval of an octave. That octave is in turn divided into a fifth and a fourth, which have the ratios of $3:2$ and $4:3$ respectively and which, when added, make an octave. If we go up a fifth from the lowest note in the octave and then up a fourth from there, we will reach the upper note of the octave. Finally the fifth can be divided into three whole tones, each corresponding to the ratio of $9:8$ and a remainder with a ratio of $256:243$ and the fourth into two whole tones with the same remainder. This is a good example of a concrete applied use of Philolaus' reasoning, recorded by Philolaus' pupil Archytas in his book "On Harmonics." In Philolaus' terms the fitting together of limiters and unlimiteds involves their combination in accordance with ratios of numbers (harmony). Similarly the cosmos and the individual things in the cosmos do not arise by a chance combination of limiters and unlimiteds; the limiters and unlimiteds must be fitted together in a "pleasing" (harmonic) way in accordance with number for an order to arise.

This philosophy of harmony also found expression in Pythagorean geometry. Geometrical figures such as triangles express mathematical ratios perhaps even more concretely than pitches in music. Mathematics arises as soon as we see one thing as separate from another, as a thing that can be counted. Somehow we recognize the apples as separate from the tree which bears them. Hence we can count them, and hence we have discovered the whole, or natural numbers. And according to Stobaeus the Pythagoreans held the view that "all the things that are known have number; for it is not possible for anything to be thought of or known without this." Countable implies knowable. Next we conceive of these as forming relations, such as the harmonic relationships in music or the harmonic ratios in geometrical constructs. In music the harmonic ratios are rational, i.e. ratios between natural numbers. It has often been argued that the Pythagoreans shunned irrational ratios, or even denied the existence of irrational numbers. Modern scholarship has revised this.

There was a story in the ancient world about how a young Pythagorean named Hippasos was drowned by his fellow Pythagoreans, following an order issued by Pythagoras himself, for divulging certain things about irrational numbers and about the pentagonal dodecahedron. You can either believe that they silenced him because what he said did not fit with the Pythagorean beliefs, or you can follow modern scholarship and take it as an example of the strictness of the oath of silence on certain matters employed in the Pythagorean society. The most often quoted ancient authority for this story actually has it that "the man who first divulged the nature of commensurability and incommensurability to men who were not worthy of being made part of this knowledge, became so much hated by the other Pythagoreans, that not only they cast him out of the community; they built a shrine for him as if he was dead, he who had once been their friend." Thus, the Pythagoreans did in all likliness not deny existence of the irrational, or incommensurable, numbers, but rather considered it too marvellous to be shared with common men. However, the Pythagoreans did probably not have the conception of the irrational numbers that we have today. The irrational numbers, indeed even the rational numbers, were not considered as numbers in the same sense as the natural numbers. The rational numbers existed as ratios between natural numbers, as expressions of harmonia. The irrationals appeared in geometric constructs, e.g. as in the length of the diagonal of a square of side-length one. One may infer that the irrationals were seen as the apeiron, drawn in and limited by the peiron of geometry, as an elusive infinitesimal betweeness needed for geometry to make sense, or as a murky infinity needed for the harmonic ratios of the cosmos to exist in the first place.

2. Plato and Euclid

Let us explain, a bit more succintly than in the last section, the way Greek geometers around the time of Plato thought. The only numbers used were $2, 3, 4, ..$ and the unity 1. What we call negative numbers and zero were not accepted as actual numbers. Geometrical quantities such as line segments, angles, areas, and volumes were called "magnitudes." Magnitudes of the same kind could be compared as to size: less, equal, or greater, and they could be added or subtracted (the lesser from the greater). They could not be multiplied, except that the operation of forming a rectangle from two line segments, or a volume from a line segment and an area, could be considered a form of multiplication of magnitudes, whose result was a magnitude of a different kind. In Euclid's "Elements" there is an undefined concept of equality (what we call congruence) for line segments, which could be tested by placing one segment on the other to see whether they coincide exactly. In this way the equality or inequality of line segments is perceived directly from the geometry without the assistance of numbers to measure their lengths. Similarly, angles form a kind of magnitude that can be compared directly as to equality or inequality without any numerical measure of size. Two magnitudes of the same kind were said commensurable if there existed a third magnitude of the same kind such that the first two were (whole number) multiples of the third. Otherwise they were said incommensurable. For example, the Pythagorean theorem states that the diagonal of a square is incommensurable with it side.

Plato loved geometry, and considered it the highest of sciences. He also had a letter correspondence with the Pythagorean Archytas, the one who preserved Philolaus' mathematical theory for musical harmony. Plato also, at one time in his life, went to Italy and Sicily to visit the Pythagoreans there, and stayed as a guest in their house for ten days. Accordingly it is not surprising that Plato's dialogues echo many Pythagorean doctrines. Indeed, it has even been suggested that Plato was himself actually a member of the Pythagoreans. In the dialogue "The Meno," written shortly after Plato returned from his stay with the Pythagoreans, Plato began to develop what is properly called the Platonism. Up in until that dialogue Plato had mainly presented the philosophy of, the by then deceased, Socrates. In "The Meno," Socrates (as Plato's mouthpiece) interrogates an unschooled slave boy about the sort of thing the real Socrates never quizzed anyone with: mathematics. He asks the slave boy how to make a square double in area to any square you might give him to start with. He incorrectly says "double the sides." By asking him the right questions "Socrates" leads the boy to discover that such a doubled square actually has four times the area of the original square. By continuing to press the issue, the boy discovers that to make a square double in area, you have to make its sides equal the diagonal of the original square. If the original square has an area element equal one, then the diagonal has the length the square root of two. Again, Plato and his contemporaries did not say the square root of two (a number) is irrational (i.e., not rational), since they had no actual such number, but only the geometric notion of incommensurable magnitudes.

Plato made further use of incommensurable magnitudes in other dialogues, in particular in his "Timaeus," wherein he describes an elaborate cosmogony and cosmology. The text was recognized already in ancient times as greatly indebted to Pythagoreanism, so much in fact that Plato was accused of plagiarizing an earlier book by Philolaus. In his account of creation Plato posits a kind of proto-physical existent that he refers to as either "the receptacle" or as "space." He then distinguishes between the eternal Forms, which serve as blueprints for creation, and the actual objects in creation which are mere copies of the forms. The world of change, of sensorial existents, is "constantly borne along, now coming to be in a certain place and then perishing out of it," as Plato writes. The receptacle is the intermediate between Form and copy, and provides a situation for all things that come into being. Things come into being in the proto-space of the receptacle. The first things to be generated in this way are the four elements fire, earth, air, and water. Plato, at great length, describes how these elements are formed into three-dimensional solids through geometrical generation out of triangles; triangles which are in turn generated out of the receptacle. Each elemental solid can be decomposed into the triangles and put together as a new kind of solid, accounting for how the elements may transform into each other. The geometrical solids corresponding to the elements are: fire-tetrahedron, earth-cube, water-icosahedron, and air-octahedron. After constructing and detailing the solids corresponding with the four elements, Plato admits there is a fifth construction, "one god used for the whole universe, embroidering figures on it [the Zodiac]." The only solid left, as there are only five that can be constructed in Euclidean space, is the dodecahedron: the twelve-sided polyhedron with pentagons for faces.

Circa one hundred years after Plato wrote his "Timaeus," Euclid wrote his monumental the "Elements." The work hardly needs any presentation. The "Elements" is divided into thirteen books. Books I to VI develope the basic axioms and results of plane geometry, while books VII to X deal with number theory and include the Euclidean algorithm, as well as proofs of the infinitude of primes and the irrationality of the square root of two. Books XI to XIII deal with solid geometry, culminating in the construction of the five regular, or Platonic, solids mentioned previously. This ordering of the material, as well as the title "Elements," suggest that Euclid wrote his work as an expansion of Plato's "Timaeus." This said, Euclid mentions nothing about elements, creation, etc., in his work. It seems the implicit connection to Plato's work was meant to be understood only by an educated audience familiar with both works. Allthough a work primarily in logic and mathematics, it is clear that the geometry Euclid tried to formalize was the intuited geometry of physical space.

We want to emphasize here that Euclid begins with the axioms of plane geometry, and only after that does he continue with number theory. The succession is entirely logical. The square root of two, for example, is discovered while doing geometry, more or less as a consequence of the Pythagorean theorem. In this text we follow Euclid, and see what the axioms of geometry imply for the numbers we use in our description (in our model). Think of the imagination shown by Euclid and his predecessors! The idea of axiomatizing the rules of geometry and mathematics is utterly, utterly strange. Calculating areas and volumes is quite natural as a practical application, but to leave that and create a mathematics with an entirely "Platonic" existence is a superbly bold leap of imagination. And to then venture to complete the project in the way Euclid did! Rarely have the Muses been more generous.

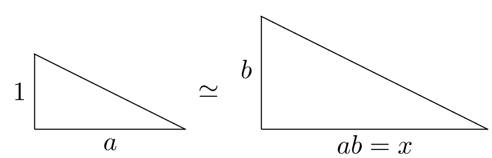
CHAPTER 2

Fields and Geometric Construction

In this chapter we discuss how to construct abstract number fields directly from the axioms of geometry. To recall the definition of a field, see the appendix entitled "Axioms and properties of groups and fields."

1. Line segment arithmetic

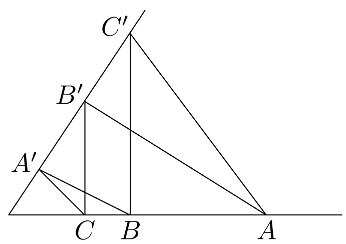
A common misconception is that analytic geometry was invented by Descartes. At least it is a misconception if we by analytic geometry mean drawing perpendicular coordinate axes and choosing an interval to serve as unit, so as to establish a one-to-one correspondence between the points of the plane and ordered pairs of real numbers. The real numbers had not yet, properly, been invented by the time of Descartes, and even the idea of representing a line segment by any sort of number was not yet clearly developed. If we read the geometry of Descartes carefully, we see that he is applying algebra to geometry. In any problem, he represents known line segments by letters $a, b, c, ...$, and unknown line segments by letters $x, y, z, ...$. Then, from the data of the problem, he seeks relations among them that can be expressed as equations with letters $a, b, c, x, y, z, ...$. These equations are solved by the usual rules of algebra. The solution gives a recipe for the geometrical construction of the unknown line segments. Throughout this process, the letters represent line segments, not numbers. Two line segments can be added by placing them end to end. Two line segments a, b can be multiplied, once one has fixed a segment 1 to act as unit, by making a right-angled triangle with sides $1, a$, and another similar triangle with sides b, x, and then defining $ab = x$. Note that we define the product ab by similarity of the triangles, invoking only purely geometrical concepts. At all times the letters should be understood to refer to line segments, not to real (or rational, for that matter) numbers. See image below.



What Descartes really did was to create an "arithmetic of line segments," or "algebra of line segments." However, nowhere in the geometry of Descartes does he explain by what right he may assume that the operations he defines on line segments obey the usual laws of arithmetic. For example, it is a nontrivial matter to show that his multiplication of line segments, defined using similar triangles as above, is commutative. This difficulty was answered in the most satisfactory way by Hilbert.

In classical geometry there is no notion of length of line segments. Axiomatic geometry only allows us to consider ratios (expressed in geometric terms) of line segments, on the basis of the undefined notion of congruence. Intuitively, we may perhaps think of congruence of segments, as meaning the segments are of equal size. The same applies to angles. There is no degree, no number, associated to angles in classical geometry. Angles are a special kind of magnitude, and two angles may be compared only in terms of congruence. We shall now describe how Hilbert in the spirit of Descartes used congruence to introduce an abstract field of line segments, by which geometry may be faithfully represented in terms of arithmetic. To explain Hilbert's "algebra of line segments" we need Pappus' theorem, which we shall state but not prove. The theorem is a classical result, discovered by the Greek mathematician Pappus of Alexandria who lived around the end of the third century AD.

CLAIM 1. *Consider a neutral geometry satisfying Playfair's axiom. Let* A, B, C *be points on a line,* A', B', C' *points on a another line, and assume the two lines intersect in a point not equal to any of aforementioned six points. If* CB' *is parallel to* BC' *and* CA' *is also parallel to* AC'*, then* BA' *is parallel to* AB'*.*



In what follows, we are assuming we are working in a neutral geometry satisfying Playfair's axiom, so that Pappus' theorem is valid. We need this for our construction to make sense. The axioms of congruence imply that congruence of line segments is an equivalence relation. Thinking of congruence of segments as meaning "of equal size," we would want congruent line segments to be represented by the same "number" in our soon to be constructed field. Hence we consider the equivalence classes of line segments. Our notion of "size" is that it is something positive, and accordingly we would like the equivalence classes of line segments to be positive elements in the field. Hopefully, the ideas are clear to the reader. The natural definition of addition in the field is that adding segments to each other corresponds to laying the segments one after the other to get a third segment.

DEFINITION 1. *Given two congruence classes* a, b *of line segments, we define their* ***sum*** *as follows. Choose points* A, B *such that the line segment* AB *is a representative of the congruence class* a*. Then, choose a point* C *collinear with* A, B *and such that* B *is between* A *and* C*, and* BC *is a representative of the congruence class* b*. Then we define the sum* $c = a + b$ *to be the congruence class represented by the segment* AC*.*

Based on this definition we may order the congruence classes, i.e. in above case we would say that a and b are smaller than c, and indicate this by writing $a, b < c$. Similarly, we may say c is greater than both a and b, and write $c > a, b$. Now, we claim the addition we have defined for the congruence classes satisfies the field axioms for addition.

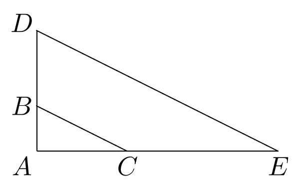
Claim 2. *In any neutral geometry, the addition of line segment classes satisfies:*

(i) $a+b$ *is well defined, i.e. independent of choice of representatives for* a *and* b.
(ii) $a+b=b+a$*, i.e. addition is commutative.*
(iii) $a+(b+c)=(a+b)+c$*, i.e. addition is associative.*
(iv) *Given any two classes* a, b*, one and only one of the following holds:* $a=b$*; There is a class* c *such that* $a+c=b$*; There is a class* d *such that* $a=b+d$.

Remark 1. *It is not too difficult to verify above properties, using Hilbert's incidence, congruence and betweeness axioms. We remark that Playfair's axiom is not needed, so addition of congruence classes is defined also in neutral geometry.*

To define multiplication properly, we need a unit congruence class. To this end, select any convenient segment, which, having been selected, shall remain constant throughout the discussion. Denote said congruence class by 1. We then define multiplication of congruence classes just the same way as did Descartes.

Definition 2. *Given line segment classes* a, b*, we define the* ***product*** ab *as follows. Choose a right angled triangle* ABC *in such a way that* AB *is a representative of* a *and* AC *is a representative of the unit segment class* 1*. Let* E *be a point collinear with* A, C*, and such that* AE *is a representative of* b*. Finally, find a point* D *collinear with* A, B *such that the triangle* ADE *is similar to the triangle* ABC*. We then define the product* ab *to be represented by the line segment* AD.

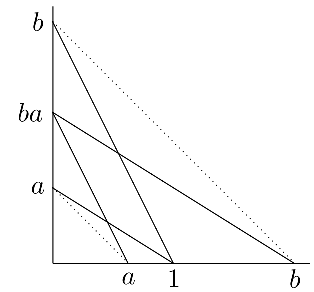


Note that we need Playfair's axiom (the parallel postulate) in order for above definition to make sense. Without it we cannot be sure that the point D exists. We shall now demonstrate that this definition satisfies all the axioms for multiplication in an algebraic field. Unlike in the case with addition, we shall give a complete proof. Not only because it is a lot less trivial, but since the proof beautifully reveals the fully geometric nature of the algebraic structure we are defining.

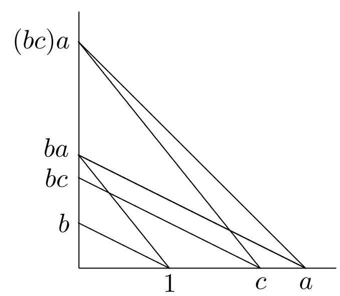
Proposition 1. *In any neutral geometry satisfying Playfair's axiom, multiplication of line segment classes has the following properties:*

(i) ab *is well defined.*
(ii) $a \cdot 1 = a$ *for all* a.
(iii) $ab = ba$ *for all* a, b.
(iv) $a(bc) = (ab)c$ *for all* a, b, c.
(v) *For any* a *there is a unique* b *such that* $ab = 1$.
(vi) $a(b+c) = ab + ac$ *for all* a, b, c.

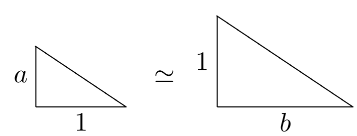
Proof. (i): The multiplication is well defined. Let A, B, C, D, E be related as in the definition of multiplication. If $A'B'C'$ is another right triangle with sides representative of $1, a$, then it is congruent to ABC by the side-angle-side axiom (C6). Hence, if $A'D'E'$ is a right triangle with side $A'E'$ a representative of b and $\angle D'E'A' \cong \angle B'C'A \cong BCA$, then $A'D'E' \cong ADE$. So, we get a congruent segment $A'D' \cong AD$. (ii): Immediate. (iii): Proving commutativity is a bit more involved, and here we need Pappus' theorem. Construct the product ab as in the definition. Furthermore, lay off from A upon the line determined by A, C the segment a, and upon the line determined by A, B the segment b. Connect by a straight line the extremity of the unit segment 1 with the extremity of b, and draw through the endpoint of a a line parallel to this line. This parallel will determine, by its intersection with the line determined by A, B, the segment ba. See the image below.



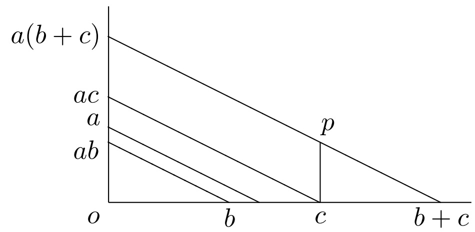
By Pappus' theorem (Claim 1), the two dotted lines are parallel, and hence the segment ba just found coincides with the segment ab as previously constructed, and we have established commutativity. (iv): In order to show the associative law $a(bc) = (ab)c$ we again use Pappus' theorem, as well as the just shown commutativity. Construct first of all the segment $d = bc$, then da, after that the segment $e = ba$, and finally ec. By virtue of Pappus' theorem, the extremities of the segments $da = (bc)a = a(bc)$ and $ce = c(ba) = (ab)c$ coincide, as may be clearly seen from below picture.



For the existence and uniqueness of a multiplicative inverse (v) we consider a right triangle with sides $1, a$. Define b in accordance with below picture:



The uniqueness of b is clear, and the picture above is merely the geometric "definition" of multiplicative inverse. Lastly, we need to verify the distributive law (vi). In order to demonstrate this, we construct the segments ab, ac, and $a(b+c)$, and draw through the extremity of the segment c (see image below) a straight line parallel to the other side of the right angle.



The right-angled triangle with vertices at $o, (ab), b$ in the picture and the triangle with vertices $c, p, (b+c)$ are congruent. The line from ac to c is by our definition of multiplication parallel to the line from $a(b+c)$ to $b+c$, whence the segment between c and p is congruent to the segment from ac to $a(b+c)$ (the theorem relating to the equality of the opposite sides of a parallelogram). The desired result follows. □

We have now shown how the naturally defined line segment arithmetic behaves just like the positive elements of an algebraic field. This means that, in order to get an algebraic field we have to add to the congruence classes of segments some negative elements. It is not obvious our segment equivalence classes can always be completed into a field in such a manner, nor whether such an extension is uniquely determined or not. Luckily, the situation is the best one could hope for, as promised by the following theorem.

PROPOSITION 2. *Given a neutral geometry satisfying Playfair's axiom, and a unit segment* 1 *has been chosen, there is an up to isomorphism unique extension of the set of congruence classes of line segments into an ordered field* $\mathbb{K}$*, with the set of congruence classes as the set of positive elements in* $\mathbb{K}$ *and arithmetic as defined by Hilbert.*

PROOF. The proof has little geometrical content, and is also quite involved if done rigorously, so we will allow ourselves to be sketchy. Let P denote our set of congruence classes of line segments, and $\mathbb{K}$ be the set of equivalence classes a, b of ordered pairs of elements of P, where $(a, b) \sim (a', b')$ if $a+b' = a'+b$. Define addition by $(a, b)+(c, d) = (a+c, b+d)$, and multiplication by $(a, b)(c, d) = (ac+bd, ad+bc)$. With these definitions the additive identity in $\mathbb{K}$ is $0 = (a, a)$, and the multiplicative identity is $1 + a, a)$. We leave it to the disbelieving reader to verify that the defined operations satisfy the field axioms. We define a mapping $\varphi : P \longrightarrow \mathbb{K}, a \mapsto (a+b, b)$, for some fixed $b \in P$, and note that this map is injective, so that we may identify P with $\varphi(P)$. It may be verified that $\varphi(P) \subset \mathbb{K}$ satisfies all the criteria of being a positive subset for $\mathbb{K}$. That $\mathbb{K}$ is uniquely determined follows from establishing that all elements $x \in \mathbb{K} \setminus P$ are of the form $x = -p$ for some $p \in \varphi(P)$. □

Let us pause and consider what we have accomplished, and what questions remain that we would hope to solve. We have shown that there is a canonical way of defining an arithmetic on purely geometric line segments, or congruence equivalence classes of such segments rather, and that the resulting set of segment classes can always be uniquely interpreted as the positive elements of an abstract field. We are not (as in high school geometry) forcing some number field (e.g. the real numbers) onto our geometry; quite the opposite. The field arises naturally.

Our construction works for a geometry that is at least neutral and satisfies Playfair's axiom. A natural question is what implications for the abstract field inclusion of further axioms into our geometry would have. Another, intimately related, question is whether the field $\mathbb{K}$ is for some geometries a familiar field, e.g. perhaps $\mathbb{K}$ will turn out to be the just real numbers after all! A priori it is not obvious at all that the field of line segment classes will be isomorphic to some familiar number field. In other words, we would like to characterize the field in terms of the geometry, and vice versa. This classification is the subject of a major theorem we will prove in our next chapter (see chapter "The main theorems"). Whereas of yet we are able to relate the field of congruence classes only to the properties of line segments in the geometry, we shall see that the field may in fact be used to represent all of the geometry in terms of arithmetic. At the end we will be able to think of $\mathbb{K} \times \mathbb{K}$ as a model of planes in our geometry, and of $\mathbb{K}^3$ as a model for the full geometric space. Before turning to the next chapter, where we will state and prove these wonderful results in detail, we shall familiarize ourselves with some fields with various algebraic properties, and get some glimpses of what geometrical properties that might be related.

2. The Hilbert field

DEFINITION 3. *A field $\mathbb{K}$ is called **Pythagorean** if it is ordered and, for every element $a \in \mathbb{K}$, the square root $\sqrt{1+a^2}$ exists in $\mathbb{K}$.*

To see why "Pythagorean" is descriptive of the kind of fields in the definition, consider the product $\mathbb{K}^2$ of a field. In the plane so constructed, mark points $(0,0), (a,0)$ and (a,b). These three points form a right angled triangle. By the Pythagorean theorem, the hypotenuse x of said triangle would satisfy $x^2 = a^2 + b^2$. We take the square root of this and obtain $x = \sqrt{a^2+b^2} = |a|\sqrt{1+(b/a)^2}$. Thus, a necessary and sufficient condition for x to exist in our field is that $\mathbb{K}$ is Pythagorean. The standard Euclidean metric used in analytic geometry makes use of the Pythagorean theorem, defining the distance between points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ to be

$$\operatorname{dist}(A,B) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}.$$

As we just saw, for this distance to exist the base field has to be Pythagorean. The real numbers is clearly a Pythagorean field, while the rational field is not. The following is a sort of intermediate field, which is Pythagorean.

DEFINITION 4. ***Hilbert's field** Ω is the set of real numbers that can be expressed starting from the rational numbers $\mathbb{Q}$ and using a finite number of operations $+, -, \cdot, \div$, and $c \mapsto \sqrt{1+c^2}$.*

The name "Hilbert's field" is not really standard, but we follow Hartshorne ("Geometry: Euclid and Beyond") and honour Hilbert, who did study it in his "Foundations of Geometry."

PROPOSITION 3. *Hilbert's field Ω is the smallest Pythagorean field.*

PROOF. If $a, b \in \Omega$, then each of a, b can be obtained in a finite number of steps using rational numbers and operations $+, -, \cdot, \div, c \mapsto \sqrt{1+c^2}$. Hence the same is true for $a \pm b$, $a \cdot b$ and a/b if $b \neq 0$. Hence Ω satisfies the field axioms. It is clear from construction that Ω is Pythagorean. ☐

We saw that over a Pythagorean field, the hypotenuse of a triangle always exists. This of course is very important for the usual notion of distance or length to make sense, and accordingly for the obvious notion of congruence between line segments. Since Ω is the smallest Pythagorean field, we may guess that the analytic geometry over Ω is in some sense the most minimalistic analytic geometry in which congruence of line segments makes sense. The details of this will be the subject of a later theorem (see "The main theorems").

3. The constructible field

Roughly, the word "constructible" refers in geometry to the geometrical objects that can be exactly drawn or constructed using only straightedge and compass. Observe that we have only a straightedge, and not an ideal marked ruler (that would be a real number line!). Tradition has it that Plato was responsible for setting this requirement. The early geometers were obsessed with construction problems, something which is only very natural if we consider the practical origins of geometry, e.g. to build a house properly one will need to know how to construct exactly a right angle. Almost all of Euclid's proofs in his "Elements" are constructive, meaning that he explicitly (with theoretical straightedge and compass) constructs the solution, not just gives an argument for the possibility to do so. Euclid just assumed validity of straightedge and compass construction, but to actually make it rigorous one has to include an axiom not postulated by Euclid. One has to assume the circle-circle intersection property: Given two circles Γ and Δ such that Γ contains a point inside Δ, and vice versa, we postulate that the circles have to intersect in two points. It is perhaps a bit surprising that this cannot be deduced from the other axioms. Particularly, it may seem an obvious consequence of Pasch's axiom, but it is actually independent of that! (See the appendix including the axioms; "Hilbert's axioms.") Every operation with a compass and straightedge can be translated mathematically into drawing circles and intersecting them with either lines or other circles. The circle-circle intersection property ensures that circles and lines intersect when the ought to, enabling us to construct angles and segments using Euclid's techniques.

Consider the analytic geometry of a xy-plane over a field $\mathbb{K}$, and in it represent two circles Γ, Δ by

$$\Gamma : (x-a)^2 + (y-b)^2 = r^2, \quad \Delta : (x-c)^2 + (y-d)^2 = s^2.$$

Suppose we want to find the coordinates of an intersection point, i.e. a point (x, y) that satisfies both equations. Subtracting one equation from the other yields the linear equation

$$2(c-a)x + 2(d-b)y + a^2 + b^2 - c^2 - d^2 = r^2 - s^2.$$

To find x and y we express one in terms of the other using above linear equation, and then substitute into one of the circle equations. This gives us a quadratic equation that is solved in terms of square roots. Thus, in terms of analytic geometry the circle-circle intersection property is equivalent to the existence of square roots for all positive elements in the base field. This motivates our next definition.

DEFINITION 5. *A field $\mathbb{K}$ is called* ***Euclidean*** *if it is ordered and, for every element $a \in \mathbb{K}$, with $a > 0$, the square root $\sqrt{a}$ exists in $\mathbb{K}$.*

Observe that being Euclidean implies being Pythagorean, but not the other way around. In the analytic geometry over an Euclidean field we can arithmetically represent all of Euclid's straightedge and compass constructions. The minimal such field we accordingly call "the constructible field."

DEFINITION 6. ***The constructible field*** $\mathbb{F}$ *is the set of all real numbers that can be obtained from the rational numbers* $\mathbb{Q}$ *by a finite number of operations* $+, -, \cdot, \div$ *and* $0 < a \mapsto \sqrt{a}$.

PROPOSITION 4. *The constructible field* $\mathbb{F}$ *is the smallest Euclidean field.*

PROOF. Just like the proof that Hilbert's field is the smallest Pythagorean field (Proposition 3). □

We remark briefly on why it is called constructible. Imagine we are given a blank paper. The paper is not a gridded so we have no coordinate grid. Instead we have a rigid line segment and a compass; enabling us to connect two known points with a straight line, and to draw a circle of any known radius (known means we already have in our possession a line segment congruent to such a radius). We want to construct a coordinate grid, as best we can. We begin by laying our line segment somewhere on the plane and mark one endpoint as $(0, 0)$ and the other as $(1, 0)$. We may now speak about a unit length. Then we use our compass and draw a circle of unit radius about the point $(1, 0)$. Then we use our line segment to extend the line through $(0, 0)$ and $(1, 0)$, so as to cut the circle. We mark the point where this line cuts the circle as $(2, 0)$. Continuing in this fashion we can generate all integer coordinates along one axis. Then, we draw circles of radius two around the points $(-1, 0)$ and $(1, 0)$. These two circles, we can be sure, intersect in two points not on our coordinate axis. Connecting these intersection points to $(0, 0)$ with a line gives us a line perpendicular to our first coordinate axis. On this new line we can, as with the first, easily mark of the integer spaced points. We extend to a $\mathbb{N}^2$-grid on our paper. Then, by drawing suitable circles and lines (the reader should be able to imagine how) we construct new line segments, congruent to rational numbers. We have shown how to define addition and multiplication of line segments geometrically. In particular, the triangles corresponding to multiplication are constructible by means of straightedge and compass. Hence we can construct not only all line segments corresponding to a rational number, but also all numbers that can be obtained from the rational numbers by a finite number of operations $+, -, \cdot, \div$. Lastly, intersecting circles correspond to finding a square root, as we saw, so we may add to the list of operations that of sending $a > 0$ to $\sqrt{a}$. In this way, we construct a $\mathbb{F}^2$-grid, with $\mathbb{F}$ the constructible field. Using only straightedge and compass, and a finite amount of time (so we are allowed only finitely many operations), this is the finest grid we can achieve. The subleties of Cartesian geometry over various fields will be the subject of our next chapter.

Note that Hilbert's field is a subfield of the constructible field, $\Omega \subset \mathbb{F}$, and that both are subfields of the real field $\mathbb{R}$, from which they inherit their ordered structure. Hilbert's field is a proper subfield of the constructible field, something that can be shown proving that $\sqrt{1 + \sqrt{2}}$ is in $\mathbb{F}$ but not in Ω.

4. The real field

To conclude our discussion of fields naturally introduced in relation to familiar geometrical properties, we turn to the field of real numbers. In the first chapter we outlined the geometrical origins of the discovery of the irrational (incommensurable) numbers; the canonical example being the square root of two, pressed upon us when considering the diagonal of a unit square. Under the name "Dedekind cut" this idea still underlies the modern axiomatization of the real numbers. In mathematics, a "Dedekind cut," named after Richard Dedekind $(1831 - 1916)$, in a totally ordered set X, is a partition of it into sets S and T such that S is closed downwards (meaning that for all s in S, $x \leq s$ implies that x is in S as well) and T is closed upwards, and S contains no greatest element. The cut itself is, conceptually, the "gap" between S and T. The original and most important cases are Dedekind cuts for rational numbers and real numbers. Dedekind used cuts to prove the completeness of the reals without using the axiom of choice (proving the existence of a complete ordered field to be independent of said axiom). The Dedekind cut resolves the contradiction between the continuous nature of the number line continuum and the discrete nature of the numbers themselves. Wherever a cut occurs and it is not on a rational number, an irrational number is inserted by the mathematician. Through this device, there is considered to be a real number, either rational or irrational, at every point on the number line continuum, with no discontinuity. In Dedekind's own words:

> Whenever, then, we have to do with a cut produced by no rational number, we create a new, an irrational number, which we regard as completely defined by this cut.... From now on, therefore, to every definite cut there corresponds a definite rational or irrational number....

This procedure lends itself very naturally to comparison with Philolaus' Pythagorean notion of an unlimited void inhaled by the limited, "separating successive terms in a series," filling in the gaps between the rational numbers. The important difference is that the ancients never concieved of these "gaps" as actual numbers, and consequently had no notion of "the set of all gaps," i.e. the field of real numbers. While oftentimes presented in a formal and set-theoretic fashion, Dedekind used the ambiguous word cut (Schnitt) in the geometric sense. That is, it is an intersection of a line with another line that crosses it. Dedekind posited one of the lines as a prototypical real number line. At that one point on the number line were the lines cut, if there is no rational number, Dedekind posits an irrational number. This results in the positioning of a real number at every point on the continuum. To be able to do this geometrically we have to include the Dedekind axiom into our geometry. Suppose the points of a line are partitioned into two non-empty collections S, T in such a way that no point of S is between two points of T, and no point of T is between two points of S. The Dedekind axiom postulates that then there exists a unique point P such that for any $A \in S$ and any $B \in T$, either $A = P$ or $B = P$ or the point P is between A and B. The content of this axiom is the "ruler postulate" found in some books, stating a one-to-one correspondence between the points of a geometric line and the real number line.

CHAPTER 3

The main theorems

In this chapter we present the main theoretical results, stating and proving an important and beautiful "main theorem."

1. The Cartesian space over a field

We return to the discussion of how the abstract field of line segment classes relates to the familiar procedures in analytic geometry; i.e. the use of coordinates and the usual parametrizations of lines and planes. To begin, we define what we mean by the analytic geometry over a field.

DEFINITION 7. *The* ***Cartesian space*** *over the field* $\mathbb{K}$*, written* $\mathcal{C}_{\mathbb{K}}$*, is the set* $\mathbb{K}^3$ *of ordered triples of elements of* $\mathbb{K}$ *together with the following geometrical interpretation: the ordered triples are called the* ***points*** *of* $\mathcal{C}_{\mathbb{K}}$*, and a* ***line*** *in* $\mathcal{C}_{\mathbb{K}}$ *is a subset of points* (x, y, z) *satisfying an equation*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

for some fixed $a, b, c, l, m, n \in \mathbb{K}$*. A* ***plane*** *in* $\mathcal{C}_{\mathbb{K}}$ *is a subset of points satisfying* $lx + my + nz$ *for some* $l, m, n \in \mathbb{K}$*.*

The name "Cartesian" is something of a misnomer, since (as we mentioned earlier) Descartes was not doing analytic geometry but rather doing algebra with line segments. Note that we should verify that notions of incidence, betweeness, congruence, etc., can be defined for a general Cartesian space $\mathcal{C}_{\mathbb{K}}$! In short, we have to verify that $\mathcal{C}_{\mathbb{K}}$ can be interpreted consistently as a geometry. However, this is tedious work and we postpone it to two later proofs. (See sections "Proof of the first theorem" and "Proof of the second theorem.") The main idea of analytic geometry is to represent, or model, the axiomatic geometry using Cartesian space. This brings us to the next topic.

2. Faithful models

We may consider it common knowledge that $\mathcal{C}_{\mathbb{R}}$ models three-dimensional Euclidean geometry. What we mean by this is that every statement in geometry can be translated into a statement of arithmetic relations between subsets of $\mathbb{R}^3$. But, the converse is not necessarily true! An example is the problem of duplicating the cube, i.e. to given a cube construct a new cube with twice the volume of the old cube. Analytically we interpret this as, given the edge a of a cube, find an x such that the cube of x is twice the cube of a. Working in $\mathcal{C}_{\mathbb{R}}$ we may wonder what the fuzz is about and simply write down the solution $x = a\sqrt[3]{2}$. However, it may be shown that this x is not an element of the constructible field, and hence the problem is actually impossible to solve using the geometer's straightedge and compass! (See chapter "Some applications" for details.) To sort this out we introduce the concept of a faithful model. To this end, we consider a geometry as an abstract set $\mathcal{G}$, whose singleton subsets are points, and with lines and planes as proper subsets.

DEFINITION 8. *Let $\mathcal{G}$ be an abstract geometry and $\mathcal{C}_{\mathbb{K}}$ a Cartesian space over a field $\mathbb{K}$. We say $\mathcal{C}_{\mathbb{K}}$ is a **faithful model** of the geometry $\mathcal{G}$ if there is a bijection $\varphi : \mathcal{G} \longmapsto \mathcal{C}_{\mathbb{K}}$ that is compatible with the undefined notions. This means:*

(i) *A subset $l \subset \mathcal{G}$ is a line if and only if $\varphi(l) \subset \mathcal{C}_{\mathbb{K}}$ is a line.*
(ii) *A subset $\sigma \subset \mathcal{G}$ is a plane if and only if $\varphi(\sigma) \subset \mathcal{C}_{\mathbb{K}}$ is a plane.*
(iii) *For any three points $A, B, C \in \mathcal{G}$, B is between A and C if and only if $\varphi(B)$ is between $\varphi(A)$ and $\varphi(C)$.*
(iv) *Given four points $A, B, C, D \in \mathcal{G}$, the line segments AB and AC are congruent if and only if the line segments $\varphi(A)\varphi(B)$ and $\varphi(C)\varphi(D)$ are congruent.*
(v) *If $\angle ABC$ is an angle in $\mathcal{G}$, then we define $\varphi(\angle ABC) = \angle\varphi(A)\varphi(B)\varphi(C)$. If α, β are two angles in $\mathcal{G}$, then α and β are congruent if and only if $\varphi(\alpha)$ and $\varphi(\beta)$ are congruent.*

*If two Cartesian spaces $\mathcal{C}_{\mathbb{K}}$ and $\mathcal{C}_{\mathbb{K}'}$ are both faithful models of the same geometry, then we refer to them as **isomorphic models**.*

With this definition we are ready to move to the statement of three important theorems.

3. Three theorems

Our first theorem is a statement about faithful models and the field of segment arithmetic.

THEOREM 1. *Consider an abstract neutral geometry satisfying Playfair's axiom. Let $\mathbb{K}$ be the field of segment arithmetic obtained from this geometry. Then the Cartesian space $\mathcal{C}_{\mathbb{K}}$ over $\mathbb{K}$ is a faithful model of the abstract geometry, and $\mathbb{K}$ is Pythagorean.*

Our hope is to obtain a dictionary between abstract geometries and known number fields. We want to empasize that the above theorem (Theorem 1) provides no real clue what the abstractly defined field $\mathbb{K}$ is. The line segment field is almost completely useless, with knowledge only of the first theorem. We know it exists, and that it theoretically can be used to analytically represent the geometry, but we cannot practically use it as long as its algebraic properties remain a mirage. Roughly, all we are able to deduce about the field of segment arithmetic is that if the geometry is neutral and satisfies Playfair's axiom, then the field is Pythagorean. Of course, from the preceeding chapter we have some clues, e.g. we suspect that if the geometry satisfies the circle-circle intersection property, then the field of segment arithmetic has to be Euclidean. And indeed, the next theorem verifies our suspicion.

THEOREM 2. *If $\mathbb{K}$ is any Pythagorean field, then the Cartesian space $\mathcal{C}_{\mathbb{K}}$ over $\mathbb{K}$ is a neutral space satisfying Playfair's axiom. Furthermore, the geometry of* *the Cartesian space* $\mathcal{C}_{\mathbb{K}}$ *also satisfies the circle-circle intersection property (i.e. is thoroughly Euclidean) if and only if* $\mathbb{K}$ *is an Euclidean field.*

For the statement of the third theorem we need to make precise what we mean by Archimedes' axiom for a field and Dedekind's axiom for a field. The geometric axioms are listed in the appendix "Hilbert's axioms." We say a field $\mathbb{K}$ is Archimedean, or satisfies Archimedes' axiom for a field, if for any $0 > a \in \mathbb{K}$ there is an an integer n such that $n > a$. We say a field $\mathbb{K}$ satisfies Dedekind's axiom if it can be written as the disjoint union of two nonempty subsets $\mathbb{K} = S \sqcup T$, such that for all $a \in S$ and $b \in T$ we have $a < b$, and there is a unique element $c \in \mathbb{K}$ such that for all $a \in S$ and $b \in T$ we have $a \leq c \leq b$.

THEOREM 3. *The Cartesian space* $\mathcal{C}_{\mathbb{K}}$ *over an ordered field* $\mathbb{K}$ *satisfies the Dedekind axiom or Archimedes' axiom if and only if* $\mathbb{K}$ *satisfies the corresponding property for a field.*

The proofs of these theorems is a very long-winded business, since we need to verify a very long list of axioms. Each proof is dedicated a section of its own. The proofs are perhaps not entirely essential, but we ask the reader to at least skim through them. There is some new theory introduced in the proof sections; for example, we discuss how to define congruence of angles in Cartesian space, and briefly discuss rigid transformations. After the proofs we shall state and prove our main theorem (see section "The main theorem").

4. Proof of the first theorem

We begin by setting up notation and defining an appropriate map $\varphi : \mathcal{G} \longrightarrow \mathcal{C}_{\mathbb{K}}$. Let $O \in \mathcal{G}$ be some fixed point in the abstract geometry. We will refer to O as the origin. Fix three perpendicular lines through O, which we shall refer to as the x-, y- and z-axis. On each axis, mark points $1_x, 1_y, 1_z$ such that $O1_x, O1_y, O1_z$ are all representatives of the unit congruence class $1 \in \mathbb{K}$. Choosing these unit points determines an orientation of the xyz-system. Now, for any point P, we drop a perpendiculars PA, PB, PC to the x-axis, y-axis and z-axis, respectively. The segment OA represents a segment class $a \in \mathbb{K}$, the segment OB an element $b \in \mathbb{K}$, and the segment OC an element $c \in \mathbb{K}$. Now, define $\varphi : \mathcal{G} \longrightarrow \mathcal{C}_{\mathbb{K}}$ by $\varphi(P) = (\pm a, \pm b, \pm c)$, where we choose the sign according to whether A (B, resp. C) is on the positive or negative side of the x-axis (y- resp. z-axis). Clearly, φ is a bijection. We have to verify that φ satisfies the criteria in the definition of a faithful model. We divide the proof into successive propositions.

PROPOSITION 5. $l \subset \mathcal{G}$ *is a line if and only if* $\varphi(l) \subset \mathcal{C}_{\mathbb{K}}$ *is a line.*

PROOF. Let l be a line in $\mathcal{G}$. We claim that we can without loss of generality consider a line in the xy-plane that is niether vertical not horizontal. Let l meet the x-axis at A. Let B be a point on the x-axis such that AB represents the unit congruence class. Draw a line from B perpendicular to the x-axis, and let C denote the intersection of this line with l. Let $k \in \mathbb{K}$ be the class of the segment BC. We will call k the slope of l. Let l intersect the y-axis at D, and let $b \in \mathbb{K}$ represent that point. Now, consider an arbitrary point (x, y) in the plane $\mathcal{C}_{\mathbb{K}}$. Let $P = \varphi^{-1}((x, y))$. Make add a point E such that DEP is a triangle with a right angle at E. Ignoring to bother about signs due to orientation, assume DE represents x and PE represents

$y - b$. Then P lies on l if and only if $\angle CAB \cong \angle PDE$. By the definition of our segment arithmetic, this condition is equivalent to saying $y - b = kx$. In other words, P lies on l if and only if $y = kx + b$, which defines a line $\varphi(l)$ in $\mathcal{C}_{\mathbb{K}}$. This settles the proposition. □

PROPOSITION 6. *A subset $\sigma \subset \mathcal{G}$ is a plane if and only if $\varphi(\sigma) \subset \mathcal{C}_{\mathbb{K}}$ is a plane.*

PROOF. We here use some results we have not yet proven. Let σ be the plane determined by three noncollinear points $A, B, C \in \mathcal{G}$. By (Proposition 10) $\varphi(A), \varphi(B), \varphi(C)$ determine a unique plane. We have to verify that for any point D contained in σ the image $\varphi(D)$ is contained in the plane determined by $\varphi(A), \varphi(B), \varphi(C)$, because then we can identify this plane with $\varphi(\sigma)$, and φ will map planes to planes (and we check similarly for φ^{-1}). But we claim it is sufficient to check that any point collinear with A, B and any point collinear with A, C both map into $\varphi(\sigma)$, and vice versa for φ^{-1}. This follows from the property that φ maps lines to lines. □

We define betweeness for points on lines in $\mathcal{C}_{\mathbb{K}}$ as follows: Let $A = (a_1, a_2, 1_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$ be three distinct but collinear points. We say that B lies between A and C if at least one of the following six double inequalities is fulfilled:

$$\begin{aligned} a_1 < b_1 < c_1; \quad a_1 > b_1 > c_1; \\ a_2 < b_2 < c_2; \quad a_2 > b_2 > c_2; \\ a_3 < b_3 < c_3; \quad a_3 > b_3 > c_3. \end{aligned}$$

With this notion of betweeness we return to our proof.

PROPOSITION 7. *For any three points $A, B, C \in \mathcal{G}$, B is between A and C if and only if $\varphi(B)$ is between $\varphi(A)$ and $\varphi(C)$.*

PROOF. Let $A, B, C \in \mathcal{G}$ be three collinear points. We shall for simplicity assume A, B, C lie above the first quadrant of the xy-plane. Let A', B', C' be the respective projections onto the x-axis. Since the lines AA', BB', CC' are parallel, B will be between A and C if and only if B' is between A' and C'. Let OA', OB', OC' be represented respectively by $a, b, c \in \mathbb{K}$. Then B is between A and C if and only if either $a < b < c$ or $a > b > c$. This is equivalent to saying $\varphi(B)$ is between $\varphi(A)$ and $\varphi(C)$. □

It would seem that to study congruence of line segments in analytic geometry, we would have to resort to the Euclidean metric. But this is not necessary. We may instead consider the squared distance

$$\rho^2(A, B) = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2,$$

for points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. We then make the following definition.

DEFINITION 9. *Two line segments AB and CD in the Cartesian space $\mathcal{C}_{\mathbb{K}}$ over an ordered field $\mathbb{K}$ are said to be* ***congruent*** *if $\rho^2(A, B) = \rho^2(C, D)$.*

It is easily verified that this will give the same notion of congruence as would equality between the unsquared distances. For the next proposition we need a lemma, stating the more most commonly known version of the Pythagorean theorem.

Lemma 1. *If ABC is a right triangle with legs of congruence class a, b and hypotenuse of class c, then*

$$a^2 + b^2 = c^2$$

in the field $\mathbb{K}$ of segment arithmetic. Furthermore, the field of segment arithmetic of a neutral geometry satisfying Playfair's axiom is Pythagorean.

Proof. Drop a perpendicular CD from the vertex C to a point D on the hypotenuse, making a right angle with the hypotenuse. Then we find that the new triangles ACD and CBA have the sime angles as the old triangle ABC. Without proof, we claim this implies all three triangles are similar. As a consequence all sides are proportional. Denote the class of BD as x. By the proportionality we obtain

$$\frac{x}{a} = \frac{a}{c}, \quad \frac{c-x}{b} = \frac{b}{c}.$$

Cross multiplying we get $cx = a^2$ and $c^2 - cx = b^2$, from which by substitution we obtain $a^2 + b^2 = c^2$. That the field of segment arithmetic is Pythagorean is an immediate consequence of the validity of this. Verily, if we construct a right triangle with legs are of class $1, a$, then by above lemma the hypotenuse is a segment of class $\sqrt{1 + a^2}$, and this is in $\mathbb{K}$. $\square$

With this lemma, we continue the proof.

Proposition 8. *Given four points $A, B, C, D \in \mathcal{G}$, the line segments AB and AC are congruent if and only if the line segments $\varphi(A)\varphi(B)$ and $\varphi(C)\varphi(D)$ are congruent.*

Proof. We may assume everything is happening in a plane. Let A, B be points in $\mathcal{G}$ and let the segmet AB represent $d \in \mathbb{K}$. Write $\varphi(A) = (a_1, a_2)$, $\varphi(B) = (b_1, b_2)$. Then, if we draw a right triangle ABC' with legs parallel to the axes we find AC' is of congruence class $b_1 - a_1$ and BC of class $b_2 - a_1$, by construction. We use the Pythagorean theorem to find

$$d^2 = (b_1 - a_1)^2 - (b_2 - a_2)^2.$$

Now, let CD be another segment, with length $d' \in \mathbb{K}$. Then, similarly,

$$d'^2 = (d_1 - c_1)^2 + (d_2 - c_2)^2.$$

We have $AB \cong CD$ if and only if $d = d'$, because $\mathbb{K}$ was constructed from the set of congruence equivalence classes. On the other hand, $d = d'$ if and only if $d^2 = d'^2$, because both are positive elements. But d^2, d'^2 are the squared distances, so then, due to our definition of congruence in $\mathcal{C}_{\mathbb{K}}$, we can conclude that $AB \cong CD$ if and only if $\varphi(A)\varphi(B) \cong \varphi(C)\varphi(D)$. $\square$

Next we will define congruence for angles in the Cartesian space. A ray originating at a point A is the subset of a line through A, consisting of A and all points of the line on some chosen side of A. An angle is the union of two rays originating at the same point and not lying on the same line. Let $l : y = kx + m$ and $l' : y = k'x + m'$ be two lines in the xy-plane of a Cartesian space $\mathcal{C}_{\mathbb{K}}$. Let r be a ray that is a subset of l and let r' be a ray that is a subset of l', and assume both rays originate at the same point. We call the set of points in the xy-plane that lie on the same side of l as r' and simultaneously also on the same side of l' as r the interior of the angle $\angle(r, r')$. We say $\angle(r, r')$ is a right angle if the slopes of the lines the rays lie on satisfy $kk' = -1$. We then say an angle is acute if it is contained in the interior of a right angle, and obtuse if it contains a right angle in its interior. As we did for congruence of line segments, we now define a function that will enable us to study congruence analytically.

DEFINITION 10. *If $\alpha = \angle(r, r')$ is an angled formed by rays r, r' lying on lines of slopes k, k', we define the **tangent** of α to be*

$$\tan\alpha = \pm\left|\frac{k'-k}{1+kk'}\right|,$$

where we take $+$ if the angle is acute and $-$ if the angle is obtuse.

The reason we use this, admittedly akward definition instead of the usual "length of opposite side over length of adjacent side," is because that formula necessitates the possibility to take square roots in our field $\mathbb{K}$, i.e. that $\mathbb{K}$ is Euclidean. It may be verified that the definition given above coincides with the usual, when the standard formula is a legal expression.

PROPOSITION 9. *If α, β are two angles in $\mathcal{G}$, then α and β are congruent if and only if $\varphi(\alpha)$ and $\varphi(\beta)$ are congruent.*

PROOF. Suppose we are given angles α, α' in $\mathcal{G}$. Let the vertices be A, A', respectively. Choose any two points B, C on the two rays of α, one point on each ray. Then find B', C' on the rays of α' such that $AB \cong A'B'$ and $AC \cong A'C'$. If $\alpha \cong \alpha'$, then by the side-angle-side axiom (C6) $ABC \cong A'B'C'$, and in particular $BC \cong B'C'$. Conversely, if $BC \cong B'C'$, then by the side-side-side theorem the two triangles are again congruent, and so $\alpha \cong \alpha'$. Next, apply φ to the six points A, B, C, A', B', C'. By (Proposition 8) $\varphi(A)\varphi(B) \cong \varphi(A')\varphi(B')$ and $\varphi(A)\varphi(C) \cong \varphi(A')\varphi(C')$. According to our later result (Theorem 2), the geometry in $\mathcal{C}_{\mathbb{K}}$ satisfies Hilbert's axioms. So by repeating the congruence argument carried out in $\mathcal{G}$ we see that $\alpha \cong \alpha'$ if and only if $\varphi(\alpha) \cong \varphi(\alpha')$. □

With above propositions proved, theorem number one (Theorem 1) follows. In the proof we assumed some properties not yet proven, assumptions regarding what axioms the geometry of $\mathcal{C}_{\mathbb{K}}$ satisfies. The proof of the next theorem (Theorem 2) fills in these gaps in our proof.

5. Proof of the second theorem

In order to prove the theorem we divide the claim into successive propositions, which we will prove in due course.

PROPOSITION 10. *If $\mathbb{K}$ is any ordered field, the Cartesian space $\mathcal{C}_{\mathbb{K}}$ satisfies Hilbert's incidence axioms* (I1 − I7) *together with the parallel axiom* (P).

PROOF. This offers no surprises and we will not give a complete proof. The proposition is actually true for any field, but we are only interested ordered fields, which a priori makes the situation a lot more simple. This is because all ordered fields must have characteristic zero. Thus, lines and planes may be treated using vector calculus (via $\mathbb{K}^3 \cong \mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}$), and we can be sure nothing pathological ever happens because the $\mathbb{K}$ has characteristic zero. The doubtful reader may verify all axioms, we content ourselves with a remark regarding Playfair's axiom. If l is a line and A is a point, then not only is there at most one line l' through A parallel to l, but the stronger result that such an l' always exists holds. □

As a useful prelimiary to the next proposition, we note without proof that it is always possible in the Cartesian space $\mathcal{C}_{\mathbb{K}}$ over a field $\mathbb{K}$ to make an affine transformation, taking the coordinate axes x, y, z to any three unequal but intersecting lines x', y', z' in $\mathcal{C}_{\mathbb{K}}$. Furthermore, we may scale these new axes however we please, i.e. we are free to choose where the unit points on the axes are assigned. This is highly useful, since it enables us to reduce many problems to questions in plane geometry.

PROPOSITION 11. *If* $\mathbb{K}$ *is an ordered field, then we may define betweeness in* $\mathcal{C}_{\mathbb{K}}$ *so as to satisfy Hilbert's betweeness axioms* (B1 − B4).

PROOF. Suppose that $\mathbb{K}$ is a given ordered field with positive subset P. Recall (section "Proof of the first theorem") that we define betweeness for points on lines as follows: Let $A = (a_1, a_2, 1_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$ be three distinct collinear points on a line l. We say that B lies between A and C if at least one of the following six double inequalities is fulfilled:

$$\begin{aligned} a_1 < b_1 < c_1; \quad a_1 > b_1 > c_1; \\ a_2 < b_2 < c_2; \quad a_2 > b_2 > c_2; \\ a_3 < b_3 < c_3; \quad a_3 > b_3 > c_3. \end{aligned}$$

The first betweeness axiom (B1) is obvious from our definition. (B2): That or any two distinct points A, B there exists a point C such that B is between A and C - This follows from the properties of the field $\mathbb{K}$. Any ordered field has zero characteristic, so $1/2 \in \mathbb{K}$. Given, say $b > d$, there exist $a, c, e \in \mathbb{K}$ such that $a < b < c < d < e$, e.g. choose $a = b - 1$, $c = (b + d)/2$ and $c = d + 1$. (B3): We have to show that given three collinear points, one and only one of the points is between the other two. This follows from the fact that in any ordered field $\mathbb{K}$, if a, b, c are three distinct elements, then one and only one of the following six possibilities can occur:

$$\begin{aligned} a < b < c; \quad a < c < b; \\ b < a < c; \quad b < c < a; \\ c < a < b; \quad c < b < a. \end{aligned}$$

Lastly, we prove validity of Pasch's axiom (B4). Let σ be a plane determined by a trangle ABC, and let l be a line in σ. By an affine transformation, we may identify σ with the xy-plane, and assume that l is a vertical line, say with equation $x = d$. Let a_1, b_1, c_1 be the respective x-coordinates of points A, B, C. Assume $a_1 < d < b_1$. Then it is clear that a will meet BC but not AC if $c_1 < d$, and will meet AC but not BC if $c > d$. This completes the proof. □

PROPOSITION 12. *Let* $\mathbb{K}$ *be a field and let* $\mathcal{C}_{\mathbb{K}}$ *be the associated Cartesian space. Then* $\mathcal{C}_{\mathbb{K}}$ *satisfies Hilbert's first five congruence axioms (C1-C5) if the base field* $\mathbb{K}$ *is Pythagorean.*

PROOF. (C1): By an affine transformation, we may assume everything is happening in the xy-plane. Suppose we are given a line $y = kx + m$ and a point $A = (a, ka + m$ on that line. To show that the first congruence axiom is fulfilled, we need to show that, for any non-negative $d \in \mathbb{R}$, we can find a point $C = (c, kc + m$ on $y = kx + m$ such that the line segment from A to C has length d. See the earlier discussion regarding Pythagorean fields. The criteria we want to fulfill is

$$\sqrt{(a-c)^2 + (ma+b-(mc+b))^2} = d,$$

which after simplification becomes $|a-c|\sqrt{1+m^2} = d$. Since $\mathbb{K}$ is Pythagorean $\sqrt{1+m^2}$ exists in $\mathbb{K}$, so we may solve for c and obtain exactly two solutions. Hence the first congruence axiom is fulfilled. The second congruence axiom (C2) asserts the transitivity of congruence of line segments, and this follows immediately from our definition of congruence using the squared distance function. For (C3), take three collinear points A, B, C with B between A and C. Using the affine freedom, assume without loss of generality that $A = (0,0), B = (1,0), C = (1+c, 0)$. We then have $\rho^2(A,C) = (1+c)^2$. Similarly, if D, E, F is another set of collinear points, with E between D and F, we may (see later discussion of rigid motions) set $D = A = (0,0)$. If $AB \cong DE$, we for an obvious choice of $f \in \mathbb{K}$ have $\rho^2(D,F) = (1+f^2)^2$. Now, adding $BC \cong EF$ implies $c^2 = f^2$, so that $\rho^2(D,F) = \rho^2(A,C)$. The fourth congruence axiom (C4) is about laying off angles. Suppose we are given an angle α and a ray emanating from a point A with slope k. We must find a line through A with slope k' such that

$$\tan\alpha = \pm\left|\frac{k'-k}{1+kk'}\right|,$$

where the sign should be adjusted according to whether α is acute or not. We solve this equation and get

$$k' = \frac{k \pm \tan\alpha}{1 \mp \tan\alpha}.$$

The two solutions give angles on either side of the given line through A, as necessitated by the axiom. Lastly, axiom (C5) is the transitivity of congruence of angles, and this is immediate from our definition of congruence using the tangent function. □

Euclid "proved" the side-angle-side axiom (C6), using a method of superposition; rigidly moving one triangle and placing it on top of the other. But, to do this one has to assume that those rigid translations exist. We shall prove that there are enough such rigid motions in a Cartesian space over a Pythagorean field, enough here meaning enough to imply the side-angle-side axiom. We first precisely define what a rigid motion is.

DEFINITION 11. *Let $\mathcal{G}$ be a geometry considered formally as a set. We define a* ***rigid motion*** *of $\mathcal{G}$ to be a mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{G}$ defined on all points, such that φ preserves the structures determined by the undefined notions. This means that:*

(i) *φ maps $\mathcal{G}$ to $\mathcal{G}$ bijectively.*
(ii) *φ maps lines to lines.*
(iii) *φ maps planes to planes.*
(iv) *φ preserves betweeness of collinear points.*
(v) *For any two points A, B we have $AB \cong \varphi(A)\varphi(B)$.*
(vi) *For any angle α we have $\alpha \cong \varphi(\alpha)$.*

Next, we make precise what we will mean by "enough" rigid motions, or existence of rigid motions.

DEFINITION 12. *By* ***existence of rigid motions*** *in a geometry $\mathcal{G}$ we mean:*

(i) *For any two points $A, B \in \mathcal{G}$ there is a rigid motion φ such that $\varphi(A) = B$.*

(ii) *For any three points* O, A, B*, there is a rigid motion* φ *such that* $\varphi(O) = O$ *and* φ *rotates the ray* OA *into the ray* OB*.*

(iii) *For any line* l *on a plane* $\sigma \subset \mathcal{G}$ *there is a rigid motion* φ *that leaves* l *invariant and reflects the plane* σ *through* l*, i.e. such that* $\varphi(P) = P$ *for all* $P \in l$ *and* φ *interchanges the two sides of* σ*.*

We now show that existence of rigid motions (henceforth abbreviated (ERM)) holds in the Cartesian space over any Pythagorean field, and that this implies validity of (C6).

PROPOSITION 13. *If* $\mathbb{K}$ *is Pythagorean the associated Cartesian space* $\mathcal{C}_{\mathbb{K}}$ *satisfies the side-angle-side axiom, i.e. Hilbert's sixth congruence axiom.*

PROOF. Let $\mathcal{C}_{\mathbb{K}}$ denote the Cartesian space over a Pythagorean field $\mathbb{K}$. As a first example of a rigid motion we take any translation τ, mapping (x, y, z) to $(x + a, y + b, z + c)$, with some fixed $a, b, c \in \mathbb{K}$. It is obvious that translations are rigid motions, so condition (i) of (ERM) is clear. For condition (ii) we consider the rotation group $SO_3(\mathbb{K})$. Since a rotation is linear, lines are transformed to lines and planes to planes. A brief calculation will show that rotations preserve the relative slopes of curves, so that angles are preseved. Since linerar transformations either preserve or reverse inequalities, rotations preserve betweeness. Finally, rotations have unit determinant, so congruence of line segments is preserved. Combining a translation and a rotation (i.e. making an affine transformation) we have (ii) satisfied. Finally, we prove condition (iii) of (ERM). Using a translation we first move the line l to pass through the origin. We then by successive rotations move σ into the xy-plane and align l with the x-axis. Then, we do the reflection defined by $(x, y) \mapsto (x, -y)$. The rotations and the translation are invertible, so we have proven condition (iii). Next, we prove that (ERM) implies (C6), using Euclid's method. Suppose we are given two triangles ABC and $A'B'C'$ such that $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$. We then move one on top of the other using rigid motions, taking AB to $A'B'$ and AC to $A'C'$ (with A to A'). A moment's thinking makes it clear that then $\angle BAC$ is "on top" of $\angle B'A'C'$, and the two angles $\angle ABC$ and $\angle BCA$ are moved similarly, and since congruence of line segments is preserved: $BC \cong B'C'$. This settles the proposition. □

PROPOSITION 14. *The Cartesian space* $\mathcal{C}_{\mathbb{K}}$ *satisfies the circle-circle intersection property, i.e. the Euclidean axiom, if and only if* $\mathbb{K}$ *is Euclidean.*

PROOF. We begin by showing that $\mathbb{K}$ Euclidean implies the circle-circle intersection propery holds in $\mathcal{C}_{\mathbb{K}}$. Let $\mathbb{K}$ be Euclidean and represent two circles Γ, Δ in the xy-plane of $\mathcal{C}_{\mathbb{K}}$ by

$$\Gamma : (x - a)^2 + (y - b)^2 = r^2, \quad \Delta : (x - c)^2 + (y - d)^2 = s^2.$$

To find an intersection we substitute one equation from the other and get a linear equation, express y in terms of x (say), and substitute back into one of the circle equations to solve for x. We get x in terms of square roots, which exist in the base field since $\mathbb{K}$ was assumed Euclidean. Hence the circle-circle intersection property is satisfied. For the converse statement, assume the circle-circle intersection property is satisfied. Take points $O = (0, 0), A = (a, 0)$ and $A' = (a + 1, 0)$. Let Γ be the circle given by

$$\Gamma : (x - (a + 1)/2)^2 + y^2 - \frac{(a + 1)^2}{4} = 0,$$

i.e. the circle centered midpoint between O and A', and containing both points. Next, we note that if $\Gamma = 0$ is the equation of a circle and $l = 0$ the equation of a line, then $\Gamma + l = 0$ is the equation of another circle, whose intersections with Γ are the points where l ought to intersect Γ. So, by the circle-circle intersection property it follows that also circles and lines intersect when they ought to. Thus, if we draw a line l perpendicular to the x-axis through A, it will intersect Γ in a point B in the positive quadrant. Solving for the coordinates of B we find $B = (a, \sqrt{a})$, so $\sqrt{a} \in \mathbb{K}$. □

With this proposition every statement in the second theorem (Theorem 2) has been verified.

6. Proof of the third theorem

The third theorem (Theorem 3) has a surprisingly short proof, in comparison to the two earlier proofs.

Proof. For Archimedes' axiom, we note that we are free to choose our coordinates such that, if given a segment AB we may decide it represents the unit. If C, D are collinear with A, B, and correspond to elements $c < d \in \mathbb{K}$, then n copies of AB will exceed CD if and only if $n > d - c$, proving that the Cartesian space $\mathcal{C}_{\mathbb{K}}$ satisfies Archimedes' axiom if and only if $\mathbb{K}$ is Archimedean. For the Dedekind axiom, choose coordinates such that the line in question is the x-axis, and identify the points on that line with elements of $\mathbb{K}$. Then the Dedekind axiom for $\mathcal{C}_{\mathbb{K}}$, applied to the x-axis, is the same statement as the Dedekind axiom for a field. □

7. The main theorem

We shall soon state our main theorem. It provides us with an almost complete dictionary between the field of segment arithmetic and axiomatically defined geometries, as long as we include Archimedes' axiom in our list. This is because for Archimedean geometries (that are at least neutral and satisfy Playfair's axiom) the field of segment arithmetic turns out to be isomorphic to a subfield of the real numbers. Our main theorem classifies the possible subfields.

Theorem 4. *The Cartesian space $\mathcal{C}_{\Omega}$ over Hilbert's field is a faithful model of an Archimedean neutral geometry satisfying Playfair's axiom. The Cartesian space $\mathcal{C}_{\mathbb{F}}$ over the constructible field is a faithful model of an Euclidean geometry satisfying Archimedes' axiom. The Cartesian space $\mathcal{C}_{\mathbb{R}}$ over the real numbers is a faithful model of an Archimedean Euclidean geometry satisfying the Dedekind axiom.*

Proof. Theorem 1 says an axiomatically defined geometry, that is at least neutral and satisfies Playfair's axiom, is faithfully represented by some Cartesian space $\mathcal{C}_{\mathbb{K}}$ over a Pythagorean field. If the geometry is also Archimedean, then by Theorem 3 $\mathbb{K}$ must be Archimedean for the model $\mathcal{C}_{\mathbb{K}}$ to be faithful. We now show that any ordered field $\mathbb{K}$ satisfying Archimedes' axiom is isomorphic, with its ordering, to a subfield of $\mathbb{R}$. Since $\mathbb{K}$ is ordered it has characteristic zero and contains a subfield isomorphic to the rational numbers $\mathbb{Q}$. To see this, consider the injection of the natural numbers $\mathbb{N}$ into $\mathbb{K}$ given by $n \mapsto 1+1+...+1$ (n times). Forming quotients we get all rational numbers, or a subfield of $\mathbb{K}$ isomorphic to the field of rational numbers. Denote this subfield of $\mathbb{K}$ as $\mathbb{K}_0$, and denote the isomorphism with the rational numbers $\varphi_0 : \mathbb{K}_0 \cong \mathbb{Q}$. We will extend φ_0 to an isomorphism $\varphi : \mathbb{K} \longrightarrow \mathbb{R}$. Take $a \in \mathbb{K}$, and let $a_0 \in \mathbb{Z}$ be the unique integer such that $a_0 \leq a < a_0 + 1$. (For notational convenience we do not distinguish between a_0 and $\varphi_0(a_0)$.) Such an integer a_0 exists because of Archimedes' axiom. Continue and define similarly $a_n \in 10^{-n}\mathbb{Z}$ such that $a_n \leq a \leq a_n + 10^{-n}$. In this way we get a sequence $a_0 \leq a_1 \leq ...$ of rational numbers. In the field of real numbers $\mathbb{R}$ this sequence converges to a certain real number, which we label $\varphi(a)$. In this way we obtain a map $\varphi : \mathbb{K} \longrightarrow \mathbb{R}$, which may easily be verified to be a homomorphism of fields, i.e. $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$. Hence $\mathbb{K}$ is isomorphic to the subfield $\varphi(\mathbb{K}) \subset \mathbb{R}$ of the real numbers. Now, by Theorem 2 we immediately conclude that $\mathcal{C}_\Omega$ faithfully models an Archimedean neutral geometry satisfying Playfair's axiom, and furthermore, $\mathcal{C}_\mathbb{F}$ is a faithful model of an Archimedean Euclidean geometry. Finally, $\mathbb{K}$ satisfies the Dedekind axiom if and only if its image $\varphi(\mathbb{K})$ in the real numbers satisfies the Dedekind axiom, since the fields are order-isomorphic. Each real number $r \in \mathbb{R}$ is characterized by the sets $S = \{a \leq r\}$ and $T = \{a > r\}$, so that $\varphi(\mathbb{K})$ satisfies the Dedekind axiom if and only if $\varphi(\mathbb{K}) = \mathbb{R}$. We immediately conclude that an Archimedean Euclidean geometry satisfying Dedekind axiom is faithfully modeled by $\mathcal{C}_\mathbb{R}$. □

Note that we have to add each and every axiom in our list to end up with something isomorphic to, or faithfully represented by, the real Cartesian space. The real numbers gives us a quite narrow sense of geometry. We would also like to point out that the real numbers can actually be defined as the field of segment arithmetic of an Archimedean Euclidean geometry satisfying the Dedekind axiom. Allthough this definition would be cumbersome for most purposes, it is true to the geometric origin of the real numbers. Without Euclid's geometry there would have been no real numbers.

Hilbert's field is not the only subfield of the real numbers that is Pythagorean but not Euclidean, and the constructible field is not the only subfield of the real numbers that is Euclidean but does not also satisfy the Dedekind axiom. Hence they are not (unlike $\mathcal{C}_\mathbb{R}$) the only Cartesian models of their respective axiomatic geometries. This means an Archimedean neutral geometry satisfying Playfair's axiom, and an Euclidean geometry satisfying Archimedes' axiom, are examples of geometries that are not categorical, i.e. allow for non-isomorphic faithful models.

With this theorem, we can with good conscience apply the techniques of analytic geometry. The calculational power of the analytic approach is far greater than that of the synthetic approach, allowing us to easily treat curves and surfaces, not to mention define area and volume in managable ways. However, from the theoretical, or inventive, perspective the axiomatic method might still be the preferable, as it is a lot more unconstricted. In the next chapter we show the power and utility of working with faithful models, proving three theorems that remained unsolved for about two thousand years, or as long as people were able to properly handle only synthetic proofs.

CHAPTER 4

Some applications

In this chapter we exhibit the power of our main theorem. One could claim that we have, in this monograph, added nothing essentially new to Euclid's geometry, only uncovered a surprisingly rich agebraic structure that was already inherent in the geometry. The field of segment arithmetic almost defines itself, rather than being created and imposed by us. However, the line segment field, along with the deep statements in our main theorem, enables us to use algebraic techniques in our study of geometry. With this expanded toolbox we obtain results not possible, or at least extremely difficult, to obtain with Euclid's synthetic methods.

Tradition has it that Plato (or Oenopides, an earlier geometer) was responsible for setting the requirement that a solution to a construction problem in geometry would count only if it was accomplished using an unmarked straightedge and a compass, without lifting the tools from the paper. By the time of Plato, Greek geometers were busy thinking about three major problems, nowadays called the three classical problems. The problems are easy to state, but notoriously difficult without advanced theory. The three problems are: to duplicate the cube, to trisect an angle, and to square a circle. Around 320 CE, Pappus of Alexandria (whose famous theorem we made extensive use of in the chapter about line segment arithmetic) declared that it was impossible to solve any of the three classical problems under the Platonic restriction, but as far as historians know Pappus did not offer a proof of this assertion. We shall see that Pappus was indeed correct - the problems are unsolvable using only Plato's idealized straightedge and compass - and we shall give full proof of this. However, there are other ways by which one can construct solutions, some of which we will discuss.

1. Duplicating the cube

The problem of duplicating the cube is to, given a cube, find a new cube whose volume is twice the volume of the old cube. The problem is for historical reasons also often called the Delian problem. According to legend, the citizens of Athens consulted the oracle of Apollo at Delos in 430 BC, in order to learn how to defeat a plague which was ravaging their lands. The oracle responded that to stop the plague, they must double the size of their altar to Apollo. The Athenians dutifully doubled each side of the altar, and the plague increased. The correct interpretation was that they must double the volume of their altar, not merely its side length; this proved to be a most difficult problem indeed. The problem was solved in one way by Plato's friend the Pythagorean Archytas, and other solutions were proposed by Archytas' student Menaechmus and the geometer Eudoxus. The only problems were that the plague was finished several decades before, and both solutions require more than a straightedge and compass.

To attack the problem we translate it into the Cartesian space $\mathcal{C}_{\mathbb{F}}$ over the constructible field. From our main theorem we know that this Cartesian space is a faithful model of the Euclidean geometry of figures constructible by means of straightedge and compass. Our reformulation of the problem is that given a cube with edge a, find an x such that the cube of x is twice the cube of a, i.e. $x^3 = 2a^3$. (We are here calculating volumes, allthough we have not properly defined such a thing in this monograph. It should be clear how it works.) The solution is of course $x = a\sqrt[3]{2}$. We are free to choose our unit length, so essentially the problem is to, with straightedge and compass, construct a segment of length $\sqrt[3]{2}$. Before attacking the problem we state and prove a lemma. For the basic theory about field extensions, see the appendix "Field extensions."

LEMMA 2. *If a is constructible but irrational, then there is a finite sequence of real numbers $a_1, ..., a_n = a$ such that $\mathbb{Q}(a_1, ..., a_k)$ is an extension of $\mathbb{Q}(a_1, ..., a_{k-1})$ of degree 2. In particular, $[\mathbb{Q}(a) : \mathbb{Q}] = 2^r$ for some integer $r > 0$.*

PROOF. If a is constructible it can be obtained by a finite number of field operations and square roots. Each time there is a square root in writing a we let this square root generate an extension of $\mathbb{Q}$. Hence, the existence of the a_k is clear. Then

$$2^n = [\mathbb{Q}(a_1, ..., a_n) : \mathbb{Q}] = [\mathbb{Q}(a_1, ..., a_n) : \mathbb{Q}(a)][\mathbb{Q}(a) : \mathbb{Q}],$$

by multiplicativity of degrees of field extensions (see appendix "Field extensions"), which completes the proof. □

PROPOSITION 15. *$\sqrt[3]{2}$ is not in the constructible field $\mathbb{F}$. Hence the duplication of the cube is impossible using straightedge and compass.*

PROOF. $\sqrt[3]{2}$ is a zero of the polynomial $x^3 - 2$. This polynomial is irreducible over the rational numbers $\mathbb{Q}$. Indeed, if factored, then at least one factor would be linear, and so have a root which we by assumption may take to be of the form a/b, with $a, b \in \mathbb{Z}$ relatively prime. Then $a^3 = 2b^3$, which implies a is even. It follows that $2^3|a$, so $2^2|b$, and hence b is even too. This contradicts the assumption that a, b are relatively prime, and proves that $x^3 - 2$ is irreducible over $\mathbb{Q}$. Thus $\mathbb{Q}(\sqrt[3]{2})$ is an extension of degree 3 of $\mathbb{Q}$. By our lemma (Lemma 2) we conclude that $\sqrt[3]{2}$ is constructible if and only if there exists an integer $r > 0$ such that $3 = 2^r$. No such integer exists. □

Archytas solved the problem in the fourth century BC, but of course he used more than straightedge and compass. His solution is an ingenious geometric construction in three dimensions, determining a certain point as the intersection of three surfaces of revolution. Solutions of this type were called mechanical. For brevity of exposition, we shall give a modernized coordinate-based presentation of Archytas tour-de-force. We know of Archytas' solution from Euclid's contemporary Eudemus, who explained it like this:

> Let the two given lines be OA and OB; it is required to construct two mean proportionals between OA and OB. Draw the circle OBA having OA as diameter where OA is the greater; and inscribe OB, and produce it to meet at C the tangent to the circle

at A. ... imagine a half-cylinder which rises perpendicularly on the semicircle OBA, and that on OA is raised a perpendicular semicircle standing on the [base] of the half-cylinder. When this semicircle is moved from A to B, the extremity O of the diameter remaining fixed, it will cut the cylindrical surface in making its movement and will trace on it a certain bold curve. Then, if OA remains fixed, and if the triangle OCA pivots about OA with a movement opposite to that of the semicircle, it will produce a conical surface by means of the line OC which, in the course of its movement, will meet the curve drawn on the cylinder at a particular point.

Suppose the three surfaces of revolution intersect at the point P. Then P, being on the half-cylinder, lies above a point N on the circle OBA. The two mean proportionals constructed by Archytas are then OP and ON. To see what is happening we rephrase Archytas' model in analytic terms, and construct three surfaces of revolution:

$$\begin{aligned} \text{the cone} \quad &: \quad x^2+y^2+z^2 = \frac{a^2}{b^2}x^2; \\ \text{the cylinder} \quad &: \quad x^2+y^2 = ax; \\ \text{the tore} \quad &: \quad x^2+y^2+z^2 = a\sqrt{x^2+y^2}. \end{aligned}$$

The correspondance with Eudemus descripition is $a = OA$ and $b = OB$. If $(x, y, z) = P$ denotes the point of intersection, we get $OP = \sqrt{x^2+y^2+z^2}$ and $ON = \sqrt{x^2+y^2}$. From the equations for the surfaces we get

$$\frac{a}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{b}.$$

This is equivalent to $OA : OP = OP : ON = ON : OB$. Compunding the ratios we find Archytas' promised mean proportion: $OA : OB = (ON : OB)^3$. This says the cube of side ON is to the cube of side OB as OA is to OB. In the particular case where $OA = 2OB$, we get $ON^3 = 2OB^3$, and the cube is doubled. How Archytas ever got the idea, and managed to visualize it, is something of a mystery.

Solutions like Archytas' above were called mechanical, and for good reason. The solutions typically involve (idealized) rotations of rigid contructions, pendulums swinging with a prescribed frequency relative to each other and thereby describing a curve, or etc. The geometrical objects so created are not constructible. However, they are almost constructible, in the sense that arbitrarily many points of the objects can be constructed by straightedge and compass. If the geometer was allowed to use infinitely many steps (i.e. apply the operations $+, -, \cdot, \div$ and $0 < a \mapsto \sqrt{1+a^2}$ an infinite number of times), then all mechanical objects could be created. Similarly put: the mechanical kind of geometry, obviously related to concepts of continuity and limits as it is, corresponds to the extension of the constructible field to the field of real numbers. Not only that, but a proper description of mechanical geometric constructions also necessitates real number calculus; developed by Newton and Leibniz precisely for the purpose of describing mechanical geometry.

2. Trisecting the angle

The problem of trisecting the angle is, given a certain angle, to construct by straightedge and compass a new angle equal to one third of the given angle. This is a quite elusive problem. Because some angles can be trisected and because all angles can be bisected, it is easy to get the idea that all angles should be possible to trisect. However, this is not the case, at all.

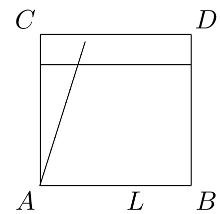
PROPOSITION 16. *The angle $\pi/3$ cannot be trisected by straightedge and compass. Hence it is not always possible to trisect an angle with straightedge and compass.*

PROOF. We are working in a plane in the Cartesian space $\mathcal{C}_{\mathbb{F}}$ over the constructible field. Since the geometry is Euclidean, the existence of the angle α in $\mathcal{C}_{\mathbb{F}}$ is equivalent to the existence of a right triangle with sides $\cos\alpha$ and $\sin\alpha$. We claim that $\cos(\pi/9)$ is not constructible, so that there is no such triangle, but that $\cos(\pi/3)$ is constructible (if the latter was not constructible our claim would amount to nothing). The number $\cos(\pi/3) = 1/2$ is clearly constructible. Next, we note that

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta)\\ &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta\\ &= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

Letting $\theta = \pi/9$ and $a = \cos(\pi/9)$, above amounts to $4a^3 - 3a = 1/2$. Thus a is a zero of $8x^3 - 6x - 1$. We want to show this polynomial has no constructible roots. We make the substitution $y = 2x$ and begin by showing $y^3 - 3y - 1$ has no rational roots. If it did, let the root be m/n with $m, n \in \mathbb{Z}$ relatively prime. Then $m^3 - 3mn^2 - n^3 = 0$. Consequently any prime factor of m also divides n, and vice versa. This can only be satisfied if $m, n = \pm 1$, since m, n are relatively prime. Hence $m/n = \pm 1$, which we by inspection see is not a root. Thus $y^3 - 3y - 1$ is irreducible over $\mathbb{Q}$, since it is of degree 3 with no rational roots. We conclude that $\mathbb{Q}(a)$ is an extension of degree 3 over $\mathbb{Q}$. But, as shown previously (Lemma 2), $\mathbb{Q}(a)$ would be an extension of degree 2 if a was constructible. We conclude that $\pi/9$ is not constructible, so that it is not possible to trisect $\pi/3$. □

The Greek Hippias of Elis was a statesman and philosopher who travelled from place to place taking money for his services. He lectured on poetry, grammar, history, politics, archaeology, mathematics and astronomy. He was Plato's senior by a generation, and is a major charachter in two of Plato's dialogues. Plato described him as vain, arrogant and of quite limited intellect, which was probably more than a little unfair as Hippias was most certainly intelligent. After all, he did find a general method for trisecting angles. We shall now indicate Hippias' solution, using a quadratrix to trisect the angle $\pi/3$. The quadratrix curve is defined as follows. Consider a square with vertices A, B, C, D. Suppose that AC rotates uniformly about A until it coincides with AB, and that in the same time CD descends uniformly to AB. (So AC and CD reach their final position AB at the same time. The quadratrix is the locus of the intersection points of both moving line segments.



Analytically, if we use a coordinate system with A as the origin, $B = (1, 0)$, $C = (0, 1)$ and $D = (1, 1)$, the quadratrix is described by the equation $x = y \cot(y\pi/2)$. For $y = 0$ the both the analytic expression and the mechanical method by wich we defined the quadratrix break down. However, the quadratrix does approach a limit: $\lim \cot(y\pi/2) = 2/\pi$. We call this limit point L.

Imagine the unit square A, B, C, D, with the quadratrix curve drawn from C to L, as described above. Find a point P on the quadratrix, such that the line AP makes an angle $\pi/3$ with the line AB. Mark a point Q between A and C, such that QP is parallel to AB (i.e. Q is on the same height as P). Next, find the point R between A and Q such that AR is one third of AQ. Lastly, find the point S on the quadtratrix such that RS is parallel to AB. The angle $\angle SAB$ is $\pi/9$. This procedure effectively translates the problem of trisecting the angle into that of trisecting a line segment, which is easy. Note that once you have constructed the quadratrix (which is done "mechanically") all steps after that can be done using only straightedge and compass.

3. Squaring the circle

It is likely that the first of the three classic problems was squaring the circle. The problem is to, given a circle, by straightedge and compass construct a square of area equal to the area enclosed by the circle. It is thought that Anaxagoras worked on solving it, about 450 BC, while he was in prison for having claimed that the sun is a giant red-hot stone and that the Moon shines by reflected light. Aforementioned Hippias is supposed to have found two different ways to square the circle, both explicitly rejected by Plato (according to Plutarch), presumably because they necessitated more than straightedge and compass. One of the methods attributed to Hippias was based on the quadratrix, and does indeed lead to a solution, but again, the mathematician will need more than a straightedge and compass. Just as the two other classical problems it is unsolvable (by dictated means), and again the proof relies on the comparably modern notion of field extensions. The trick is that π is not a constructible number. The Scottish mathematician James Gregory attempted a proof of the impossibility of squaring the circle, using the algebraic properties of π in his "The True Squaring of the Circle and of the Hyperbola" in 1667. Although his proof was incorrect, it did contain the right ideas. It was not until 1882 that Ferdinand von Lindemann rigorously proved its impossibility, in the same year as π was proven trancendental.

PROPOSITION 17. *Squaring the circle is impossible by means of straightedge and compass.*

Proof. Choose units such that the given circle is of unit radius. Thus formulated, our problem is to construct a line segment of length $\sqrt{\pi}$. But π is trancendental over $\mathbb{Q}$, so $\sqrt{\pi}$ is also trancendental over $\mathbb{Q}$. But every element of the constructible field is algebraic over $\mathbb{Q}$. □

Archimedes solved two of the classic problems, trisection of the angle and squaring the circle, with his famous Archimedean spiral (a curve invented by Conon of Alexandria, but studied in detail by Archimedes). Archimedes also discovered that all that is needed to trisect an angle is a marked straightedge (ruler) instead of an unmarked one. Here we shall briefly indicate how Archimedes squared the circle, using the Archimedean spiral. Archimedes gives the following definition of the "Archimedean" spiral in his work "On spirals":

> If a straight line drawn in a plane revolves uniformly any number of times about a fixed extremity until it returns to its original position, and if, at the same time as the line revolves, a point moves uniformly along the straight line beginning at the fixed extremity, the point will describe a spiral in the plane.

To square the circle Archimedes gives the following construction. Let O be the center of the spiral, and let P be the point on the spiral when it has completed one turn. Let the tangent at P cut the line perpendicular to OP at T. Then Archimedes proves in Proposition 19 of his "On spirals" that OT is the length of the circumference of the circle with radius OP. Now it may not be clear that this is solved the problem of squaring the circle, but Archimedes had already proved as the first proposition of "Measurement of the circle" that the area of a circle is equal to a right-angled triangle having the two shorter sides equal to the radius of the circle and the circumference of the circle. So the area of the circle with radius OP is equal to the area of the triangle OPT. The square is then easily doubled - take the diagonal of the original square as your new edge.

APPENDIX A

Hilbert's axioms

In 1899 David Hilbert, in his "Foundations of Geometry," proposed a list of twenty axioms as the foundation for a modern treatment of Euclid's geometry. His axioms were revised updates of Euclid's, bringing the Euclidean axiomatic geometry to a new level of mathematical rigor.

The primitive terms, or elements of the geometry of space, are points, lines, and planes. We denote points by capital letters $A, B, C, ...$, lines by letters $l, l', l'', ..$, and planes by letters $\sigma, \tau, \omega....$ Space is consituted of planes, planes are constituted of lines, and lines are composed of points. If two points are part of the same line we say the two points are collinear. If two lines or two points lie in the same plane we refer to the two linesor the two points as coplanar. The axioms posit how the primitive terms relate to each other.

Axiom 1. *The axioms of* ***incidence*** *are:*

- I1 *Given any two points, there exists a line containing both of them.*
- I2 *Given two points, there is no more than one line containing both of them.*
- I3 *Given any three noncollinear points, there exists a plane containing all three points.*
- I4 *Given any three noncollinear points, there exists no more than one plane containing all three points.*
- I5 *If two points on a line l lie in some plane σ, then σ contains every point of l.*
- I6 *If the planes σ and τ both contain a point A, then σ and τ have at least one other point in common.*
- I7 *Every line contains at least two points, every plane contains at least three noncollinear points, and there exist at least four noncoplanar points.*

If two lines have no point in common we say they are parallel.

Axiom 2. *The parallel postulate, or* ***Playfair's axiom****, asserts that:*

- P *For each point A and each line l, there is at most one line containing A that is parallel to l.*

For three points A, B, C we postulate an undefined relation of betweeness.

Axiom 3. *The axioms of* ***betweeness*** *are:*

- B1 *If B is between A and C, then B is also between C and A, and A, B, C are three distinct collinear points.*
- B2 *For any two distinct points A, B, there exists a point C such that B is between A and C.*
- B3 *Given three collinear points, one and only one of the points is between the other two.*

B4 *(Pasch's axiom.) Let A, B, C be three noncollinear points, and let l be a line not containing any of A, B, C. If l contains a point D lying between A and B, then l must also contain either a point lying between A and C or a point lying between B and C, but both possibilities are never satisfied simultaneously.*

For two points A, B we define the line segment AB to be the collection of A and B together with all points lying between A and B. We posit an undefined notion of congruence between line segments, subject to the axioms of congruence of line segments. If A is a point on a line l, we refer to the collection of points on some fixed side of A along l as a half-line, or ray originating at A. Given three noncolinear points A, B, C we refer to the union of the ray originating at A and passing through B with the ray originating at A and passing through C as the angle $\angle BAC$. We postulate an undefined notion of congruence for angles, written $\cong$, subject to the three axioms of congruence of angles.

Axiom 4. *The axioms of* ***congruence of line segments*** *are:*

C1 *Given a line segment AB, and a point C on some line l, there exists unique points D, E on l such that C is between D and E, and AB is congruent with both CD and CE, written $AB \cong CD$, $AB \cong CE$.*

C2 *If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.*

C3 *(Addition.) Given three collinear points A, B, C such that B is between A and C, and given three further points D, E, F such that E is between D and F. If $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.*

The axioms of ***congruence of angles*** *are:*

C4 *Given an angle $\angle BAC$ and a ray originating at a point D and passing through a point E, there, at a given side of the line through D, E, exists a unigue point F such that $\angle BAC \cong \angle EDF$.*

C5 *For any three angles $\alpha, \beta\gamma$, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.*

C6 *(SAS) Given triangles ABC and DEF, suppose that $AB \cong DE$ and $AC \cong DF$, and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely, then also $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.*

In order to justify the ruler and compass constructions employed by Euclid and classical geometers we must, by introduction of a further axiom, ensure that lines and circles intersect when they "ought" to. Given two distinct point O and A, we define a circle with center O and radius OA to be the collection of all points B such that $OA \cong OB$. We use capital Greek letters $\Gamma, \Delta, \Lambda, ...$ for circles.

Axiom 5. *The circle-circle intersection property, or* ***Euclidean axiom****, states:*

E *(CCI.) Given two circles Γ, Δ, if Γ contains at least one point inside Δ, and Γ contains at least one point outside Δ, then Δ and Γ will meet.*

There is a stronger axiom, called Dedekind's axiom, that implies the Euclidean axiom. It allows us to consider geometrical Dedekind cuts.

Axiom 6. *The* ***Dedekind axiom*** *postulates that:*

D *Suppose the points of a line are partitioned into two non-empty collections S, T in such a way that no point of S is between two points of T, and no* *point of T is between two points of S. Then there exists a unique point P such that for any $A \in S$ and any $B \in T$, either $A = P$ or $B = P$ or the point P is between A and B.*

Lastly, there is an axiom necessary to make rigorous Euclid's theory of proportion (or Archytas' theorey of proportion rather, as Euclid copied it from him).

AXIOM 7. ***Archimedes' axiom*** *postulates that:*

A *Given line segments AB and CD, there is a natural number n such that n copies of AB added together will be greater than CD.*

A geometry satisfying axioms (I1-I7), (B1-B4), (C1-C6) but no more is called a neutral space. A neutral geometry of space satisfying (P) and (E) is called an Euclidean geometry. If we want to specify if a geometry satisfies Archimedes' axiom (A), we refer to it as Archimedean, resp. non-Archimedean.

APPENDIX B

Axioms and properties of groups and fields

Definition 13. *A **group** is a set G together with a binary operation $\cdot$ called the group law, satisfying*

(i) *Associativity, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in G$.*
(ii) *There is a unique element $e \in G$, called the identity, such that $e \cdot a = a \cdot e = a$ for every $a \in G$.*
(iii) *For every $a \in G$ there is a unique element a^{-1}, called the inverse of a, such that $a^{-1} \cdot a = a \cdot a^{-1} = e$.*

*If $a \cdot b = b \cdot a$ for all $a, b \in G$, then the group is said to be **Abelian**, or, less commonly, commutative.*

Definition 14. *A **field** is a set $\mathbb{K}$, together with a binary operation $+$ called additon and a binary operation $\cdot$ called multiplication, i.e. for $a, b \in \mathbb{K}$ there are given elements $a + b \in \mathbb{K}$ and $a \cdot b \in \mathbb{K}$, subject to the following conditions:*

(i) *The set $\mathbb{K}$ is an Abelian group under addition.*
(ii) *The set $\mathbb{K} \setminus \{0\}$, with 0 the identity of the additive group $(\mathbb{K}, +)$, forms an Abelian group under multiplication.*
(iii) *The operations $+$ and $\cdot$ are related by the distributive law $a(b+c) = ab+ac$, for $a, b, c \in \mathbb{K}$.*

Definition 15. *An **ordered field** is a field $\mathbb{K}$, together with a subset P (whose elements are called positive), satisfying:*

(i) *If $a, b \in P$, then $a + b \in P$ and $ab \in P$.*
(ii) *For any $a \in \mathbb{K}$, one and only one of the following properties holds: $a \in P; a = 0; -a \in P$.*

In an ordered field $\mathbb{K}, P$, we employ the notation $a > b$ for $a - b \in P$, and we write $a < b$ if $b - a \in P$. For the real numbers with the usual notion of positive elements, or any subset of the real numbers, this notation coincides with the usual notation for "greater than" resp. "lesser than."

APPENDIX C

Field extensions

Let $\mathbb{K}$ and $\mathbb{E}$ be fields, with $\mathbb{K}$ a subfield of $\mathbb{E}$. For elements $z_1, ..., z_n \in \mathbb{E}$, the notation $\mathbb{K}(z_1, ..., z_n)$ means all elements that can be written on the form $a_0 + a_1 z_1 + ... + a_n z_n$ with $a_k \in \mathbb{K}$, i.e. $\mathbb{K} \subset \mathbb{K}(z_1, ..., z_n) \subset \mathbb{E}$. As an example, $\mathbb{Q}(\sqrt{2}) = \mathbb{Q} \oplus \mathbb{Q}\sqrt{2} \subset \mathbb{R}$. We call $\mathbb{K}(z_1, ..., z_n)$ a field extension of $\mathbb{K}$, and we say the field extension $\mathbb{K}(z_1, ..., z_n)$ is generated by $z_1, ..., z_n$ over $\mathbb{K}$.

DEFINITION 16. *The* ***degree of an extension*** $\mathbb{K}(z_1, ..., z_n)$ *over a field* $\mathbb{K}$, *written* $[\mathbb{K}(z_1, ..., z_n) : \mathbb{K}]$, *is the dimension of* $\mathbb{K}(z_1, ..., z_n)$ *as a vector space over* $\mathbb{K}$.

As an example of above definition, the field $\mathbb{Q}(\sqrt{2})$ is an extension field of degree 2 over the rational numbers $\mathbb{Q}$. Note that it is in general not true that $[\mathbb{K}(z_1, ..., z_n) : \mathbb{K}] = n + 1$. From now on, all degrees of extensions will be assumed finite.

PROPOSITION 18. *If* $\mathbb{K}$ *is an extension field of a field* $\mathbb{F}$, *and* $\mathbb{E}$ *is an extension of* $\mathbb{K}$, *then* $\mathbb{E}$ *is an extension of* $\mathbb{F}$, *and*

$$[\mathbb{E} : \mathbb{F}] = [\mathbb{E} : \mathbb{K}][\mathbb{K} : \mathbb{F}].$$

PROOF. If $\alpha_1, ..., \alpha_n$ is a vector space basis for $\mathbb{K}$ over $\mathbb{F}$, and if $\beta_1, ..., \beta_m$ is a basis for $\mathbb{E}$ over $\mathbb{K}$, then $\{\alpha_i \beta_j ; i = 1, ..., n; j = 1, ..., m\}$ is a basis for $\mathbb{E}$ over $\mathbb{F}$. □

Recall that some element ξ is said to be algebraic over a field $\mathbb{K}$ if there exists a polynomial $p \in \mathbb{K}[T]$ (coefficients in $\mathbb{K}$) such that $p(\xi) = 0$.

PROPOSITION 19. *An extension field* $\mathbb{E}$ *of a field* $\mathbb{K}$ *is an algebraic extension, i.e. every* $\xi \in \mathbb{E}$ *is algebraic over* $\mathbb{K}$.

PROOF. Take $\xi \in \mathbb{E}$. If $[\mathbb{E} : \mathbb{K}] = n$, then $\{1, \xi, ..., \xi^n\}$ cannot be linearly independent. Hence there exist $a_0, ..., a_n \in \mathbb{K}$, not all zero, such that

$$a_0 + a_1 \xi + ... + a_n \xi^n = 0.$$

Thus ξ is a root of the polynomial $p = a_0 + a_1 T + ... + a_n T^n \in \mathbb{K}[T]$. □

We note without proof that for each $\xi \in \mathbb{E}$ there is an up to multiplication with a constant factor in $\mathbb{K}$ unique monic polynomial $p \in \mathbb{K}[T]$ such that $p(\xi) = 0$. This polynomial is called the irreducible polynomial for ξ over $\mathbb{K}$.

DEFINITION 17. *Let* $\mathbb{E}$ *be an extension of a field* $\mathbb{K}$. *The* ***degree*** *of* $\xi \in \mathbb{E}$ *over* $\mathbb{K}$, *written* $\deg(\xi, \mathbb{K})$, *is the degree of the irreducible polynomial for* ξ *over* $\mathbb{K}$.

Lastly, we note without proof a foundational result.

CLAIM 3. *Let* $\mathbb{E}$ *be a field extension of a field* $\mathbb{K}$, *and take some* $\xi \in \mathbb{E}$. *Then* $[\mathbb{K}(\xi) : \mathbb{K}] = \deg(\xi, \mathbb{K})$.

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