NON-ARCHIMEDEAN GEOMETRY AND THE FRENET FRAME

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Abstract. This is a short exposition on how the three-dimensional geometry over a non-Archimedean field of formal power series may be naturally related to the classical differential geometry of curves, specifically the Frenet frame. We claim no originality in results, only originality in presentation.

1. Introduction

The nature of infinity is this! That every thing, has its
Own Vortex; and when once a traveller thro Eternity,
Has passed that Vortex, he perceives it roll backwards behind
His path, into a globe itself infolding: like a sun:
Or like a moon, or like a universe of starry majesty.

–William Blake

When the foundations of classical geometry were seriously investigated during the 19th century, it became clear that not only are there perfectly viable geometries in which the parallel postulate is not true, but there are also non-Archimedean geometries. A non-Archimedean geometry is characterized by the property that given three collinear points A, B and C segments, there may fail to exist a natural number n such that the line segment $n \cdot AB$ exceeds the segment AC. If this is the case, AB is said to be infinitesimal compared to AC, or reciprocally, AC is infinite compared to AB.

Perhaps the first to attempt the construction of a model for a non-Archimedean geometry was the Italian geometer G. Veronese [1] around 1890. However, famous fellow Italian mathematician G. Peano strongly criticized Veronese's notion of a non-Archimedean geometry, due to the lack of rigour of Veronese's description and also for the fact that he did not justify his use of infinitesimal and infinite segments. Nevertheless, the resulting argument was extremely useful to mathematics since it helped to clarify the notion of the continuum, both in geometry and among the real numbers. A decade or so after Veronese's paper on the subject, Hilbert removed any fears that a non-Archimedean system would not be consistent, by giving a detailed and explicit model for such a geometry [2].

A non-Archimedean geometry may seem forced, a pathological counterexample to its own non-existence, mainly interesting as a proof of the necessity of adding an axiom that rules out its possibility. We beg to disagree, for two reasons. First of all, a non-Archimedean geometry is full of beautiful surprises challenging the imagination. Secondly, it might not be so freakish after all, but a quite useful

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tool in physical description and modeling. Notions of the infinitesimal and infinite has historically been used extensively by mathematicians in intuitive arguments regarding limits, continuity and differentiability. For example, Archimedes and his contemporaries treated smooth curves as constituted of infinitely many infinitesimal straight segments. Leibniz and Newton used infinitesimals as the very basic constructs in the calculus they developed. Leibniz, for example created the notation $\int dx$, referring to an infinite sum of infinitesimal elements dx, or numbers "smaller than any assignable quantity." These notions were criticized, already by Newton's and Leibniz' contemporaries, for being vaguely defined or even contradictory. Even so, mathematicians kept referring to these "numbers smaller than any assignable quantity." For example, in Riemann's work we can read statements such as

> *...and is consequently an infinitely small quantity of the fourth order, so that one obtains a finite quantity if one divides it by the square of the infinitely small triangle whose vertices have the values* $(0, 0, 0, ...)$, $(x_1, x_2, x_3, ...)$, $(dx_1, dx_2, dx_3, ...)$.

These infinitesimals, and their reciprocals the infinite elements, have no existence within the field of real numbers. In non-Archimedean number fields such elements do have a well-defined existence. Non-Archimedean geometry provides a framework in which we may rigorously do geometry with infinitesimals.

In this short text we review some geometric results obtained when the real numbers are embedded in a non-Archimedean field. For the results to work, we need to be able to extend smooth real functions to functions defined on, and with values in, the non-Archimedean extension field. We also want to be able to define a norm. There are several such non-Archimedean extensions of the real numbers, and most of our results would hold for all of them, but we have chosen to base the presentation on a field introduced by D. Laugwitz [3]. A less restrictive choice would be the hyperreal numbers invented by A. Robinson [8].

2. Preliminary results

2.1. **Laugwitz' field.** We will consider the ordered field $\mathbf{L}$ of (generalized) power series in an indeterminate t with real coefficients and real increasing exponents. That is, an element of $\mathbf{L}$ is a formal expression

$$a = \sum_{k=1}^{\infty} a_k t^{\nu_k}, \quad a_k, \nu_k \in \mathbb{R}, \quad \nu_0 < \nu_1 \nu_2 < ...$$

Two series of above form are equal in $\mathbf{L}$ if they are equal as series (i.e. as formal expressions). Addition and multiplication are carried out as for ordinary absolutely convergent power series. We order $\mathbf{L}$ by definining $a \in \mathbf{L}$ to be positive ($a > 0$) if and only if the nonvanishing coefficient a_m with lowest subscript m is positive. A detailed verification that $\mathbf{L}$ indeed is an ordered field may be found in [4, 3].

The field $\mathbf{L}$ is non-Archimedean: since e.g. $nt < t^{-1}$ for all $n \in \mathbb{N}$. We also note that the real numbers may be embedded into $\mathbf{L}$ by $\mathbb{R} \to \mathbf{L}, r \mapsto rt^0$. In correspondence with this identification we say that $a \in \mathbf{L}$ is infinitesimal if $|a| < r$ for all $r \in \mathbb{R}$, infinite if $|a| > r$ for all $r \in \mathbb{R}$, or finitely bounded if $|a| < r$ for some $r \in \mathbb{R}$. We shall employ the notation $\mathbf{L}_0$ for all finitely bounded elements of $\mathbf{L}$, and write $\mathbf{L}_\mu$ for the subset of all infinitesimal numbers, and $\mathbf{L}_\infty$ for the infinite numbers.

The finite elements $\mathbf{L}_0$ constitute a subring $\mathbf{L}$. The infinitesimal elements $\mathbf{L}_\mu$ constitute a proper maximal ideal within $\mathbf{L}_0$, as can be seen by the following argument. Suppose that $\mathbf{L}_\mu \subset J$, with a proper inclusion and J an ideal in $\mathbf{L}_0$. Then $a \in J \backslash \mathbf{L}_0$ is not infinitesimal, so that a^{-1} is finitely bounded, whence $aa^{-1} = 1 \in J$. But then J is be definition equal to $\mathbf{L}_\mu$, and it follows that the set of infinitesimals is a maximal ideal in $\mathbf{L}_0$. Consequently $\mathbf{L}_0/\mathbf{L}_\mu$ is a field, and looking at the form of the power series it may be readily concluded that $\mathbf{L}_0/\mathbf{L}_\mu \cong \mathbb{R}$. Inspired by the similarity with Robinson's analysis of hyperreal numbers [8], we write $\mathtt{st} : \mathbf{L}_0 \to \mathbb{R}$ for the quotient projection, and refer to $\mathtt{st}(a)$ as the standard part of $a \in \mathbf{L}_0$. We remark that with the usual ordering of the real numbers, $\mathtt{st}(a) > 0$ if and only if $a > 0$, so that the quotient projection is order preserving. The cosets $a + \mathbf{L}_\mu$ (with $a \in \mathbf{L}$) will in what follows be called monads. For $a \in \mathbf{L}$ we write $\mu(a) = a + \mathbf{L}_\mu$ for the monad of a. In particular, note that $\mu(\mathtt{st}(a)) = \mathtt{st}^{-1}(a)$, and that $\mu(0) = \mathbf{L}_\mu$.

The intuitive image we propose is that $\mathbf{L}_0$ is the real numbers, plus an infinitesimal neighborhood (a monad) about each real number. In this way each real point gets equipped with a kind of internal geometry, similar to the notion of a tangent space. Since the real numbers may be embedded $r \mapsto rt^\nu$ for any fixed real number ν (we may think of the choice of ν as the "level of the real world"), we see that the situation repeats itself like a Russian doll, indefinitely.

The definitions above are easily extended to Cartesian products. If $a = (a_1, ..., a_n)$ is a vector in $\mathbf{L}_0^n$, the monad $\mu(a)$ of a is the set of points $b \in \mathbf{L}$ such that $|a - b|$ is infinitesimal. The projection $\mathtt{st} : \mathbf{L}_0^n \to \mathbb{R}^n$ is also immediately extended, mapping $(a_1, ..., a_n)$ to $(\mathtt{st}(a_1), ..., \mathtt{st}(a_n))$. A vector $a \in \mathbf{L}_0^n$ will be infinitesimal if each coordinate is infinitesimal, finitely bounded if each coordinate is finitely bounded, and infinite if at least one coordinate is infinite.

Proposition 1. *Let u and v be two finitely bounded and linearly independent vectors in $\boldsymbol{L}$. Then there are infinitesimal vectors u' and v' such that $u = \mathtt{st}(u) + u'$ and $v = \mathtt{st}(v) + v'$. Moreover, if $\mathtt{st}(u)$ and $\mathtt{st}(v)$ are linearly independent over $\mathbb{R}$, then u and v are linearly independent over $\boldsymbol{L}$.*

Proof. Our proof is adapted from [5]. The decomposition of the two vectors u and v is evident. Suppose $\mathtt{st}(u)$ and $\mathtt{st}(v)$ are linearly independent, and assume u and v are not linearly independent, so that there are numbers $a, b \in \mathbf{L}_0$ such that $au + bv = 0$. One of the coefficients a/b or b/a is not infinitesimal. Let us suppose that $a/b \notin \mathbf{L}_\mu$. If $a/b \notin \mathbf{L}_0$, then $b/a \in \mathbf{L}_0$. In this case $au + bv = 0$ is equivalent to $u + bv/a = 0$. This implies that $\mathtt{st}(v) = 0$, and this is impossible. Hence $a/b \notin \mathbf{L}_0 \setminus \mathbf{L}_\mu$. Thus, if there exists a linear relation between u and v, there exists a linear relation with coefficients in $textbf L_0 \setminus \mathbf{L}_\mu$, i.e. we can suppose that $au + bv = 0$ with $a, b \in \mathbf{L}_0 \setminus \mathbf{L}_\mu$. This implies

$$\mathtt{st}(au + bv) = \mathtt{st}(a)\mathtt{st}(u) + \mathtt{st}(b)\mathtt{st}(v) = 0,$$

which in turn implies that $\mathtt{st}(a) = \mathtt{st}(b) = 0$. This is impossible, which proves the proposition. □

Our next result concerns the axiomatic properties of the analytic goemetry over Laugwitz' field and subsets thereof. Allthough interesting, it will be left as a claim (without proof).

Claim 1. *With the obvious interpretation of the geometric notions of points, lines and surfaces, $\boldsymbol{L}^3$ satisfies all of Hilbert's axioms of betweeness and congruence,* *the circle-circle intersection property and the parallel axiom. Let points, lines and surfaces in* $\boldsymbol{L}_0^3$ *and* $\boldsymbol{L}_\mu^3$ *be the intersections of points, lines and surfaces in* $\boldsymbol{L}^3$ *with the respective subset. Then both* $\boldsymbol{L}_0^3$ *and* $\boldsymbol{L}_\mu^3$ *are non-Archimedean geometries satisfying all of Hilbert's axioms of betweeness and congruence together with the circle-circle intersection property, but niether satisfy the parallel axiom. Furthermore,* $\boldsymbol{L}_0^3$ *but not* $\boldsymbol{L}_\mu^3$ *has the property that there is a line segment* AB *such that for every other segment* CD *there is an integer* n *such that* $CD \leq n \cdot AB$*.*

The reason we leave above claim without proof is that a proof would require a long digression on Hilbert's axioms, plus the claim is not essential to our later developements. The reader is referred to [2, 6] for details on the axiomatic geometry. We content ourselves with remarking that the parallel axiom is invalid since two in $\mathbf{L}^3$ non-parallel lines may both pass through $\mathbf{L}_0^3$ ($\mathbf{L}_\mu^3$) but intersect outside of $\mathbf{L}_0^3$ ($\mathbf{L}_\mu^3$). The circle-circle intersection property is equivalent to the existence of $\sqrt{a} \in \mathbf{L}$ for each positive $a \in \mathbf{L}$.

We proceed to show how a differentiable function $f : D \to \mathbb{R}$, with $D \subset \mathbb{R}$, can be extended to a function ${}^L f : {}^L D \to \mathbf{L}_0$, with ${}^L D \subset \mathbf{L}_0$ and $\mathtt{st}({}^L f(x)) = f(\mathtt{st}(x))$. Let f be a real-valued infinitely differentiable function of a real variable, defined on an open interval $D = (a, b) \subset \mathbb{R}$. On passing from $\mathbb{R}$ to $\mathbf{L}$, define ${}^L D$ to be the set of points x satisfying $a < x < b$. (The ordering in the latter case is the ordering in $\mathbf{L}$). We find that ${}^L D$ consists of points $x = \mathtt{st}(x) + \sum_{k=1}^{\infty} a_k t^{\nu_k}$, $0 < \nu_1 < \nu_2 < \ldots$ of three kinds:

$$a < \mathtt{st}(x) < b;$$

$$\mathtt{st}(x) = a, \quad \sum_{k=1}^{\infty} a_k t^{\nu_k} > 0;$$

$$\mathtt{st}(x) = b, \quad \sum_{k=1}^{\infty} a_k t^{\nu_k} < 0.$$

Using the fact that f is assumed smooth, we follow Laugwitz [3] and in analogy with the usual rules for Taylor series expansion define

$$\begin{aligned} {}^L f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\mathtt{st}(x)) \Big(\sum_{k=1}^{\infty} a_k t^{\nu_k} \Big)^n \\ &= f(\mathtt{st}(x)) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(\mathtt{st}(x)) \Big(x - \mathtt{st}(x) \Big)^n. \end{aligned}$$

This is well-defined, as the expression for ${}^L f(x)$ may be arranged into a series expression in powers of t. This hinges on $\nu_0 > 0$, but this is ensured since $x - \mathtt{st}(x)$ by definition is infinitesimal. The promised property $\mathtt{st}({}^L f(x)) = f(\mathtt{st}(x))$ is also evident. We shall refer to ${}^L f$ as the Laugwitz extension of f.

2.2. **Monad decomposition.** The first result is a theorem on how monads may be decomposed, in a way similar to how vector spaces are coordinatized. The theorem was discovered by M. Goze, and is discussed by him in the context of deformations of Lie algebras in [5]. We restate both the theorem and its proof, in terms pertinent to the subject-matter discussed in this text.

Theorem 1. *For every $a \in \boldsymbol{L}_\mu^3$ there exists three in $\mathbb{R}^3$ linearly independent vectors v_1, v_2, v_3, and infinitesimals $\epsilon_1, \epsilon_2, \epsilon_3 \in \boldsymbol{L}_\mu$ such that*

$$a = \epsilon_1 v_1 + \epsilon_1 \epsilon_2 v_2 + \epsilon_1 \epsilon_2 \epsilon_3 v_3.$$

Moreover, the decomposition is unique up to equivalence in the following sense. If

$$\epsilon_1' v_1' + \epsilon_1' \epsilon_2' v_2' + \epsilon_1' \epsilon_2' \epsilon_3' v_3'$$

is another such decomposition, then there are nonzero real numbers ${c_i}^j$ such that

$$v_i = \sum_{j=1}^{i} {c_i}^j v_j', \quad \epsilon_1' \ldots \epsilon_j = \sum_{j=1}^{i} {c_i}^j \epsilon_1 \ldots \epsilon_j$$

for $j = 1, 2, 3$, i.e. the flag generated by the ordered family v_1, v_2, v_3 is equal to the flag generated by the ordered family v_1', v_2', v_3'.

Proof. Let $a = (a_1, a_2, a_3)$. With no essential loss of generality, assume that a_1 is the component with greatest absolute value. Let $\epsilon_1 = a_1$. We then have

$$a = \epsilon_1 \Big(1, \frac{a_2}{\epsilon_1}, \frac{a_3}{\epsilon_1}\Big),$$

with all components of the vector on the right hand side finitely bounded. We may then write

$$a = \epsilon_1 (v_1 + w_1)$$

with v_1 real and w_1 infinitesimal, and the two vectors linearly independent (since the first component of w_1 is in any case zero). We iterate the procedure and obtain $w_1 = \epsilon_2 (v_2 + w_2)$, with v_2 real and w_2 and ϵ_2 infinitesimals. The linear indepence of v_1 and w_1 implies the linear independence of v_1 and v_2. By construction v_1, w_1, w_2 are linearly independent, and so v_1, v_2, w_2 are linearly independent. Let $\epsilon_3 = |w_2|$. We then iterate once more and obtain

$$a = \epsilon_1 v_1 + \epsilon_1 \epsilon_2 v_2 + \epsilon_1 \epsilon_2 \epsilon_3 v_3,$$

which is the promised decomposition.

For uniqueness, assume $a = \epsilon_1' v_1' + \epsilon_1' \epsilon_2' v_2' + \epsilon_1' \epsilon_2' \epsilon_3' v_3'$ is another such decomposition. By hypothesis, either $\epsilon_1/\epsilon_1' \in \mathbf{L}_0$ or $\epsilon_1'/\epsilon_1 \in \mathbf{L}_\mu$. Accordingly, we may assume $\epsilon_1/\epsilon_1' \in \mathbf{L}_0$. There then exist $a \in \mathbb{R}$ and $\delta \in \mathbf{L}_\mu$ such that $\epsilon_1/\epsilon_1' = a + \delta$, or equivalently:

$$\epsilon_1 = a\epsilon_1' + \delta\epsilon_1'.$$

Inserting this into the decompositions we get

$$\begin{aligned}(a\epsilon_1' + \delta\epsilon_1') v_1 + (a\epsilon_1' + \delta\epsilon_1') \epsilon_2 v_2 + (a\epsilon_1' + \delta\epsilon_1') \epsilon_2 \epsilon_3 v_3 \\ = \epsilon_1' v_1' + \epsilon_1' \epsilon_2' v_2' + \epsilon_1' \epsilon_2' \epsilon_3' v_3'.\end{aligned}$$

Dividing ϵ_1' out, above equation is of the form

$$av_1 + u = v_1' + v,$$

with u and v infinitesimal vectors. By Proposition 1, we have that if v_1 and v_1' are linearly independent, then $av_1 + u$ and $v_1' + v$ are linearly independent, unless $a = 0$. Thus we are forced to put $v_1' = av_1$. Inserting this into the decomposition of a and repeating the process we get the uniqueness property. □

The theorem entails a corollary [7] that we will be crucial to our later developements.

Corollary 1. *Any vector $a \in \boldsymbol{L}_0^3$ can be written on the form $a = \mathtt{st}(a) + a'$, with $a' \in \mu(a) \cong \boldsymbol{L}_\mu^3$ decomposed as*

$$a' = \epsilon_1 v_1 + \epsilon_1\epsilon_2 v_2 + \epsilon_1\epsilon_2\epsilon_3 v_3,$$

with $v_1, v_2, v_3 \in \mathbb{R}^3$. Furthermore, the vectors v_1, v_2, v_3 may be chosen as an orthonormal basis of positive orientation, and in this case the decomposition of a' is unique up to an overall sign.

Proof. An expansion of the stated form exists by Theorem 1. Using the equivalence property, a decomposition using vectors v_1, v_2, v_3 is related to a decomposition using vectors v_1', v_2', v_3' by some set of real scalars ${c_i}^j$ such that

$$v_i = \sum_{j=1}^{i} {c_i}^j v_j', \quad j = 1, 2, 3.$$

Choose ${c_1}^1 = \pm 1/|v_1'|$. Then, choose ${c_2}^1$ and ${c_2}^2$ such that $v_2 = {c_2}^1 v_1' + {c_2}^2 v_2'$ and v_1 form an orthonormal basis with positive orientation. Finally, choose ${c_3}^1, {c_3}^2, {c_3}^3$ such that $v_3 = v_1 \times v_2$. ☐

3. Main result

The theorem and its proof are adapted from a similar result obtained in [7] in the context of nonstandard analysis.

Theorem 2. *Let $\gamma : D \to \mathbb{R}^3$ be a regular curve. Take a point x on the Laugwitz extension ${}^L\gamma : {}^LD \to \boldsymbol{L}_0^3$ of γ, with $x \neq \mathtt{st}(x)$ and a decomposition*

$$x = \mathtt{st}(x) + \epsilon_1 v_1 + \epsilon_1\epsilon_2 v_2 + \epsilon_1\epsilon_2\epsilon_3 v_3$$

in terms of an orthonormal basis v_1, v_2, v_3 with positive orientation. Then, with an appropriate choice of sign, v_1, v_2, v_3 is the Frenet frame of γ at the point $\mathtt{st}(x)$. Moreover, the curvature and torsion of γ are at $\mathtt{st}(x)$ subject to the relations:

$$\kappa = 2\mathtt{st}(\epsilon_2/\epsilon_1), \quad \tau = 3\mathtt{st}(\epsilon_3/\epsilon_1).$$

Proof. For notational convenience, and without any true loss of generality, we may assume that $\gamma(t)$ passes through the origin at time $t = 0$, and that the point x is infinitesimal, i.e. $x \in \mathbf{L}_\mu^3$. Suppose $x = {}^L\gamma(\delta)$, with $\delta \in \mathbf{L}_\mu$. We then have

$$x = \gamma'(0)\delta + \frac{\gamma''(0)}{2}\delta^2 + \frac{\gamma^{(3)}(0)}{6}\delta^3 + ...$$

Since δ^n is infinitesimal compared to δ, for $n > 1$, we divide by δ and get

$$x = \delta\Big(\gamma'(0) + \frac{\gamma''(0)}{2}\delta + \frac{\gamma^{(3)}(0)}{6}\delta^2 + ...\Big),$$

in analogue with the construction in the proof of Theorem 1. It is immediate that the vector v_1 in the decomposition of x will be parallel to $\gamma'(0)$. Scaling v_1 to unit length, as in Corollary 1, and choosing the sign appropriately, we get $v_1 = \gamma'(0)/|\gamma'(0)|$. Taking v_1, v_2, v_3 orthonormal with positive orientation (Corollary 1) then implies that v_1, v_2, v_3 coincides with the Frenet frame.

Without loss of generality, assume γ is parametrized by arc length. Let e_1, e_2, e_3 denote the Frenet frame of γ at the origin. The Frenet formulas [9] state that

$$\gamma'(0) = e_1, \quad \gamma''(0) = \kappa e_2, \quad \gamma^{(3)}(0) = -\kappa^2 e_1 + \dot\kappa e_2 + \kappa\tau e_3.$$

Inserting this into ${}^{L}\gamma(\delta)$ yields

$$\Big(\delta - \frac{\kappa^2\delta^3}{6} + ...\Big)e_1 + \Big(\frac{\kappa\delta^2}{2} + \frac{\dot{\kappa}\delta^3}{6} + ...\Big)e_2 + \Big(\frac{\kappa\tau\delta^3}{6} + ...\Big)e_3.$$

We identify this with the monad decomposition of $x = {}^{L}\gamma(\delta)$ and get

$$\begin{aligned} \frac{\epsilon_2}{\epsilon_1} &= \frac{\kappa/2 + \dot{\kappa}\delta/6 + ...}{(1-\kappa^2\delta^2/6 + ...)^2}; \\ \frac{\epsilon_3}{\epsilon_1} &= \frac{\kappa\tau/6 + ...}{(1-\kappa^2\delta^2/6 + ...)(\kappa/2 + \kappa\delta/6 + ...)}. \end{aligned}$$

Knowing that each qoutient has to be finitely bounded, we may without to careful an analysis conclude that $\mathtt{st}(\epsilon_2/\epsilon_1) = \kappa/2$, and $\mathtt{st}(\epsilon_3/\epsilon_1) = \tau/3$. This proves the theorem. □

3.1. **Concluding remarks.** Curvature and torsion of a curve are undefined in singular points of a curve. However, the values $\mathtt{st}(\epsilon_2/\epsilon_1)$ and $\mathtt{st}(\epsilon_3/\epsilon_1)$ are defined also in singular points on the curve, and hence, these numbers provide a generalized notion of curvature and torsion.

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