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DIFFERENTIAL GEOMETRY MN1 FALL 2001

PROBLEMS

1. Consider the circular helix

$$c(t) = (a \cos t, a \sin t, bt), \quad a > 0,\, b \neq 0.$$

Show that its curvature κ and torsion τ are given by the formulae

$$\kappa = \frac{a}{a^2 + b^2}, \qquad \tau = \frac{b}{a^2 + b^2}$$

Hint: Use the formulae in Exercise **1.6.3** in our paper on curves.

Note that if $b > 0$ (so that $\tau > 0$) the helix is a "right-handed" curve and if $b < 0$ (so that $\tau < 0$) it is "left-handed". Draw a picture to illustrate the meaning of right and left-handed. Be careful with the choice of orientations of the curves. Prove that it is not possible to find an isometry of $\mathbf{R}^3$ which maps a right-handed helix onto a left-handed.

2. Prove that a regular, nice curve $c : I \to \mathbf{R}^3$ is a plane curve if and only if its torsion vanishes identically.

Hint: It is no restriction to assume the curve is parameterized by arc-length. Let $(e_1(s), e_2(s), e_3(s))$ be the distinguished Frenet frame. Use the Frenet equations to conclude that $e_3(s) \equiv v_0$, a constant vector, and from the fact that $\dot{c}$ and v_0 are orthogonal prove that $c(s) \cdot v_0 \equiv \mathit{constant}$ by integration.

3. Prove that a regular, nice curve $c : I \to \mathbf{R}^3$ is a straight line if $\dot{c}(t)$ and $\ddot{c}(t)$ are linearly dependent for all t.

Hint: What is the torsion and the curvature of such a curve? We also need a certain characterization of straight lines to be found in our paper on curves.

4. Find the most general function $f(t)$ so that the curve $c(t) = (a \cos t, a \sin t, f(t))$ will be a plane curve.

Answer: $f(t) = A \sin t + B \cos t + C, \quad A, B, C = \mathit{constants}.$

5. Let $c(t)$ be a curve in R^3 parametrized by arc length and let $t_0 = 0 \in I$. Let the Frenet-frame at $c(0)$ be $e_i(0) = e_i$ and let $\kappa(0) = \kappa_0$, $\tau(0) = \tau_0$. We then have the following well known series expansion for t close to 0

$$c(t) - c(0) = te_1 + \frac{1}{2}\kappa_0 t^2 e_2 + \frac{1}{6}\kappa_0 \tau_0 t^3 e_3 + o(t^3).$$

The projections of the curve in a small neighbourhood of $c(0)$ in the planes of the Frenet-frame at that point are therefore approximated by the following curves:

a) the projection onto the $(e_1, e_3)-$plane, the rectifying plane, is described by the cubical parabola

$$x = t, \quad y = 0, \quad z = \frac{1}{6}\kappa_0 \tau_0 t^3;$$

b) the projection onto the $(e_2, e_3)-$plane, the normal plane, is described by the semi-cubical parabola with a cusp at origo

$$x = 0, \quad y = \frac{1}{2}\kappa_0 t^2, \quad z = \frac{1}{6}\kappa_0 \tau_0 t^3;$$

c) the projection onto the $(e_1, e_2)-$plane, the osculating plane, is decribed by the parabola

$$x = t, \quad y = \frac{1}{2}\kappa_0 t^2, \quad z = 0.$$

i) Draw these projektions and their orientations in the case $\tau_0 > 0$ and $\tau_0 < 0$.

ii) (For VG only) We shall now study what the curve looks like along the negative e_1-axis if we raise or lower our eyes somewhat above or under the osculating plane. This means that we shall find the projection of the curve in a $(f_2, f_3)-$plane where the $ON-$system (f_1, f_2, f_3) is created from the system (e_1, e_2, e_3) by letting the e_1, e_3-plane rotate a small angle α $(-\varepsilon < \alpha < \varepsilon)$ with the e_2-axis as axis of rotation. Derive the analytical expression of the projection and draw pictures of how it looks for different values of α in the cases $\tau_0 > 0$ and $\tau_0 < 0$.