Uppsala Universitet
Matematiska Institutionen
Thomas Erlandsson

DIFFERENTIAL GEOMETRY MN1 FALL 2001

PROBLEMS

6. Let S^2 be the sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. Let Δ be the angle a vector has turned in relation to its original position after parallel translation of the vector around the latitude circle with latitude angle λ. Show that
$$\Delta = 2\pi(1 - \sin \lambda).$$
In which direction has the vector turned when it returns to its original position? What is the area of the spherical surface bounded by the latitude circle on the upper half sphere?

7. The surface of revolution
$$f(u, v) = (h(u) \cos v, h(u) \sin v, k(u))$$
is generated by revolving the curve $(h(u), 0, k(u))$ about the z-axis. We assume that $f_v^2 = h'^2 + k'^2 = 1$ and hence that $k'k'' = -h''h'$. Prove that
$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = h^2, \quad h_{11} = -k'h'' + h'k'', \quad h_{12} = 0, \quad h_{22} = hk'.$$
Use these results to prove that the Gauss curvature
$$K = \frac{h'k'k'' - k'k'h''}{h} = -\frac{h''}{h}$$
The requirement that f has constant Gauss curvature K_0 means that h must satisfy
$$h''(u) + K_0 h(u) = 0.$$
Conversely, a function $h(u)$ satisfying $h'^2 \leq 1$ will enable us to construct interesting surfaces of constant Gauss curvature K_0. Consider the surface of revolution with $h(u) = a \cos u$, where $a > 0$ and $a^2 \sin^2 u \leq 1$, which implies that $k(u) = \int_0^u \sqrt{1 - a^2 \sin^2 t}\, dt$. Prove that this surface has constant Gauss curvature $K_0 = 1$. For what values of a is the surface a sphere? **Answer:** $a = 1$. If we let
$$h(u) = e^{-u}, \ u \geq 0, \quad k(u) = \int_0^u \sqrt{1 - e^{-2t}}\, dt$$
we obtain a surface of revolution with $K_0 = -1$. The curve generating the surface is a *tractrix* and the surface is called s *pseudosphere.*

8. Let M be a Riemann surface with constant Gauss curvature $K = K_0$.

a) Calculate the circumference and area of a circle with radius r.

b) Calculate the geodesic curvature $k_g(r)$ of a circle with radius r. Determine $\lim_{r\to\infty} k_g(r)$.

c) Let $c(t)$ be a geodesic and $\gamma(t)$ a curve whose distance to $c(t)$ is constant $= d$. Calculate the geodesic curvature k_g of the equidistant curve γ.

RESULTS

WARNING! THERE ARE VERY LIKELY MISPRINTS HERE!

a)

$$L(C(r)) = \begin{cases} \frac{2\pi}{\sqrt{K_0}} \sin(r\sqrt{K_0}), & K_0 > 0 \\ 2\pi r, & K_0 = 0 \\ \frac{2\pi}{\sqrt{-K_0}} \sinh(r\sqrt{-K_0}), & K_0 < 0. \end{cases}$$

$$A(c(r)) = \begin{cases} \frac{\pi}{K_0}(1 - \cos(r\sqrt{K_0})), & K_0 > 0 \\ \pi r^2, & K_0 = 0 \\ -\frac{\pi}{K_0}(\cosh(r\sqrt{-K_0}) - 1), & K_0 < 0. \end{cases}$$

b)

$$k_g = \begin{cases} \sqrt{K_0} \cot(r\sqrt{K_0}), & K_0 > 0 \\ \frac{1}{r}, & K_0 = 0 \\ -\sqrt{-K_0} \coth(r\sqrt{-K_0}), & K_0 < 0. \end{cases}$$

c)

$$k_g = \begin{cases} -\sqrt{K_0} \tan(d\sqrt{K_0}), & K_0 > 0 \\ 0, & K_0 = 0 \\ \sqrt{-K_0} \tanh(d\sqrt{-K_0}), & K_0 < 0. \end{cases}$$