Circumference and area of ellipses in non-Euclidean geometries

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Abstract

In this paper we derive formulas for circumference and area of an ellipse in hyperbolic geometry. Corresponding formulas for elliptic geometry are given without proof. We also find a relation connecting the formulas for circumference and area.

1 Preliminaries

This is a term paper for the course Geometry D given at Uppsala University. Due to this nature of the document, derivation of the main formulas will be carried out in quite some detail.

We are deriving the formulas under the simplified assumption that the curvature constant $k = 1$.

All real numbers in this paper will be non-negative; there will be several simplifications of the type $\sqrt{x^2} = x$ without further comments about the correctness of such equations.

A well-known property of ellipses in Euclidean geometry is used to define a non-Euclidean ellipse:

Definition 1. *Given two points* P *and* Q*, and a positive real number* $l > d = \overline{\mathrm{PQ}}$*. The set of all points* R *such that* $\overline{\mathrm{PR}} + \overline{\mathrm{QR}} = l$ *is called an ellipse with focal points* P *and* Q*, and eccentricity* d/l*.*

2 Ellipses in hyperbolic geometry

Given points P and Q, and a real number l, satisfying the conditions for an ellipse. Let O be the midpoint of PQ. Let X be any point distinct from O, and let $\alpha = \sphericalangle\mathrm{QOX}$. Dedekind's axiom gives that $\overrightarrow{\mathrm{OX}}$ cuts the ellipse in exactly one point R=R(α). Let $r = r(\alpha) = \overline{\mathrm{OR}}$ and $s = \overline{\mathrm{QR}}$, then $\overline{\mathrm{PR}} = l - s$, see Figure 1. Choose one of the sides of the line $\overleftrightarrow{\mathrm{PQ}}$, and consider rays emanating from O on that side, with $0 \leqslant \alpha \leqslant \pi/2$.

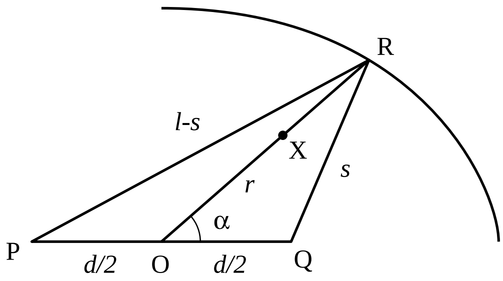

Figure 1

We have

$$
\begin{aligned}
\cosh s &= \cosh r \cosh \frac{d}{2} - \sinh r \sinh \frac{d}{2} \cos \alpha \\
\cosh (l-s) &= \cosh r \cosh \frac{d}{2} - \sinh r \sinh \frac{d}{2} \cos (\pi - \alpha) \\
&= \cosh r \cosh \frac{d}{2} + \sinh r \sinh \frac{d}{2} \cos \alpha,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\cosh (l-s) + \cosh s &= 2 \cosh r \cosh \frac{d}{2} \\
\cosh (l-s) - \cosh s &= 2 \sinh r \sinh \frac{d}{2} \cos \alpha.
\end{aligned}
$$

Combining this with the trigonometrical identities

$$
\begin{aligned}
\cosh (l-s) + \cosh s &= 2 \cosh \frac{l}{2} \cosh \frac{l-2s}{2} \\
\cosh (l-s) - \cosh s &= 2 \sinh \frac{l}{2} \sinh \frac{l-2s}{2},
\end{aligned}
$$

we get

$$
\begin{aligned}
\cosh \frac{l-2s}{2} &= \frac{\cosh r \cosh \frac{d}{2}}{\cosh \frac{l}{2}} \\
\sinh \frac{l-2s}{2} &= \frac{\sinh r \sinh \frac{d}{2} \cos \alpha}{\sinh \frac{l}{2}},
\end{aligned}
$$

from which follows

$$
\begin{aligned}
1 &= \cosh^2 \frac{l-2s}{2} - \sinh^2 \frac{l-2s}{2} \\
&= \frac{\sinh^2 \frac{l}{2} \cosh^2 r \cosh^2 \frac{d}{2} - \cosh^2 \frac{l}{2} \sinh^2 r \sinh^2 \frac{d}{2} \cos^2 \alpha}{\sinh^2 \frac{l}{2} \cosh^2 \frac{l}{2}}
\end{aligned}
$$

$$
\Rightarrow \sinh^2 \frac{l}{2} \cosh^2 \frac{l}{2} = \sinh^2 \frac{l}{2} (\sinh^2 r + 1) \cosh^2 \frac{d}{2} - \cosh^2 \frac{l}{2} \sinh^2 r \sinh^2 \frac{d}{2} \cos^2 \alpha
$$

$$\Rightarrow \sinh^2 r = \frac{\sinh^2 \frac{l}{2}(\cosh^2 \frac{l}{2} - \cosh^2 \frac{d}{2})}{\sinh^2 \frac{l}{2}\cosh^2 \frac{d}{2} - \cosh^2 \frac{l}{2}\sinh^2 \frac{d}{2}\cos^2\alpha}. \tag{1}$$

We define the major and minor axes of the ellipse in analogy with the Euclidean case: set $a = r(0)$, $b = r(\pi/2)$, then

$$\sinh^2 a = \frac{\sinh^2 \frac{l}{2}(\cosh^2 \frac{l}{2} - \cosh^2 \frac{d}{2})}{(\cosh^2 \frac{l}{2} - 1)\cosh^2 \frac{d}{2} - \cosh^2 \frac{l}{2}(\cosh^2 \frac{d}{2} - 1)} = \sinh^2 \frac{l}{2}$$

$$\cosh^2 b = \sinh^2 b + 1 = \frac{\sinh^2 \frac{l}{2}(\cosh^2 \frac{l}{2} - \cosh^2 \frac{d}{2})}{\sinh^2 \frac{l}{2}\cosh^2 \frac{d}{2}} + 1 = \frac{\cosh^2 \frac{l}{2}}{\cosh^2 \frac{d}{2}}.$$

So we have

$$l = 2a, \qquad \cosh \frac{d}{2} = \frac{\cosh a}{\cosh b}.$$

It is easy to see that a and b can be use as parameters instead of l and d. Then equation (1) becomes

$$\sinh^2 r = \frac{\sinh^2 a \left(\cosh^2 a - \frac{\cosh^2 a}{\cosh^2 b}\right)}{\sinh^2 a \frac{\cosh^2 a}{\cosh^2 b} - \cosh^2 a \left(\frac{\cosh^2 a}{\cosh^2 b} - 1\right)\cos^2\alpha},$$

which, after multiplying numerator and denominator with the factor $\frac{\cosh^2 b}{\sinh^2 a \cosh^2 a}$, becomes

$$\sinh^2 r = \frac{\cosh^2 b - 1}{1 - \frac{\cosh^2 a - \cosh^2 b}{\sinh^2 a}\cos^2\alpha} = \frac{\sinh^2 b}{1 - \frac{\sinh^2 a - \sinh^2 b}{\sinh^2 a}\cos^2\alpha} = \frac{\sinh^2 b}{1 - k_1^2\cos^2\alpha} \tag{2}$$

where

$$k_1 = \frac{\sqrt{\sinh^2 a - \sinh^2 b}}{\sinh a}.$$

Note that $b \leqslant r(\alpha) \leqslant a$, and that $r(\alpha)$ is strictly decreasing for $0 \leqslant \alpha \leqslant \frac{\pi}{2}$ when $b < a$.

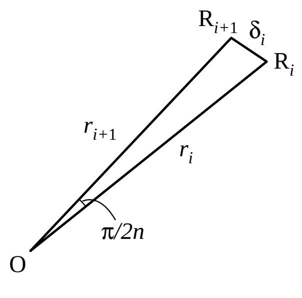

Figure 2

We will first derive a formula for the circumference of an ellipse. Let n be a positive integer, $r_i = r(\frac{\pi i}{2n})$, $\mathrm{R}_i = \mathrm{R}(\frac{\pi i}{2n})$, and $\delta_i = \overline{\mathrm{R}_i\mathrm{R}_{i+1}}$, see Figure 2. Then

$$r_{i+1} = r\left(\frac{\pi i}{2n} + \frac{\pi}{2n}\right) = r_i + \frac{\pi}{2n}r_i' + O\left(\frac{1}{n^2}\right),$$

which gives

$$\sinh r_{i+1} = \sinh r_i + O\left(\frac{1}{n}\right),$$

and

$$\begin{aligned}
\cosh \delta_i &= \cosh r_i \cosh r_{i+1} - \sinh r_i \sinh r_{i+1} \cos \frac{\pi}{2n} \\
&= \cosh r_i \cosh r_{i+1} - \sinh r_i \sinh r_{i+1} \left(1 - 2\sin^2 \frac{\pi}{4n}\right) \\
&= 2 \sinh r_i \sinh r_{i+1} \sin^2 \frac{\pi}{4n} + \cosh\left(r_{i+1} - r_i\right) \\
&= 2 \sinh^2 r_i \sin^2 \frac{\pi}{4n} + \cosh\left(r_{i+1} - r_i\right) + O\left(\frac{1}{n^3}\right) \\
&= 2\left(\frac{\pi}{4n}\right)^2 \sinh^2 r_i + \cosh\left(\frac{\pi}{2n} r_i' + O\left(\frac{1}{n^2}\right)\right) + O\left(\frac{1}{n^3}\right) \\
&= 2\left(\frac{\pi}{4n}\right)^2 \sinh^2 r_i + 1 + \frac{1}{2}\left(\frac{\pi}{2n} r_i'\right)^2 + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

From $\cosh \delta_i = 2\sinh^2 \frac{\delta_i}{2} + 1$, we get

$$\begin{aligned}
\sinh^2 \frac{\delta_i}{2} &= \left(\frac{\pi}{4n}\right)^2 \left(\sinh^2 r_i + (r_i')^2\right) + O\left(\frac{1}{n^3}\right) \\
\Rightarrow \sinh \frac{\delta_i}{2} &= \frac{\pi}{4n}\sqrt{\sinh^2 r_i + (r_i')^2} + O\left(\frac{1}{n^2}\right) \\
\Rightarrow \delta_i &= \frac{\pi}{2n}\sqrt{\sinh^2 r_i + (r_i')^2} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Now sum all δ_i, $i = 0, \ldots, n-1$, and pass to the limit as $n \to \infty$. This gives, by symmetry, one fourth of the circumference C of the ellipse. Hence

$$C = 4 \int_0^{\pi/2} \sqrt{\sinh^2 r(\alpha) + r'(\alpha)^2}\, d\alpha.$$

From (2) we get

$$\cosh r(\alpha) r'(\alpha) = \frac{d}{d\alpha} \sinh r(\alpha) = -k_1^2 \sinh b \sin\alpha \cos\alpha \left(1 - k_1^2 \cos^2\alpha\right)^{-\frac{3}{2}},$$

whence

$$\begin{aligned}
&\sinh^2 r(\alpha) + r'(\alpha)^2 \\
&= \frac{\sinh^2 b}{1 - k_1^2 \cos^2\alpha} + \frac{k_1^4 \sinh^2 b \sin^2\alpha \cos^2\alpha}{\left(1 - k_1^2\cos^2\alpha\right)^3 \cosh^2 r(\alpha)} \\
&= \frac{\sinh^2 b}{1 - k_1^2 \cos^2\alpha} + \frac{k_1^4 \sinh^2 b \left(1 - \cos^2\alpha\right) \cos^2\alpha}{\left(1 - k_1^2\cos^2\alpha\right)^3 \left(\frac{\sinh^2 b}{1 - k_1^2\cos^2\alpha} + 1\right)} \\
&= \frac{\sinh^2 b \left(1 - k_1^2\cos^2\alpha\right)\left(\cosh^2 b - k_1^2 \cos^2\alpha\right) + k_1^4 \sinh^2 b \left(1 - \cos^2\alpha\right)\cos^2\alpha}{\left(1 - k_1^2\cos^2\alpha\right)^2 \left(\cosh^2 b - k_1^2\cos^2\alpha\right)} \\
&= \sinh^2 b \frac{\cosh^2 b - k_1^2\left(\cosh^2 b + 1 - k_1^2\right)\cos^2\alpha}{\left(1 - k_1^2\cos^2\alpha\right)^2 \left(\cosh^2 b - k_1^2\cos^2\alpha\right)},
\end{aligned}$$

and hence

$$C = 4\sinh b \int_0^{\pi/2} \frac{\sqrt{\cosh^2 b - k_1^2\left(\cosh^2 b + 1 - k_1^2\right)\cos^2\alpha}}{\left(1 - k_1^2\cos^2\alpha\right)\sqrt{\cosh^2 b - k_1^2\cos^2\alpha}}\, d\alpha.$$

This integral can be transformed into a more standardized form, by applying a change of variable

$$\alpha = \arctan(u\tan\beta),$$

where the real number u will be chosen later in some suitable way. Then

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha} = \frac{1}{1+u^2\tan^2\beta} = \frac{\cos^2\beta}{\cos^2\beta + u^2\sin^2\beta} = \frac{1-\sin^2\beta}{1-(1-u^2)\sin^2\beta}.$$

For convenience, we define

$$c_\beta = \frac{1}{1-(1-u^2)\sin^2\beta}, \qquad m_1 = \frac{\tanh^2 a - \tanh^2 b}{\tanh^2 a}.$$

We will also make use of the following equation

$$\begin{aligned} m_1 &= \frac{1}{\tanh^2 a}\left(\frac{1}{\cosh^2 b} - \frac{1}{\cosh^2 a}\right) = \frac{\cosh^2 a - \cosh^2 b}{\sinh^2 a\cosh^2 b} \\ &= \frac{\sinh^2 a - \sinh^2 b}{\sinh^2 a\cosh^2 b} = \frac{k_1^2}{\cosh^2 b}. \end{aligned}$$

To avoid getting huge expressions; we apply the change of variable to the different parts of the integral in turn.

i) $\cosh^2 b - k_1^2\left(\cosh^2 b + 1 - k_1^2\right)\cos^2\alpha$

$$\begin{aligned} &= \cosh^2 b - k_1^2\left(\cosh^2 b + 1 - k_1^2\right)\frac{1-\sin^2\beta}{1-(1-u^2)\sin^2\beta} \\ &= c_\beta\left(\cosh^2 b\left(1-(1-u^2)\sin^2\beta\right) - k_1^2(\cosh^2 b + 1 - k_1^2)(1-\sin^2\beta)\right) \\ &= c_\beta\left((1-k_1^2)(\cosh^2 b - k_1^2) + (k_1^2(1-k_1^2) - (1-k_1^2-u^2)\cosh^2 b)\sin^2\beta)\right) \\ &\left\{\text{Now choose } u \text{ so that the term containing } \sin^2\beta \text{ vanishes}\right\} \\ &= c_\beta\frac{\sinh^2 b}{\sinh^2 a}\cosh^2 b(1-m_1) = c_\beta\frac{\sinh^2 b\cosh^2 b\tanh^2 b}{\sinh^2 a\tanh^2 a} = c_\beta\frac{\sinh^4 b}{\sinh^2 a\tanh^2 a}. \end{aligned}$$

That is, u has been chosen such that

$$\begin{aligned} &(1-k_1^2-u^2)\cosh^2 b = k_1^2(1-k_1^2) \\ &\Rightarrow u = \sqrt{\frac{(1-k_1^2)\cosh^2 b - k_1^2(1-k_1^2)}{\cosh^2 b}} = \sqrt{(1-k_1^2)(1-m_1)} \\ &\quad = \sqrt{\frac{\sinh^2 b\tanh^2 b}{\sinh^2 a\tanh^2 a}} = \frac{\sinh b\tanh b}{\sinh a\tanh a}. \end{aligned}$$

ii) $1 - k_1^2 \cos^2\alpha = 1 - k_1^2 \dfrac{1-\sin^2\beta}{1-(1-u^2)\sin^2\beta} = c_\beta \left(1 - k_1^2 - (1 - k_1^2 - u^2)\sin^2\beta\right)$

$$= c_\beta \left(1 - k_1^2 - \frac{k_1^2(1-k_1^2)}{\cosh^2 b}\sin^2\beta\right) = c_\beta \frac{\sinh^2 b}{\sinh^2 a}(1 - m_1 \sin^2\beta).$$

iii) $\cosh^2 b - k_1^2\cos^2\alpha = \cosh^2 b - k_1^2 \dfrac{1-\sin^2\beta}{1-(1-u^2)\sin^2\beta}$

$$= c_\beta \left(\cosh^2 b - k_1^2 - ((1-u^2)\cosh^2 b - k_1^2)\sin^2\beta\right)$$
$$= c_\beta \left(\cosh^2 b - k_1^2 - (k_1^2(1-k_1^2) + k_1^2\cosh^2 b - k_1^2)\sin^2\beta\right)$$
$$= c_\beta \left(\cosh^2 b - k_1^2\right)\left(1 - k_1^2\sin^2\beta\right) = c_\beta \cosh^2 b(1-m_1)(1-k_1^2\sin^2\beta)$$
$$= c_\beta \cosh^2 b \frac{\tanh^2 b}{\tanh^2 a}(1-k_1^2\sin^2\beta) = c_\beta \frac{\sinh^2 b}{\tanh^2 a}(1-k_1^2\sin^2\beta).$$

iv) $d\alpha = \dfrac{u(1+\tan^2\beta)}{1+(u\tan\beta)^2}d\beta = \dfrac{u(\cos^2\beta + sin^2\beta)}{\cos^2\beta + u^2\sin^2\beta}d\beta = \dfrac{u}{1-(1-u^2)\sin^2\beta}d\beta$

$$= c_\beta \frac{\sinh b \tanh b}{\sinh a \tanh a} d\beta.$$

Putting the pieces together, we get

$$C = 4\sinh b \int_0^{\pi/2} \frac{\sqrt{c_\beta \frac{\sinh^4 b}{\sinh^2 a \tanh^2 a}}\; c_\beta \sinh b \tanh b}{c_\beta \frac{\sinh^2 b}{\sinh^2 a}(1-m_1\sin^2\beta)\sqrt{c_\beta \frac{\sinh^2 b}{\tanh^2 a}(1-k_1^2\sin^2\beta)}\; \sinh a \tanh a}\, d\beta$$

which, after cancelling factors, finally gives

Proposition 1. *The circumference C of an ellipse, having major and minor axis with lengths $2a$ and $2b$ respectively, is given by*

$$C = 4\sinh b\, \frac{\tanh b}{\tanh a} \int_0^{\pi/2} \frac{d\alpha}{(1-m_1\sin^2\alpha)\sqrt{1-k_1^2\sin^2\alpha}},$$

where

$$m_1 = \frac{\tanh^2 a - \tanh^2 b}{\tanh^2 a}, \qquad k_1 = \frac{\sqrt{\sinh^2 a - \sinh^2 b}}{\sinh a}. \qquad \square$$

Of course, the integral is a complete elliptic integral of the third kind; using a common notation, we can write:

$$C = 4\sinh b\, \frac{\tanh b}{\tanh a} \prod (k_1, m_1).$$

Next we derive a formula for the area of an ellipse. Choose an even n, large enough so that $\frac{\pi}{2n}$ is smaller than the angle of parallelism at O, with respect to the perpendicular to $\overrightarrow{\mathrm{OR}_0}$ through R_0. Let m_i be the perpendicular to $\overrightarrow{\mathrm{OR}_i}$ through R_i.

Then, since $r_i \leqslant r_0 = a$, the two rays forming angles $\frac{\pi}{2n}$ with $\overrightarrow{\mathrm{OR}_i}$ cut m_i, making up an isosceles triangle, see Figure 3. Let s_i and $2b_i$ be the lengths of its sides and base respectively. By symmetry, one fourth of the area of the ellipse is given by summing the areas of all such triangles, $i = 1, 3, 5 \ldots, n-1$, and then letting $n \to \infty$.

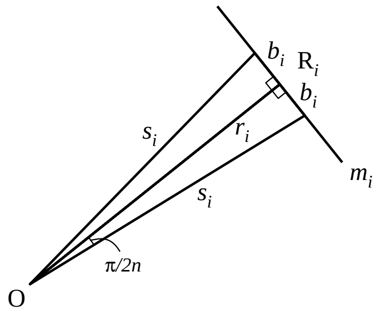

Figure 3

Let A_i be the area of each of the two right triangles that make up the isosceles triangle, then

$$\tanh\frac{A_i}{2} = \tanh\frac{r_i}{2}\tanh\frac{b_i}{2}$$

$$\cosh s_i = \cosh b_i \cosh r_i$$

$$\sinh b_i = \sin\frac{\pi}{2n}\sinh s_i$$

$$\begin{aligned}\Rightarrow\ \sinh^2 b_i &= \sin^2\frac{\pi}{2n}\sinh^2 s_i = \sin^2\frac{\pi}{2n}\left(\cosh^2 s_i - 1\right)\\ &= \sin^2\frac{\pi}{2n}\left(\cosh^2 b_i\cosh^2 r_i - 1\right)\\ &= \sin^2\frac{\pi}{2n}\left(\sinh^2 b_i\cosh^2 r_i + \sinh^2 r_i\right)\end{aligned}$$

$$\Rightarrow\ \sinh^2 b_i = \frac{\sin^2\frac{\pi}{2n}\sinh^2 r_i}{1-\sin^2\frac{\pi}{2n}\cosh^2 r_i} = \left(\frac{\pi}{2n}\right)^2\sinh^2 r_i + O\left(\frac{1}{n^4}\right)$$

$$\Rightarrow\ \sinh b_i = \frac{\pi}{2n}\sinh r_i + O\left(\frac{1}{n^3}\right)$$

$$\Rightarrow\ \cosh b_i = 1 + O\left(\frac{1}{n^2}\right)$$

$$\Rightarrow\ \tanh\frac{b_i}{2} = \frac{\sinh b_i}{\cosh b_i + 1} = \frac{\pi}{4n}\sinh r_i + O\left(\frac{1}{n^3}\right)$$

$$\Rightarrow\ \tanh\frac{A_i}{2} = \frac{\pi}{4n}\tanh\frac{r_i}{2}\sinh r_i + O\left(\frac{1}{n^3}\right) = \frac{\pi}{4n}(\cosh r_i - 1) + O\left(\frac{1}{n^3}\right)$$

$$\Rightarrow\ A_i = \frac{\pi}{2n}(\cosh r_i - 1) + O\left(\frac{1}{n^3}\right).$$

Summing and passing to the limit as $n \to \infty$ gives the area A:

$$A = 4 \int_0^{\pi/2} \cosh r(\alpha)\, d\alpha \; - \; 2\pi,$$

which, by (2), is

$$A = 4 \int_0^{\pi/2} \sqrt{\frac{\sinh^2 b}{1 - k_1^2 \cos^2\alpha} + 1}\, d\alpha \; - \; 2\pi = 4 \int_0^{\pi/2} \sqrt{\frac{\cosh^2 b - k_1^2 \cos^2\alpha}{1 - k_1^2 \cos^2\alpha}}\, d\alpha \; - \; 2\pi.$$

Now we make a change of variable

$$\alpha = \arctan(v \cot\beta).$$

Then

$$\cos^2\alpha = \frac{1}{1 + \tan^2\alpha} = \frac{1}{1 + v^2\cot^2\beta} = \frac{\sin^2\beta}{\sin^2\beta + v^2\cos^2\beta} = \frac{\sin^2\beta}{v^2 + (1 - v^2)\sin^2\beta}$$

and

$$d\alpha = \frac{-v(1 + \cot^2\beta)}{1 + (v\cot\beta)^2} d\beta = -\frac{v(sin^2\beta + \cos^2\beta)}{sin^2\beta + v^2\cos^2\beta} d\beta = -\frac{v}{v^2 + (1 - v^2)\sin^2\beta} d\beta$$

which gives

$$\begin{aligned}
A &= -4v \int_{\pi/2}^{0} \frac{\sqrt{\cosh^2 b - k_1^2 \frac{\sin^2\beta}{v^2 + (1 - v^2)\sin^2\beta}}}{\sqrt{1 - k_1^2 \frac{\sin^2\beta}{v^2 + (1 - v^2)\sin^2\beta}}(v^2 + (1 - v^2)\sin^2\beta)}\, d\beta \; - \; 2\pi \\
&= 4v \int_0^{\pi/2} \frac{\sqrt{\cosh^2 b(v^2 + (1 - v^2)\sin^2\beta) - k_1^2\sin^2\beta}}{(v^2 + (1 - v^2)\sin^2\beta)\sqrt{v^2 + (1 - v^2)\sin^2\beta - k_1^2\sin^2\beta}}\, d\beta \; - \; 2\pi \\
&= 4v \int_0^{\pi/2} \frac{\sqrt{v^2\cosh^2 b + ((1 - v^2)\cosh^2 b - k_1^2)\sin^2\beta}}{(v^2 + (1 - v^2)\sin^2\beta)\sqrt{v^2 - (k_1^2 + v^2 - 1)\sin^2\beta}}\, d\beta \; - \; 2\pi \\
&= \frac{4}{v} \int_0^{\pi/2} \frac{\sqrt{\cosh^2 b + ((1 - v^2)\cosh^2 b - k_1^2)/v^2 \sin^2\beta}}{(1 + (1/v^2 - 1)\sin^2\beta)\sqrt{1 - (1 - (1 - k_1^2)/v^2)\sin^2\beta}}\, d\beta \; - \; 2\pi.
\end{aligned}$$

As before, the constant v is chosen so that $\sin^2\beta$ vanishes from the numerator, that is

$$v = \sqrt{1 - \frac{k_1^2}{\cosh^2 b}} = \sqrt{1 - m_1} = \frac{\tanh b}{\tanh a} = \frac{\coth a}{\coth b}.$$

Then

$$\frac{1 - k_1^2}{v^2} = \frac{\sinh^2 b \tanh^2 a}{\sinh^2 a \tanh^2 b} = \frac{\cosh^2 b}{\cosh^2 a},$$

and we can formulate

Proposition 2. *The area A of an ellipse, having major and minor axis with lengths $2a$ and $2b$ respectively, is given by*

$$A = 4\cosh b \, \frac{\coth b}{\coth a} \int_0^{\pi/2} \frac{d\alpha}{\left(1 - m_2 \sin^2\alpha\right)\sqrt{1 - k_2^2 \sin^2\alpha}} \; - \; 2\pi,$$

where

$$m_2 = \frac{\coth^2 a - \coth^2 b}{\coth^2 a}, \qquad k_2 = \frac{\sqrt{\cosh^2 a - \cosh^2 b}}{\cosh a}. \qquad \square$$

The formula has been written in this particular way to emphasize the similarity with the formula for circumference. From this similarity one can easily derive the following relation:

$$C(a,b) = i\left(A\left(\tfrac{\pi i}{2} - a, \tfrac{\pi i}{2} - b\right) + 2\pi\right). \tag{3}$$

See next section for some comments about this equation, and the corresponding formula for elliptic geometry.

As always, it is good practice to check formulas for special cases and limits, especially since these formulas differ significantly from their counterparts in Euclidean geometry. In particular, we would expect the following relations to hold:

i) $C(a,a) = 2\pi \sinh a$

ii) $A(a,a) = 4\pi \sinh^2 \frac{a}{2}$

iii) $C(a,b) \to 4a$, as $b \to 0$

iv) $A(a,b) \to 0$, as $b \to 0$

v) $C(a,ta) \approx 4a \int_0^{\pi/2} \sqrt{1-(1-t^2)\sin^2\alpha}\; d\alpha,\ 0 < t < 1,\ a$ small

vi) $A(a,ta) \approx \pi t a^2,\ 0 < t < 1,\ a$ small

The first two conditions are equalities with formulas for circles in hyperbolic geometry (see [1], pp. 407–410). The last two show that, for infinitesimal ellipses, the hyperbolic formulas reduce to their Euclidean counterparts.

Proof. We will not prove all these conditions in detail, for some only an indication of a proof is given.

i) $$C(a,a) = 4\sinh a \int_0^{\pi/2} 1 \; d\alpha = 2\pi \sinh a$$

ii) $$A(a,a) = 4\cosh a \int_0^{\pi/2} 1 \; d\alpha - 2\pi = 2\pi(\cosh a - 1) = 4\pi \sinh^2 \frac{a}{2}$$

iii) $$\lim_{b\to 0} C(a,b) = \left\{\text{substitute } \ \alpha = \arctan\left(\tfrac{\sinh a}{\sinh b}\cot\beta\right)\right\}$$

$$= \lim_{b\to 0}\ 4\tanh a\ \cosh b \int_0^{\pi/2} \frac{\sqrt{1-k_1^2\sin^2\beta}}{1-k_1^2\tanh^2 a\sin^2\beta}\ d\beta$$

$$= 4\tanh a \int_0^{\pi/2} \frac{\cos\beta}{1-\tanh^2 a\sin^2\beta}\ d\beta = \{\tanh a\ \sin\beta = x\}$$

$$= 4\int_0^{\tanh a} \frac{dx}{1-x^2} = 4a.$$

iv) $$\lim_{b\to 0} A(a,b) = \left\{\alpha = \arctan\left(\tfrac{\tanh b}{\tanh a}\cot\beta\right)\right\}$$

$$= \lim_{b\to 0}\ 4\cosh b \int_0^{\pi/2} \sqrt{\frac{\tanh^2 b + \left(\tanh^2 a - \tanh^2 b\right)\sin^2\beta}{\left(1-k_2^2\right)\tanh^2 b + k_2^2\sin^2\beta}}\ d\beta\ -\ 2\pi$$

$$= 4\int_0^{\pi/2} 1\ d\beta\ -\ 2\pi = 0.$$

v) $$C(a,ta) \approx 4t^2 a \int_0^{\pi/2} \frac{d\alpha}{(1-(1-t^2)\sin^2\alpha)^{3/2}}\ d\alpha$$

$$= \left\{\alpha = \arctan\left(\frac{\cot\beta}{t}\right)\right\} = 4a\int_0^{\pi/2} \sqrt{1-(1-t^2)\sin^2\beta}\ d\beta.$$

vi) Using series expansion one gets

$$A(a,ta) = \frac{4}{t}\left(1+\frac{5t^2-2}{6}a^2\right)\int_0^{\pi/2} \frac{1+\frac{1-t^2}{2}a^2\sin^2\alpha}{1+\frac{1-t^2}{t^2}\left(1-\frac{2a^2}{3}\right)\sin^2\alpha}\ d\alpha - 2\pi + O(a^4)$$

$$= \frac{4}{t}\left(1+\frac{5t^2-2}{6}a^2\right)\frac{\pi}{2}t\left(1-\frac{5t^2-3t-2}{6}a^2\right) - 2\pi + O(a^4)$$

$$= \pi t a^2 + O(a^4).$$

□

3 Ellipses in elliptic geometry

The corresponding formulas for elliptic geometry are derived with very similar calculations. Only the resulting equations are given here:

$$C(a,b) = 4\sin b\,\frac{\tan b}{\tan a}\int_0^{\pi/2}\frac{d\alpha}{\left(1-\frac{\tan^2 a-\tan^2 b}{\tan^2 a}\sin^2\alpha\right)\sqrt{1-\frac{\sin^2 a-\sin^2 b}{\sin^2 a}\sin^2\alpha}}$$

$$A(a,b) = 2\pi \;-\; 4\cos b\,\frac{\cot b}{\cot a}\int_0^{\pi/2}\frac{d\alpha}{\left(1-\frac{\cot^2 a-\cot^2 b}{\cot^2 a}\sin^2\alpha\right)\sqrt{1-\frac{\cos^2 a-\cos^2 b}{\cos^2 a}\sin^2\alpha}}$$

$$C(a,b) + A\left(\tfrac{\pi}{2}-a, \tfrac{\pi}{2}-b\right) = 2\pi \tag{4}$$

The formulas were derived under the assumption $b \leqslant a$. But with a change of variable $\alpha = \frac{\pi}{2} - \beta$ one can show that $C(a,b) = C(b,a)$ and $A(a,b) = A(b,a)$. So equation (4), the counterpart of (3) in hyperbolic geometry, gives a relation between circumference and area of two related ellipses; the two terms add up to the area of the elliptic room. On a sphere with radius R the equation becomes

$$C(a,b)\,R + A\left(\tfrac{\pi R}{2}-a, \tfrac{\pi R}{2}-b\right) = 2\pi R^2,$$

which equals (3) for $R = i$, illustrating the fact that a sphere with imaginary radius can be used as a model for hyperbolic geometry (see [1], p. 186).

Note that the formula for area should be written as $A(b,a)$ to make its integral an elliptic of the third kind. The form chosen was for the sake of symmetry with the hyperbolic case.

It is not difficult to show that, in the standard unit sphere model of elliptic geometry, the orthogonal projection of an ellipse onto the plane tangent to the middle point O, is a Euclidean ellipse – and conversely for Euclidean ellipses with major axis $\leqslant 1$. Or equivalently: ellipses are parallel projections of circles in three-space, having radius $\leqslant 1$ and centre coinciding with the centre of the sphere, onto the sphere. This in turn shows that, for the extremal case when $a = \pi/2$, the ellipse consists of two lines.

Formulas similar to (4) can be found for other objects in elliptic geometry. Let $C(h)$ and $A(h)$ be the circumference and area respectively of an equilateral triangle with altitude h. Then one can show that $C(h)+A(\pi-h) = 2\pi$. It would be interesting to investigate such equations further, but that is beyond the scope of this document.

References

[1] Marvin J. Greenberg. *Euclidean and non-Euclidean geometries*. W. H. Freeman and Company, New York, 1999.