Term paper

Geometrisk topologi D 6p, HT – 02

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Motivation

This term paper is based on a set of three questions regarding the properties of the three-sphere and its decomposition into two congruent torii. These questions were presented in »Problem sheet B« in the course »Geometrisk topologi D« given at the department of mathematics of Uppsala during the fall of 2002. The main focus is placed on explicit calculations leading to topological results. The paper may hence be regarded as an example of how explicit results can provide insight in more abstract topological properties of fundamental topological spaces.

Abstract

The unit three-sphere can be decomposed into two congruent solid torii. The intersection of these torii, when transformed by stereographic projection, is a torus whose meridian and longitude lines may be interchanged by a rigid motion in the three-sphere considered as a space of its own. Finally, it is proved that this torus of intersection in the three sphere may be identified with a hyperboloid of one sheet in the projective three space.

Problem sheet B, exercise 1

Prove that the unit 3-sphere of Euclidean 4-space $\mathbf{R}^4$, which has the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \tag{1}$$

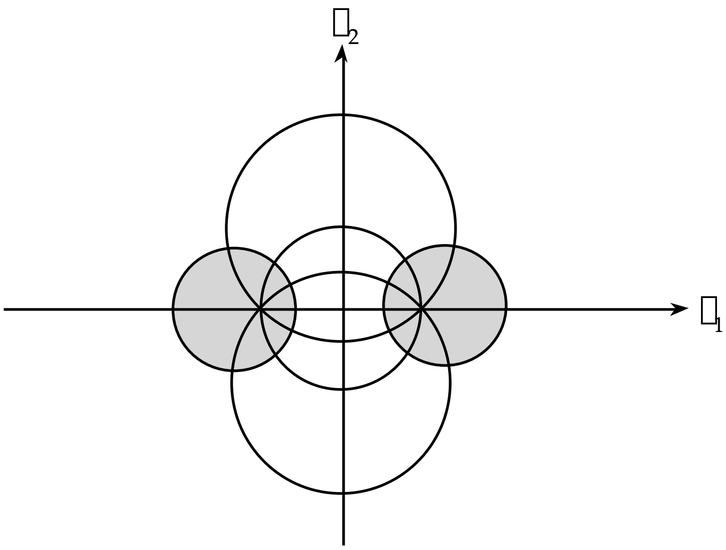
can be decomposed into two solid tori (topological product of the closed disk and the circle):

$$x_1^2 + x_2^2 \le x_3^2 + x_4^2 \ \text{ and } \ x_1^2 + x_2^2 \ge x_3^2 + x_4^2 \tag{2+3}$$

These solid tori are congruent, i.e., they can be transformed to one another by a rigid Euclidean motion about the midpoint of the sphere. The common boundary surface, whose points satisfy equation (3) and also

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 \tag{4}$$

can be transformed, by stereographic projection, to a torus of revolution. The stereographic projection defined by projecting from the north pole (0, 0, 0, 1) to the equatorial hyperplane $x_4 = 0$. The hyperplane is taken to have Cartesian coordinates x_1, x_2, x_3 and the torus of revolution has the x_3-axis as its axis of rotation. One sees in this way that the 3-sphere (which is obtained by adding the image of the north pole to the equatorial hyperplane) may be decomposed into the union of the two solid tori. The "core" of the first solid is the unit circle in the (x_1, x_2)-plane, and the core the second solid torus is the x_3-axis plus the point at infinity.



Solution

Equation (2) and (3) yields:

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}$$

$$x_1^2 = \frac{1}{2} - x_1^2 \quad ; \quad x_2^2 = \frac{1}{2} - x_1^2 \quad ; \quad x_3^2 = \frac{1}{2} - x_1^2 \quad ; \quad x_4^2 = \frac{1}{2} - x_1^2$$

$$x_1 = \pm\sqrt{\frac{1}{2} - x_1^2} \quad ; \quad x_2 = \pm\sqrt{\frac{1}{2} - x_1^2} \quad ; \quad x_3 = \pm\sqrt{\frac{1}{2} - x_1^2} \quad ; \quad x_4 = \pm\sqrt{\frac{1}{2} - x_1^2} \qquad (5)$$

Consider the stereographic projection of the unit three-sphere, projecting from the point $(x_1,x_2,x_3,x_4)=(0,0,0,1)$ down to the equatorial hyperplane $x_4=0$. Let the space obtained by transforming all of the Euclidean four-space by this stereographic projection, have the Cartesian coordinates (ξ_1, ξ_2, ξ_3) defined by:

$$\xi_1 = \frac{1}{1-x_4}x_1 \quad ; \quad \xi_2 = \pm\frac{1}{1-x_4}x_2 \quad ; \quad \xi_3 = \pm\frac{1}{1-x_4}x_3 \qquad (6)$$

(For a general formula see Seifert, Threllfall page XXXX, english edition)

Consider the x_1,x_2 - plane ($x_3 = 0$):

$$\xi_1 = \underbrace{\frac{1}{1-x_4}}_{k}x_1 = k \cdot x_1 \qquad (7)$$

$$\xi_2 = \pm\underbrace{\frac{1}{1-x_4}}_{k}x_2 = \pm k \cdot \underbrace{x_2}_{(5)} = \pm k \cdot \sqrt{\frac{1}{2} - x_1^2} \qquad (8)$$

$$\xi_2^2 = \left(\pm k \cdot \sqrt{\frac{1}{2} - x_1^2}\right)^2 = k^2 \cdot \left(\frac{1}{2} - x_1^2\right) = \frac{k^2}{2} - \underbrace{\left(k \cdot x_1\right)^2}_{(7)} = \left(\frac{k}{\sqrt{2}}\right)^2 - \xi_1^2$$

$$\xi_1^2 + \xi_2^2 = \left(\frac{k}{\sqrt{2}}\right)^2 \qquad (9)$$

Equation (9) forms circles in the $\xi_1\xi_2$ -plane that depend on k, which in turn depend on x_4. By simple inspection of (9), the maximum resp. minimum circle formed by (9) is simply the minimum resp. maximum of x_4.

Find the boundary points of x_4:

$$\xi_3 = 0 \quad \Rightarrow \quad \pm\frac{1}{1-x_4}\underbrace{x_3}_{(5)} = 0$$

$$\pm\frac{1}{1-x_4}\sqrt{\frac{1}{2} - x_4^2} = 0$$

$$\underline{\underline{x_4 = \pm\frac{1}{\sqrt{2}}}}$$

Case 1: $x_4 = \frac{1}{\sqrt{2}} \Rightarrow$

$$\square_1 = \frac{1}{1 - x_4} x_1 = \frac{1}{1 - \frac{1}{\sqrt{2}}} x_1 = \frac{\sqrt{2}}{\sqrt{2} - 1} x_1$$

$$\square_1^2 = \left(\frac{\sqrt{2}}{\sqrt{2} - 1} x_1 \right)^2 \tag{10}$$

$$\square_2 = \pm \frac{1}{1 - x_4} \underbrace{x_2}_{(5)} = \pm \frac{1}{1 - x_4} \sqrt{\frac{1}{2} - x_1^2} = \frac{1}{1 - \frac{1}{\sqrt{2}}} \sqrt{\frac{1}{2} - x_1^2} = \pm \frac{\sqrt{2}}{\sqrt{2} - 1} \sqrt{\frac{1}{2} - x_1^2}$$

$$\square_2^2 = \left(\pm \frac{\sqrt{2}}{\sqrt{2} - 1} \sqrt{\frac{1}{2} - x_1^2} \right)^2 = \left(\frac{\sqrt{2}}{\sqrt{2} - 1} \right)^2 \left(\frac{1}{2} - x_1^2 \right) = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{2} - 1} \right)^2 - \underbrace{\left(\frac{\sqrt{2}}{\sqrt{2} - 1} x_1 \right)^2}_{(10)}$$

$$\square_1^2 + \square_2^2 = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{2} - 1} \right)^2$$

$$(\text{radius})^2 = \left(\frac{1}{\sqrt{2}} \right)^2 \left(\frac{\sqrt{2}}{\sqrt{2} - 1} \right)^2$$

$$\text{radius} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2} - 1} \right) = \frac{\sqrt{2}}{2 - \sqrt{2}} = \underline{\underline{\frac{1}{\sqrt{2} - 1}}}$$

Case 2: $x_4 = -\frac{1}{\sqrt{2}} \Rightarrow$

$$\square_1 = \frac{1}{1 - x_4} x_1 = \frac{1}{1 + \frac{1}{\sqrt{2}}} x_1 = \frac{\sqrt{2}}{\sqrt{2} + 1} x_1$$

$$\square_1^2 = \left(\frac{\sqrt{2}}{\sqrt{2} + 1} x_1 \right)^2 \tag{11}$$

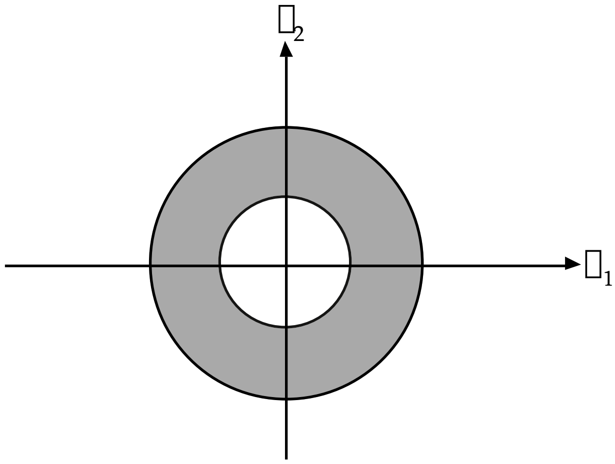
$$\square_2 = \pm \frac{1}{1 - x_4} \underbrace{x_2}_{(5)} = \pm \frac{1}{1 - x_4} \sqrt{\frac{1}{2} - x_1^2} = \frac{1}{1 + \frac{1}{\sqrt{2}}} \sqrt{\frac{1}{2} - x_1^2} = \pm \frac{\sqrt{2}}{\sqrt{2} + 1} \sqrt{\frac{1}{2} - x_1^2}$$

$$\square_2^2 = \left(\pm \frac{\sqrt{2}}{\sqrt{2} + 1} \sqrt{\frac{1}{2} - x_1^2} \right)^2 = \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \right)^2 \left(\frac{1}{2} - x_1^2 \right) = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \right)^2 - \underbrace{\left(\frac{\sqrt{2}}{\sqrt{2} + 1} x_1 \right)^2}_{(11)}$$

$$\square_1^2 + \square_2^2 = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \right)^2$$

$$(\text{radius})^2 = \left(\frac{1}{\sqrt{2}} \right)^2 \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \right)^2$$

$$\text{radius} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \right) = \frac{\sqrt{2}}{2 + \sqrt{2}} = \underline{\underline{\frac{1}{\sqrt{2} + 1}}}$$



Now, consider the $\square_1,\square_3$ - plane in a similar fashion ($x_2 = 0$):

$$\square_2 = 0 \quad \square \quad \pm\frac{1}{1\square x_4}\underbrace{x_2}_{(5)} = 0$$

$$\pm\frac{1}{1\square x_4}\sqrt{\frac{1}{2}\square x_1^2} = 0$$

$$\underline{\underline{x_1 = \pm\frac{1}{\sqrt{2}}}}$$

Case 1: $x_1 = \frac{1}{\sqrt{2}} \quad \square$

$$\square_1 = \frac{1}{1\square x_4}\cdot\frac{1}{\sqrt{2}}$$

$$\sqrt{2}\cdot\square_1 = \frac{1}{1\square x_4} \tag{12}$$

$$\sqrt{2}\cdot\square_1\cdot(1\square x_4) = 1$$

$$1\square x_4 = \frac{1}{\sqrt{2}\cdot\square_1}$$

$$x_4 = 1\square\frac{1}{\sqrt{2}\cdot x_1}$$

$$x_4^2 = \square 1\square\frac{1}{\sqrt{2}\cdot\square_1}\square^2 = \square 1+\frac{1}{2\cdot\square_1^2}\square\frac{2}{\sqrt{2}\cdot\square_1}\square$$

$$2x_4^2 = 2\square 1+\frac{1}{2\cdot\square_1^2}\square\frac{2}{\sqrt{2}\cdot\square_1}\square = 2+\frac{1}{\square_1^2}\square\frac{4}{\sqrt{2}\cdot\square_1} = 2+\frac{1}{\square_1^2}\square\frac{2\sqrt{2}}{\square_1} \tag{13}$$

$$\square_3 = \pm\underbrace{\frac{1}{1\square x_4}}_{(12)}\underbrace{\sqrt{\frac{1}{2}\square x_4^2}}_{(5)} = \pm\sqrt{2}\cdot\square_1\cdot\sqrt{\frac{1}{2}\square x_4^2}$$

$$\square_3^2 = \left(\pm\sqrt{2}\cdot\square_1\cdot\sqrt{\frac{1}{2}\square x_4^2}\right)^2 = 2\cdot\square_1^2\cdot\left(\frac{1}{2}\square x_4^2\right) = \square_1^2\square\underbrace{2x_4^2}_{(13)}\cdot\square_1^2 = \square_1^2\square\left(2+\frac{1}{\square_1^2}\square\frac{2\sqrt{2}}{\square_1}\right)\cdot\square_1^2$$

$$\square_3^2 = \square_1^2\square\left(2\square_1^2+\frac{\square_1^2}{\square_1^2}\square\frac{2\sqrt{2}\cdot\square_1^2}{\square_1}\right)$$

$$\square_3^2+\square_1^2\square 2\sqrt{2}\cdot\square_1 = 1$$

$$\underline{\square_3^2+\left(\square_1\square\sqrt{2}\right)^2 = 1} \qquad (14)$$

Case 2: $x_1 = \square\frac{1}{\sqrt{2}}$ $\square$

$$\square_1 = \frac{1}{1\square x_4}\cdot\square\frac{1}{\sqrt{2}}$$

$$\square\sqrt{2}\cdot\square_1 = \frac{1}{1\square x_4} \qquad (15)$$

$$\square\sqrt{2}\cdot\square_1\cdot(1\square x_4) = 1$$

$$1\square x_4 = \frac{1}{\square\sqrt{2}\cdot\square_1}$$

$$x_4 = 1+\frac{1}{\sqrt{2}\cdot x_1}$$

$$x_4^2 = \left(1+\frac{1}{\sqrt{2}\cdot\square_1}\right)^2 = \left(1+\frac{1}{2\cdot\square_1^2}+\frac{2}{\sqrt{2}\cdot\square_1}\right.$$

$$2x_4^2 = 2\left(1+\frac{1}{2\cdot\square_1^2}+\frac{2}{\sqrt{2}\cdot\square_1}\right) = 2+\frac{1}{\square_1^2}+\frac{4}{\sqrt{2}\cdot\square_1} = 2+\frac{1}{\square_1^2}+\frac{2\sqrt{2}}{\square_1} \qquad (16)$$

$$\square_3 = \pm\underbrace{\frac{1}{1\square x_4}}_{(15)}\underbrace{\sqrt{\frac{1}{2}\square x_4^2}}_{(5)} = \pm\sqrt{2}\cdot\square_1\cdot\sqrt{\frac{1}{2}\square x_4^2}$$

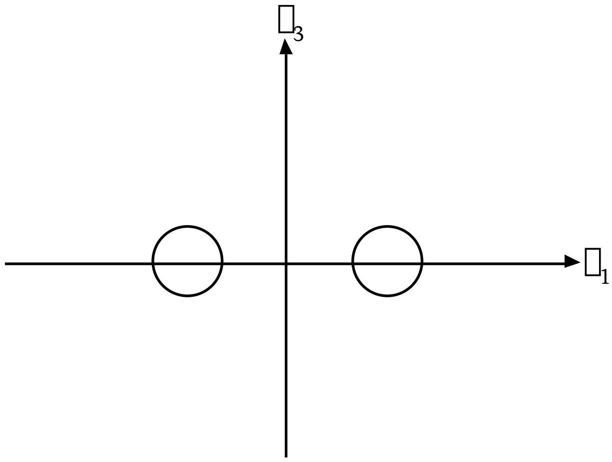
$$\square_3^2 = \left(\pm\sqrt{2}\cdot\square_1\cdot\sqrt{\frac{1}{2}\square x_4^2}\right)^2 = 2\cdot\square_1^2\cdot\left(\frac{1}{2}\square x_4^2\right) = \square_1^2\square\underbrace{2x_4^2}_{(16)}\cdot\square_1^2 = \square_1^2\square\left(2+\frac{1}{\square_1^2}+\frac{2\sqrt{2}}{\square_1}\right)\cdot\square_1^2$$

$$\square_3^2 = \square_1^2\square\left(2\square_1^2+\frac{\square_1^2}{\square_1^2}+\frac{2\sqrt{2}\cdot\square_1^2}{\square_1}\right) = \square_1^2\square 2\square_1^2\square 1\square 2\sqrt{2}\cdot\square_1$$

$$\square_3^2+\square_1^2+2\sqrt{2}\cdot\square_1 = \square 1$$

$$\underline{\square_3^2+\left(\square_1+\sqrt{2}\right)^2 = 1} \qquad (17)$$

Equation (14) and (17) form circles with radius one and centers at opposite sides of the $\square_3$
See figure below.



Problem sheet B, exercise 2

Consider the unit three-sphere S^3 embedded in Euclidean four-space R^3, described by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \tag{1}$$

and let it be decomposed into two congruent solid tori with common boundary

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2} \tag{2}$$

Consider the stereographic projection p projecting from the point $(x_1,x_2,x_3,x_4)=(0,0,0,1)$ to the hyperplane $x_4=0$. Let the space obtained by transforming all of the Euclidean four-space by this stereographic projection, i.e. $p(R^4)$, have the Cartesian coordinates (ξ_1, ξ_2, ξ_3) defined by:

$$\xi_i = \frac{x_i}{1 - x_4} \qquad (i=1, 2, 3) \tag{3}$$

Conversely:

$$x_i = \frac{2\xi_i}{\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2} \qquad (i=1, 2, 3) \tag{4}$$

$$x_4 = \frac{\xi_1^2 + \xi_2^2 + \xi_3^2 - 1}{\xi_1^2 + \xi_2^2 + \xi_3^2 + 1} \tag{5}$$

The boundary surface defined by (2) forms a torus of revolution around the ξ_3-axis after stereographic projection. Show that this torus of revolution can be mapped topologically onto itself by a rigid motion in spherical three-space such that the meridian circles and the longitude circles are interchanged.

Solution:

By equation (2):

$$x_2 = \pm\sqrt{\frac{1}{2} - x_1^2} \quad \text{and} \quad x_3 = \pm\sqrt{\frac{1}{2} - x_4^2} \tag{6}$$

This in combination with (3) yields

$$\xi_1 = \frac{x_1}{1 - x_4} \qquad \xi_2 = \pm\frac{\sqrt{\frac{1}{2} - x_1^2}}{1 - x_4} \qquad \xi_3 = \pm\frac{\sqrt{\frac{1}{2} - x_4^2}}{1 - x_4} \tag{7}$$

The surface defined by (2) is a torus of revolution around the $\square_3$-axis, it has the following equation:

$$\left(c_1 \square \sqrt{\square_1^2 + \square_2^2}\right) \square \square_3^2 = c_2 \tag{8}$$

where c_1 and c_2 are constants.

A longitude line of the torus is defined to be a circle consisting of the set of points in the torus with $\square_3$ = constant. A meridian is a circle consisting of the set of points where either $\square_1$ or $\square_2$ is constant (but not both since it is homeomorphic to S^1). An example of a meridian and a longitude line are shown in figure 1.

A rigid motion in S^3 is an isomorphic mapping from S^3 to $S^1 x S^1$ can be embedded in R^4 in an obvious way and a rotation of the space R^4 is an isomorphic mapping of S^3, and hence a rigid motion of that space. If a rotation of R^4 is found that changes the meridians and the longitude lines of the torus, the problem is solved.

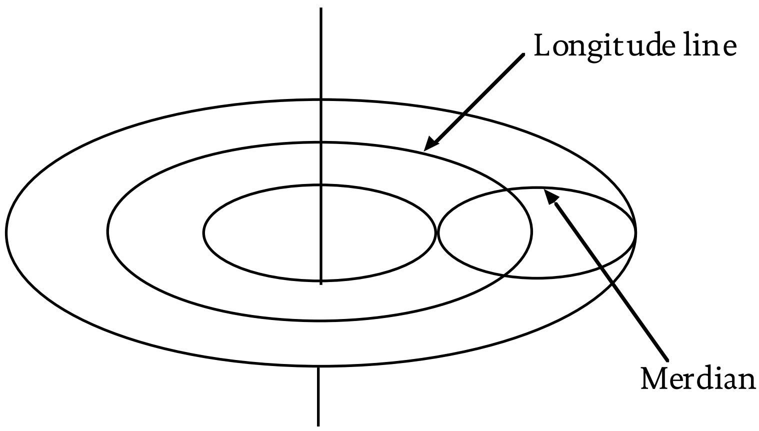


Figure 1, x_3-axis indicated in center

Consider the properties of a longitude line of the torus of revolution in S^3 as a set of points in R^4. If $\square_3$=constant, then

$$\text{constant} = \square_3 = \pm \frac{\sqrt{\frac{1}{2} \square x_4^2}}{1 \square x_4} \quad \square \quad x_4 = \text{constant} \quad \square \quad x_3 = \text{constant} \tag{9}$$

by (6) and (7). Hence a meridian of the torus, expressed as coordinates in R^4 is the set of points satisfying (2) where x_3 and x_4 are constant.

Now, consider the properties of a longitude line of the same torus embedded in R^4. The longitude lines are distinguished by the property ;

$$\text{constant} = \frac{\square_1}{\square_2}$$

Expressed as coordinates in R^4 this is equivalent to

$$\text{constant} = \frac{\xi_1}{\xi_2} = \frac{\dfrac{x_1}{(1-x_4)}}{\dfrac{x_2}{(1-x_4)}} = \frac{x_1}{x_2} \qquad (8)$$

Reconsider the demands on a meridian circle: x_4=constant and x_3=constant. This implies that

$$\text{constant} = \frac{x_3}{x_4} \qquad (10)$$

This means that interchanging x_1 and x_2 with x_3 and x_4 changes the meridians with the longitude lines. Above, it was noted that a rotation in R^4 is an isomorphism from S^3 to S^3, and a rigid motion in S^3. There are two obvious rotations in R^4 that performs the desired interchange of coordinates, described by multiplication with the following matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad (11+12)$$

The rigid motions induced by multiplication in R^4 of matrix (11) or (12) can certainly be transformed, using equations (4) and (5), into isomorphisms from S^3 to S^3, i.e. rigid motions in S^3. This proves that there exist topological mappings from the torus of revolution onto itself such that the meridian circles and the longitude circles are interchanged. This solves the problem.

Problem sheet B, exercise 3

The projective three-space P^3 may be constructed by identifying opposite points of the three-sphere S^3. The three-sphere can be decomposed into two congruent solid torii whose surface of intersection forms a torus. Show that this torus of intersection is transformed into a hyperboloid of one sheet when S^3 is transformed into P^3 as indicated above.

Solution:

Identify S^3 with the set

$$\left\{(x_1, x_2, x_3, x_4) \,\middle|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\} \subset R^4 \qquad (1)$$

Let this copy of S^3 be decomposed into the two solid torii

$$x_1^2 + x_2^2 \leq x_3^2 + x_4^2 \qquad \text{and} \qquad x_1^2 + x_2^2 \geq x_3^2 + x_4^2 \qquad (2+3)$$

The torus of intersection is the set

$$\left\{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}\right\} \subset R^4 \qquad (4)$$

The identification of opposite points of S^3 will be performed as follows; consider the three-sphere as a subset of R^4 as above and identify the coordinates

$$[x_1, x_2, x_3, x_4] = [\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4] \qquad \lambda \neq 0 \qquad (5)$$

That is, constructing P^3 from R^4 but still keeping the restrictions on the set of R^4 that constitutes the space S^3 and the torus of intersection (i.e. equations (1) and (4)). Note that the projective space P^3 is described by homogeneous coordinates as in equation (5), together with an improper plane corresponding to one of the coordinates in R^4 being zero.

Without loss of generality, choose $x_4=0$ to define the improper plane. This improper plane corresponds in some aspects to the point at infinity. The special case $(x_1, x_2, x_3, x_4)=(0, 0, 0, 0) \in R^4$ is not of any interest since it is not an element of S^3, embedded in R^4 as shown by equation (1).

The torus of intersection defined by equation (4) is transformed with the three-sphere into the set of points $[x_1 : x_2 : x_3 : x_4] \in P^3$ satisfying the following ($x_4 \neq 0$):

$$\left\{x_1^2 + x_2^2 = x_3^2 + x_4^2\right\} = \left\{\frac{x_1^2}{x_4^2} + \frac{x_2^2}{x_4^2} = \frac{x_3^2}{x_4^2} + \frac{x_4^2}{x_4^2}\right\} \qquad (x_4 \neq 0) \qquad (6)$$

which is equivalent to

$$\left\{\frac{x_1^2}{x_4^2} + \frac{x_2^2}{x_4^2} = \frac{x_3^2}{x_4^2} + 1\right\} = \left\{\frac{x_1^2}{x_4^2} + \frac{x_2^2}{x_4^2} - \frac{x_3^2}{x_4^2} + 1\right\} \qquad (x_4 \neq 0) \qquad (7)$$

Let $x = \frac{x_1}{x_4},\ y = \frac{x_2}{x_4},\ z = \frac{x_3}{x_4}$.

Then equation (7) becomes

$$x^2 + y^2 - z^2 = 1 \qquad (8)$$

which is the equation of a hyperboloid of one sheet.

In the case when x_4 is zero, the elements of S^3satisfying equation (4) belong to the improper plane of P^3 that closes the Euclidean space R^4 to give the projective plane P^3. This plane may well be regarded as a point at infinity were the hyperboloid of one sheet degenerates. For a more thorough investigation of this geometrical properties, consult the book of Seifert and Threllfall.

References

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