

FINDING LINKING SETS

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Abstract

We present the most general definition of the linking of sets in a Banach space and discuss necessary and sufficient conditions for sets to link.

1 Introduction

Many nonlinear problems in the physical and social sciences can be reduced to finding critical points (minima, maxima and minimax points) of functionals (real-valued functions on various spaces). The first critical points to be studied were maxima and minima, and much of the activity in the calculus of variations has been devoted to the finding of such points. A more difficult problem is to find critical points that are neither maxima nor minima.

It is desired to find sets with the following properties:

Definition 1.1. *Let A, B be subsets of a Banach space E . We shall say that A links B if the class of $G \in C_U1(E; \mathbb{R})$ satisfying*

$$(1.1) \quad a_0 := \sup_A G < b_0 := \inf_B G$$

*Supported in part by a NSF grant.

†Supported by Veteskapsådet. *AMS Subject Classification:* Primary 35J65, 58E05, 49B27. *Key words and phrases:* Critical point theory, variational methods, saddle point theory.

is non-empty, and for every such G there is a sequence $\{u_k\} \subset E$ and a constant c such that

$$(1.2) \quad b_0 \leq c < \infty$$

and

$$(1.3) \quad G(u_k) \rightarrow c, \quad G'(u_k) \rightarrow 0.$$

(In this definition, as well as in the rest of the paper, we consider functionals of the class $C_U^m \subset C^m$, i.e. functionals whose Fréchet derivatives up to the order m are uniformly continuous on bounded sets.)

The importance of the concept of linking stems from the fact that in many applications, a sequence satisfying (1.3) (called a Palais-Smale sequence) leads to a critical point of G . In particular, this is true if any sequence satisfying (1.3) has a convergent subsequence. (If this is the case, then G is said to satisfy the Palais-Smale condition.) Thus, a useful method of finding critical points of G is to find two subsets A, B of E such that A links B and (1.3) holds.

There are several sufficient conditions given in the literature which imply that a set A links a set B in this sense (cf., e.g., [BBF, BN, MW, N, R, Sc1, Si, ST, T]). The most comprehensive is that of [ST].

In [ST] our goal was to prove a theorem of the form:

Theorem 1.2. *Let G be a C^1 -functional on a Banach space E , and let A, B be subsets of E such that A has a certain relationship to B . Assume that*

$$(1.4) \quad a_0 := \sup_A G < b_0 := \inf_B G.$$

Then there is a sequence $\{u_k\} \subset E$ and an $a \geq b_0$ such that

$$(1.5) \quad G(u_k) \rightarrow a, \quad G'(u_k) \rightarrow 0.$$

The reason for the requirement that A have a relationship to B is that the theorem is obviously false if A and B are two arbitrary sets. The problem facing us was to find a very general relationship that would make the theorem true. We defined a relationship which indeed made the theorem true and was the most general known hypothesis that did so. We termed this relationship “linking.” (Previous authors used this term to describe more restrictive definitions.) We suspect that our definition of linking is the most general possible. In this paper we discuss two conditions, one slightly stronger than the other; the weaker one is necessary and the stronger one is sufficient for Theorem 1.2 to hold.

In the present paper, we define two relationships which are close to each other. The stronger one is sufficient for linking while the weaker one is necessary. We use the following maps.

Definition 1.3. We shall say that a map $\varphi : E \rightarrow E$ is of class Λ , if it is a homeomorphism onto E , and both φ, φ^{-1} are bounded on bounded sets. If, furthermore, $\varphi, \varphi^{-1} \in C1_U(E; E)$, we shall say that $\varphi \in \Lambda_U$.

Definition 1.4. We shall say that a bounded set A is **chained** to a set B if $A \cap B = \emptyset$ and

$$(1.6) \quad \inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\|, \quad \varphi \in \Lambda_U.$$

Definition 1.5. We shall say that a bounded A is **strongly chained** to B if $A \cap B = \emptyset$ and (1.6) holds for every $\varphi \in \Lambda$.

Definition 1.6. For $A \subset E$, we define

$$\mathcal{K}_U(A) = \{\varphi^{-1}(B_R) : \varphi \in \Lambda_U, \varphi(A) \subset B_R\},$$

where

$$B_R = \{u \in E : \|u\| < R\}.$$

Our main results are:

Theorem 1.7. If a bounded set A is strongly chained to B , then A links B .

Theorem 1.8. Let G be a $(C2 \cap C_U)$ -functional on E , and let A, B be nonempty subsets of E such that A is bounded and chained to B and

$$(1.7) \quad a_0 := \sup_A G \leq b_0 := \inf_B G.$$

Let

$$(1.8) \quad a := \inf_{K \in \mathcal{K}_U(A)} \sup_K G.$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$(1.9) \quad G(u_k) \rightarrow a, \quad G'(u_k) \rightarrow 0.$$

If $a = b_0$, then we can also require that

$$(1.10) \quad d(u_k, B) \rightarrow 0.$$

Theorem 1.9. If E is a Hilbert space and A links B , then it is chained to B .

Theorem 1.9 with additional assumptions is due to the authors elsewhere [ST1]. There it was proved that in a Hilbert space a compact set links another set if and only if it is chained to it.

2 The strong case

Definition 2.1. For $A \subset E$, we define

$$\Lambda(A) = \{\varphi \in \Lambda : \varphi(u) = u, \quad u \in A\}$$

and

$$\Lambda_U(A) = \{\varphi \in \Lambda_U : \varphi(u) = u, \quad u \in A\}$$

Lemma 2.2.

$$\begin{aligned} \varphi_1, \varphi_2 \in \Lambda &\implies \varphi_1 \circ \varphi_2 \in \Lambda. \\ \varphi_1, \varphi_2 \in \Lambda_U &\implies \varphi_1 \circ \varphi_2 \in \Lambda_U. \\ \varphi_1, \varphi_2 \in \Lambda(A) &\implies \varphi_1 \circ \varphi_2 \in \Lambda(A). \\ \varphi_1, \varphi_2 \in \Lambda_U(A) &\implies \varphi_1 \circ \varphi_2 \in \Lambda_U(A). \end{aligned}$$

Lemma 2.3.

$$\begin{aligned} \varphi \in \Lambda &\text{ iff } \varphi^{-1} \in \Lambda. \\ \varphi \in \Lambda_U &\text{ iff } \varphi^{-1} \in \Lambda_U. \\ \varphi \in \Lambda(A) &\text{ iff } \varphi^{-1} \in \Lambda(A). \\ \varphi \in \Lambda_U(A) &\text{ iff } \varphi^{-1} \in \Lambda_U(A). \end{aligned}$$

Definition 2.4. For $A \subset E$, we define

$$\mathcal{K}(A) = \{\varphi^{-1}(B_R) : \varphi \in \Lambda, \quad R > \sup_{x \in A} \|\varphi(x)\|\}$$

Lemma 2.5. If A is strongly chained to B and $K \in \mathcal{K}(A)$, then $K \cap B \neq \emptyset$.

Proof. There is a $\varphi \in \Lambda$ such that $K = \varphi^{-1}(B_R)$ with $R > \sup_{x \in A} \|\varphi(x)\|$. Since A is strongly chained to B , we have

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\| < R.$$

Hence, there is a $v \in B$ such that $\varphi(v) \in B_R$. Hence, $v = \varphi^{-1}\varphi(v) \in \varphi^{-1}B_R$. \square

Lemma 2.6. If $K \in \mathcal{K}(A)$ and $\sigma \in \Lambda(A)$, then $\sigma(K) \in \mathcal{K}(A)$.

Proof. There is a $\varphi \in \Lambda$ such that $K = \varphi^{-1}(B_R)$ with $R > \sup_{x \in A} \|\varphi(x)\|$. Let

$$\tilde{\varphi} = \varphi \circ \sigma^{-1}.$$

Then $\tilde{\varphi} \in \Lambda$ (Lemma 2.2). Now,

$$\tilde{\varphi}^{-1} = \sigma \circ \varphi^{-1}.$$

Hence,

$$\sigma(K) = \sigma[\varphi^{-1}(B_R)] = \tilde{\varphi}^{-1}(B_R)$$

Moreover,

$$\tilde{\varphi}(u) = \varphi(u), \quad u \in A.$$

Hence,

$$\sup_{x \in A} \|\tilde{\varphi}(x)\| = \sup_{x \in A} \|\varphi(x)\| < R.$$

Therefore, $\sigma(K) \in \mathcal{K}(A)$. □

We can now give the proof of Theorem 1.7.

Proof. Let G be a $(C1 \cap C_U)$ -functional on E , and let A, B be nonempty subsets of E such that A is strongly chained to B and (1.7) holds. Then

$$(2.1) \quad a := \inf_{K \in \mathcal{K}(A)} \sup_K G$$

is finite. By Lemma 2.5, $K \cap B \neq \emptyset$ for all $K \in \mathcal{K}(A)$. Hence, $a \geq b_0$. Assume first that $b_0 < a$. If (1.9) were false, there would exist a positive constant δ such that $3\delta < a - b_0$ and

$$(2.2) \quad \|G'(u)\| \geq 3\delta$$

whenever

$$(2.3) \quad u \in Q = \{u \in E : |G(u) - a| \leq 3\delta\}.$$

Since $G \in C1(E, \mathbb{R})$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into E such that

$$(2.4) \quad \|Y(u)\| \leq 1, \quad u \in \hat{E}$$

and

$$(2.5) \quad (G'(u), Y(u)) \geq 2\delta$$

whenever u satisfies (2.3). Let

$$\begin{aligned} Q_0 &= \{u \in E : |G(u) - a| \leq 2\delta\} \\ Q_1 &= \{u \in E : |G(u) - a| \leq \delta\} \\ Q_2 &= E \setminus Q_0 \\ \eta(u) &= d(u, Q_2) / [d(u, Q_1) + d(u, Q_2)]. \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1, u \in Q_1; \eta(u) = 0, u \in \bar{Q}_2; 0 < \eta(u) < 1, \text{ otherwise.}$$

Consider the differential equation

$$(2.6) \quad \sigma'(t) = W(\sigma(t)), \quad t \in \mathbf{R}, \sigma(0) = u,$$

where

$$(2.7) \quad W(u) = -\eta(u)Y(u).$$

The mapping W is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. Hence by a well known existence theorem for ordinary differential equations in a Banach space, (2.6) has a unique solution for all $t \in \mathbf{R}$. Let us denote the solution of (2.6) by $\sigma(t)u$. The mapping $\sigma(t)$ is in $C(E \times \mathbf{R}, E)$ and is called the flow generated by (2.7). Note that

$$(2.8) \quad \begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma(t)u), \sigma'(t)u) \\ &= -\eta(\sigma(t)u)(G'(\sigma(t)u), Y(\sigma(t)u)) \\ &\leq -2\delta\eta(\sigma(t)u). \end{aligned}$$

Thus,

$$\begin{aligned} G(\sigma(t)u) &\leq G(u), \quad t \geq 0, \\ G(\sigma(t)u) &\leq a_0, \quad u \in A, t \geq 0, \end{aligned}$$

and

$$\sigma(t)u = u, \quad u \in A, t \geq 0.$$

This follows from the fact that

$$G(\sigma(t)u) \leq a_0 \leq b_0 < a - 3\delta, \quad u \in A, t \geq 0.$$

Hence, $\eta(\sigma(t)u) = 0$ for $u \in A, t \geq 0$. This means that

$$\sigma'(t)u = 0, \quad u \in A, t \geq 0,$$

and

$$\sigma(t)u = \sigma(0)u = u, \quad u \in A, t \geq 0.$$

Let

$$(2.9) \quad E_\alpha = \{u \in E : G(u) \leq \alpha\}.$$

There is a $T > 0$ such that

$$(2.10) \quad \sigma(T)E_{a+\delta} \subset E_{a-\delta}.$$

In fact, we can take $T = 1$. Let u be any element in $E_{a+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(T)u) \leq G(\sigma(t_1)u) < a - \delta$$

by (2.8). Hence $\sigma(T)u \in E_{a-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all t , and (2.8) yields

$$(2.11) \quad G(\sigma(T)u) \leq G(u) - 2\delta T \leq a - \delta.$$

Hence (2.10) holds. Now by (2.1) there is a $K \in \mathcal{K}(A)$ such that

$$(2.12) \quad K \subset E_{a+\delta}.$$

Note that $\sigma(T) \in \Lambda(A)$. Let $\tilde{K} = \sigma(T)(K)$. Then $\tilde{K} \in \mathcal{K}(A)$ by Lemma 2.6. But

$$\sup_{\tilde{K}} G = \sup_{u \in \tilde{K}} G(\sigma(T)u) < a - \delta,$$

which contradicts (2.1), proving the theorem for the case $b_0 < a$.

Now assume $b_0 = a$. If there did not exist a sequence satisfying both (1.9) and (1.10), then there would be positive numbers ϵ, δ, T such that $\delta < \epsilon T$ and (2.2) holds whenever

$$u \in Q = \{u \in E : d(u, B) \leq 4T, |G(u) - a| \leq 3\delta\}.$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : d(u, B) \leq 3T, |G(u) - a| \leq 2\delta\} \\ Q_1 &= \{u \in E : d(u, B) \leq 2T, |G(u) - a| \leq \delta\}. \end{aligned}$$

Since $a = b_0$, we see that $Q_1 \neq \emptyset$. Define Q_2 and $\eta(u)$ as before and let $\sigma(t)$ be the flow generated by the mapping (2.7) with everything now with respect to the new sets Q_j . Let u be any element in $E_{a+\delta}$. If there is a $t_1 \leq T$ such that $\sigma(t_1)u \notin Q_1$, then either

$$(2.13) \quad G(\sigma(t_1)u) < a - \delta$$

or

$$(2.14) \quad d(\sigma(t_1)u, B) > 2T.$$

Since

$$\|\sigma(t)u - \sigma(t')u\| \leq |t - t'|$$

by (2.4), (2.14) implies

$$(2.15) \quad d(\sigma(t)u, B) > T, \quad 0 \leq t \leq T.$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$(2.16) \quad G(\sigma(T)u) \leq G(u) - 2\epsilon T \leq a + \delta - 2\delta = a - \delta.$$

Thus we have either

$$(2.17) \quad G(\sigma(T)u) < a - \delta$$

or (2.15) holds. Since $b_0 = a$, this shows that

$$(2.18) \quad \sigma(T)E_{a+\delta} \cap B = \phi.$$

We also note that

$$(2.19) \quad \sigma(t)A \cap B = \phi, \quad 0 \leq t \leq T.$$

For we have by (2.8)

$$G(\sigma(t)u) \leq a_0 - 2\epsilon \int_0^t \eta(\sigma(\tau)u) d\tau, \quad u \in A.$$

If $\sigma(t)u \in B$, we must have $G(\sigma(t)u) \geq b_0 \geq a_0$. The only way this can happen is if

$$\eta(\sigma(\tau)u) \equiv 0, \quad 0 \leq \tau \leq t.$$

But this implies $\sigma(\tau)u \in \bar{Q}_2$ for such τ , and this in turn implies either

$$G(\sigma(\tau)u) < a - \delta, \quad 0 \leq \tau \leq t$$

or

$$d(\sigma(\tau)u, B) > 2T, \quad 0 \leq \tau \leq t.$$

In either case we cannot have $\sigma(t)u \in B$. Thus (2.19) holds. Let $K \in \mathcal{K}(A)$ satisfy (2.12), and let $\tilde{K} = \sigma(T)K$. Then $\tilde{K} \in \mathcal{K}(A)$. But (2.18) and (2.19) imply that $\tilde{K} \cap B = \phi$, contradicting the fact that A is strongly chained to B . This completes the proof of the theorem. \square

3 The remaining proofs

Lemma 3.1. *If A is chained to B and $K \in \mathcal{K}_U(A)$, then $K \cap B \neq \phi$.*

Proof. There is a $\varphi \in \Lambda_U$ such that $K = \varphi^{-1}(B_R)$ with $\varphi(A) \subset B_R$. Since A is chained to B , we have

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\| < R.$$

Hence, there is a $v \in B$ such that $\varphi(v) \in B_R$. Hence, $v = \varphi^{-1}\varphi(v) \in \varphi^{-1}B_R$. \square

Lemma 3.2. *If $K \in \mathcal{K}_U(A)$ and $\sigma \in \Lambda_U(A)$, then $\sigma(K) \in \mathcal{K}_U(A)$.*

We can now give the proof of Theorem 1.8.

Proof. Let G be a $(C^2 \cap C_U)$ -functional on E , and let A, B be nonempty subsets of E such that A is chained to B and (1.7) holds. Then a given by (1.8) is finite. By Lemma 3.1, $K \cap B \neq \emptyset$ for all $K \in \mathcal{K}_U(A)$. Hence, $a \geq b_0$. Assume first that $b_0 < a$. If (1.9) were false, there would exist a positive constant δ such that $3\delta < a - b_0$ and

$$(3.1) \quad \|G'(u)\| \geq 3\delta$$

whenever

$$(3.2) \quad u \in Q = \{u \in E : |G(u) - a| \leq 3\delta\}.$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : |G(u) - a| \leq 2\delta\} \\ Q_1 &= \{u \in E : |G(u) - a| \leq \delta\} \\ Q_2 &= E \setminus Q_0 \\ \eta(u) &= d(u, Q_2) / [d(u, Q_1) + d(u, Q_2)]. \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1, \quad u \in Q_1; \quad \eta(u) = 0, \quad u \in \bar{Q}_2; \quad 0 < \eta(u) < 1, \quad \text{otherwise.}$$

Consider the differential equation

$$(3.3) \quad \sigma'(t) = W(\sigma(t)), \quad t \in \mathbf{R}, \sigma(0) = u,$$

where

$$(3.4) \quad W(u) = -\eta(u)G'(u)/\|G'(u)\|.$$

Since $G \in C^2(E, \mathbb{R})$, the mapping W is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. Hence by a well known existence theorem for ordinary differential equations in a Banach space, (3.3) has a unique solution for all $t \in \mathbf{R}$. Let us denote the solution of (3.3) by $\sigma(t)u$. The mapping $\sigma(t)$ is in $C^1_U(E \times \mathbb{R}, E)$ and is called the flow generated by (3.4). Note that

$$(3.5) \quad \begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma(t)u), \sigma'(t)u) \\ &= -\eta(\sigma(t)u)\|G'(\sigma(t)u)\| \\ &\leq -2\delta\eta(\sigma(t)u). \end{aligned}$$

Thus,

$$\begin{aligned} G(\sigma(t)u) &\leq G(u), \quad t \geq 0, \\ G(\sigma(t)u) &\leq a_0, \quad u \in A, t \geq 0 \end{aligned}$$

and

$$\sigma(t)u = u, \quad u \in A, t \geq 0.$$

Again, this follows from the fact that

$$G(\sigma(t)u) \leq a_0 \leq b_0 < a - 3\delta, \quad u \in A, t \geq 0.$$

Hence, $\eta(\sigma(t)u) = 0$ for $u \in A, t \geq 0$. This means that

$$\sigma'(t)u = 0, \quad u \in A, t \geq 0,$$

and

$$\sigma(t)u = \sigma(0)u = u, \quad u \in A, t \geq 0.$$

Let

$$(3.6) \quad E_\alpha = \{u \in E : G(u) \leq \alpha\}.$$

We note that there is a $T > 0$ such that

$$(3.7) \quad \sigma(T)E_{a+\delta} \subset E_{a-\delta}.$$

(In fact, we can take $T = 1$.) Let u be any element in $E_{a+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(T)u) \leq G(\sigma(t_1)u) < a - \delta$$

by (3.5). Hence $\sigma(T)u \in E_{a-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all t , and (3.5) yields

$$(3.8) \quad G(\sigma(T)u) \leq G(u) - 2\delta T \leq a - \delta.$$

Hence (3.7) holds. Now by (1.8) there is a $K \in \mathcal{K}_U(A)$ such that

$$(3.9) \quad K \subset E_{a+\delta}.$$

Note that $\sigma(T) \in \Lambda_U(A)$. Let $\tilde{K} = \sigma(T)(K)$. Then $\tilde{K} \in \mathcal{K}_U(A)$ by Lemma 3.2. But

$$\sup_{\tilde{K}} G = \sup_{u \in \tilde{K}} G(\sigma(T)u) < a - \delta,$$

which contradicts (1.8), proving the theorem for the case $b_0 < a$.

Now assume that $b_0 = a$. If there did not exist a sequence satisfying both (1.9) and (1.10), then there would be positive numbers ϵ, δ, T such that $\delta < \epsilon T$ and (2.2) holds whenever

$$u \in Q = \{u \in E : d(u, B) \leq 4T, |G(u) - a| \leq 3\delta\}.$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : d(u, B) \leq 3T, |G(u) - a| \leq 2\delta\} \\ Q_1 &= \{u \in E : d(u, B) \leq 2T, |G(u) - a| \leq \delta\}. \end{aligned}$$

Since $a = b_0$, we see that $Q_1 \neq \phi$. Define Q_2 and $\eta(u)$ as before and let $\sigma(t)$ be the flow generated by the mapping (3.4) with everything now with respect to the new sets Q_j . Let u be any element in $E_{a+\delta}$. If there is a $t_1 \leq T$ such that $\sigma(t_1)u \notin Q_1$, then either

$$(3.10) \quad G(\sigma(t_1)u) < a - \delta$$

or

$$(3.11) \quad d(\sigma(t_1)u, B) > 2T.$$

Since

$$\|\sigma(t)u - \sigma(t')u\| \leq |t - t'|,$$

(3.11) implies

$$(3.12) \quad d(\sigma(t)u, B) > T, \quad 0 \leq t \leq T.$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$(3.13) \quad G(\sigma(T)u) \leq G(u) - 2\epsilon T \leq a + \delta - 2\delta = a - \delta.$$

Thus we have either

$$(3.14) \quad G(\sigma(T)u) < a - \delta$$

or (3.12) holds. Since $b_0 = a$, this shows that

$$(3.15) \quad \sigma(T)E_{a+\delta} \cap B = \phi.$$

We also note that

$$(3.16) \quad \sigma(t)A \cap B = \phi, \quad 0 \leq t \leq T.$$

For we have by (3.5)

$$G(\sigma(t)u) \leq a_0 - 2\epsilon \int_0^t \eta(\sigma(\tau)u) d\tau, \quad u \in A.$$

If $\sigma(t)u \in B$, we must have $G(\sigma(t)u) \geq b_0 \geq a_0$. The only way this can happen is if

$$\eta(\sigma(\tau)u) \equiv 0, \quad 0 \leq \tau \leq t.$$

But this implies $\sigma(\tau)u \in \bar{Q}_2$ for such τ , and this in turn implies either

$$G(\sigma(\tau)u) < a - \delta, \quad 0 \leq \tau \leq t$$

or

$$d(\sigma(\tau)u, B) > 2T, \quad 0 \leq \tau \leq t.$$

In either case we cannot have $\sigma(t)u \in B$. Thus (3.16) holds. Let $K \in \mathcal{K}_U(A)$ satisfy (3.9), and let $\tilde{K} = \sigma(T)K$. Then $\tilde{K} \in \mathcal{K}_U(A)$. But (3.15) and (3.16) imply that $\tilde{K} \cap B = \emptyset$, contradicting the fact that A is chained to B . This completes the proof of the theorem. \square

Now we give the proof of Theorem 1.9.

Proof. Assume that $\varphi \in \Lambda_U$ does not satisfy (1.6), and let $G(u) = \|\varphi(u)\|^2$. Then by the definition of the class Λ_U , $G \in C^1_U(E, \mathbb{R})$, $\sup G(A) < \inf G(B)$ and G has no critical level $c \geq \inf_{x \in B} G(x) > 0$. To show the latter, assume that there is a sequence u_k satisfying (1.3). Then we have, for any bounded sequence $v_k \in E$,

$$(3.17) \quad (\varphi'(u_k)v_k, \varphi(u_k)) \rightarrow 0.$$

Let $v_k := (\varphi'(u_k))^{-1}(\varphi(u_k))$. Then $(\varphi'(u_k)v_k, \varphi(u_k)) = G(u_k) \rightarrow c$. However, the sequence v_k is bounded: $G(u_k) \rightarrow c$ implies that $\varphi(u_k)$ is bounded, which implies that u_k , and consequently that $(\varphi'(u_k))^{-1}$ and v_k are also bounded. Hence, $G'(u_k) \rightarrow 0$ implies $G(u_k) \rightarrow 0$, showing that $c = 0$. Thus, A does not link B . \square

4 Equivalent linking conditions

In this section we relate our present definition of linking to the one of [ST].

Let $A, B \subset E$ be disjoint sets and assume that A is bounded. Let Φ be a set of all maps $\psi \in C_U([0, 1] \times E; E)$ such that $\psi(0, \cdot) = id_E$ and that there exists a point $\xi \in E$ such that $\psi(t, \cdot)$ converges to ξ as $t \rightarrow 1$ uniformly on bounded sets. Let $\Psi \subset \Phi$. We shall say that A is Ψ -chained to B if for every $\psi \in \Psi$,

$$(4.1) \quad \psi(A, [0, 1]) \cap B \neq \emptyset.$$

Proposition 4.1. *If A is Φ -chained to B then A is strongly chained to B .*

Proof. Assume that A is not strongly chained to B . Then there is a map $\varphi \in \Lambda$ and a $R > 0$ such that $\varphi(A) \subset B_R(0)$ and $\varphi(B) \cap B_R(0) = \emptyset$. Let $\psi(t, u) = (1 - t)u$. Then $\psi \in \Phi$ and $[\psi(t, \cdot) \circ \varphi](A) \subset B_R(0)$. Therefore $[\psi(t, \cdot) \circ \varphi](A) \cap \varphi(B) = \emptyset$. This implies that $[\varphi^{-1} \circ \psi(t, \cdot) \circ \varphi](A) \cap B = \emptyset$. Since $\varphi^{-1} \circ \psi(t, \cdot) \circ \varphi \in \Phi$, this means that A is not Φ -chained to B , a contradiction. \square

Corollary 4.2. *If A is Φ -chained to B then A links B .*

Let $\Phi_0 := \{\psi \in \Phi : \psi(t, \cdot) \in C_U^1, \exists \varepsilon_t, \|\psi'(t, \cdot)\| \geq \varepsilon_t, t \in (0, 1)\}$

Proposition 4.3. *If E is a Hilbert space, A is strongly chained to B , $d(A, B) > 0$, then A is Φ_0 -chained to B .*

Proof. If A is strongly chained to B then A links B (Theorem 1.7). Assume that A is not Φ_0 -chained to B . Then there is a map $\psi \in \Phi_0$ and a point $\xi \in E$ such that $\psi(t, \cdot) \rightarrow_{t \rightarrow 1} \xi$ and $\psi([0, 1], A) \cap B = \emptyset$. Since A is bounded, $\psi(t, \cdot)$ converges to ξ uniformly on A . Then there is a $\tau \in (0, 1)$ and a $r > 0$ such that $\psi(\tau, A) \subset B_r(\xi)$ and $B_{2r}(\xi) \cap B = \emptyset$. Let $G(u) = \|\psi(\tau, u) - \xi\|$. The functional G is in C^v and it does not have a critical level $c > r$ since $\|\psi'(\tau, \cdot)\|$ is bounded away from zero. \square

Corollary 4.4. *If E is a Hilbert space and A links B , then A is Φ_0 -chained to B .*

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