GEOMETRY/TOPOLOGY – LECTURE 2

1. Homotopy and the fundamental group

All maps considered are assumed to be continuous. The material discussed makes sense for all topological spaces. The reader unfamiliar with abstract topological spaces may restrict to topological spaces that are subsets of some finite dimensional Euclidean space, or of a Banach space, or some more general metric space.

1.1. Homotopy.

Let X and Y be topological spaces, and let $I = [0,1]$.

1.1.A. *Definition and basic properties.*

Definition 1. *Two maps* $f_0, f_1 \colon X \to Y$ *are* homotopic *if there exists a map*

$$F \colon X \times I \to Y$$

such that $F|_{X\times\{0\}} = f_0$ *and* $F|_{X\times\{1\}} = f_1$. *The map* F *is called a* homotopy *between* f_0 *and* f_1.

Note that homotopy is an equivalence relation on the set $C^0(X,Y)$ of continuous maps $X \to Y$ and if f is homotopic to g we write $f \simeq g$. We next observe that maps induce functions between homotopy classes by composition. More precisely, we have the following.

Proposition 2. *If* $h \colon X' \to X$ *and* $k \colon Y \to Y'$ *are maps and if* $f \simeq g \colon X \to Y$ *then* $h \circ f \circ k \simeq h \circ g \circ k$.

1.1.B. *Homotopy equivalence and contractible spaces.*

Definition 3. *A map* $f \colon X \to Y$ *is a* homotopy equivalence *with homotopy inverse* $g \colon Y \to X$ *if* $f \circ g \simeq \mathrm{id}_X$ *and* $g \circ f \simeq \mathrm{id}_Y$. *In case there exits such a map one says* X *and* Y *are* homotopy equivalent.

Examples of homotopy equivalent spaces:

- $\mathbb{R}^n - \{0\} \simeq S^{n-1}$,
- band $\simeq$ Möbius band,
- $GL(n) \simeq O(n)$,
- Let Q denote the space of positive definite quadratic forms on $\mathbb{R}^2$. Then $Q \simeq$ point.
- If Q' denote the space of positive definite quadratic forms on $\mathbb{R}^2$ with distinct eigenvalues then $Q' \simeq S^1$.

Definition 4. *A space that is homotopy equivalent to a point is called* contractible.

Definition 5. *If* $A \subset X$ *then a homotopy* $F \colon X \times I \to Y$ *is a homotopy* relative A *if it is fixed on* A, *i.e. if* $F(a,t) = F(a,0)$ *for all* $a \in A$ *and* $t \in I$.

Definition 6. *A subset* $A \subset X$ *is a* strong deformation retract *of* X *is there is a homotopy rel* A *from* 1_X *to a map with image in* A. *(It is a deformation retract if the restriction to* A *is required to be the identity only for* $t = 1$.*)*

1.1.C. *Concatenation.* Let $F, G\colon X\times I\to Y$ be homotopies with $F|_{X\times\{1\}} = G_{X\times\{0\}}$. Then the *concatentation* $F\star G\colon X\times I\to Y$ is the homotopy F followed by the homotopy G. In formulas

$$F\star G = \begin{cases} F(x,2t) & 0\le t\le \frac{1}{2}, \\ G(x,2t-1) & \frac{1}{2}\le t\le 1. \end{cases}$$

Here the exact parameterization of I does not matter very much as the following result shows:

Lemma 7. *Let $\phi_1,\phi_2\colon (I,\partial I)\to(I,\partial I)$ be maps equal on ∂I. Let $F\colon X\times I\to Y$ be a homotopy and define*

$$G_i = F(x,\phi_i(t)),\quad i=1,2,$$

then $G_1\simeq G_2$ rel $X\times\partial I$.

Proof.

$$H(x,t,s) = F(x, s\phi_2(t) + (1-s)\phi_1(t))$$

gives a homotopy. □

To prepare for the definition of the fundamental group we make three observations.

First, let $C\colon X\times I\to Y$ denote a constant homototopy (i.e. constant in $t\in I$). Assume that $F\colon X\times I\to Y$ is a homotopy then

$$F\star C\simeq F \text{ rel } X\times\partial I,$$
$$C\star F\simeq F \text{ rel } X\times\partial I,$$

provided C is such that the concatenations in the left hand sides are defined.

Second, the homotopy $F^{-1}(x,t) = F(x,1-t)$ is called the *inverse* homotopy of F and

$$F\star F^{-1}\simeq C.$$

Third if $F, G, H\colon X\times I\to Y$ are homotopies such that $F\star G$ and $G\star H$ are defined then

$$(F\star G)\star H \;\simeq\; F\star(G\star H) \quad \text{rel } X\times\partial I$$