GEOMETRY/TOPOLOGY – LECTURE 3

1. The fundamental group

Let X be a topological space and fix $x \in X$. A *loop based a x* is a map $\gamma\colon I \to X$ with $\gamma(0) = \gamma(1) = x$. Two loops γ and β based at x are homotopic if

$$\gamma \;\simeq\; \beta \quad \text{rel } \partial I.$$

We write $[\gamma]$ for the homotopy class of γ.

1.1. **Definition and basic properties.**

Theorem 1. *The set of homotopy classes of loops based at x form a group with multiplication*

$$[\gamma] \cdot [\beta] = [\gamma \star \beta]$$

and with inverse

$$[\gamma]^{-1} = [\gamma^{-1}],$$

where we view γ as a homotopy of a 1-point space, the unit is the class of the constant loop at x.

Proof. This follows from the three observations about concatentaion made at the end of Lecture 2. □

The group in the above theorem is called the fundamental group (or the Poincaré group) and is denoted $\pi_1(X, x)$.

Theorem 2. *If X is path connected then $\pi_1(X, p) \approx \pi_1(X, q)$ for any $p, q \in X$.*

Proof. Let $\omega\colon I \to X$ be a path with $\omega(0) = p$ and $\omega(1) = q$. Then concatenation with ω defines a maps between $\pi_1(X, p)$ and $\pi_1(X, q)$. More precisely, we have

$$\phi\colon \pi_1(X, p) \to \pi_1(X, q), \quad \phi([\gamma]) = [\omega \star \gamma \star \omega^{-1}]$$

and

$$\psi\colon \pi_1(X, q) \to \pi_1(X, p), \quad \psi([\beta]) = [\omega^{-1} \star \beta \star \omega].$$

First,

$$\begin{aligned}\phi([\gamma][\alpha]) &= \phi([\gamma \star \alpha]) = [\omega \star \gamma \star \alpha \star \omega^{-1}] \\ &= [\omega \star \gamma \star \omega^{-1} \star \omega \star \alpha \star \omega^{-1}] = [\omega \star \gamma \star \omega^{-1}] \cdot [\omega \star \alpha \star \omega^{-1}] \\ &= \phi([\gamma]) \cdot \phi([\alpha]),\end{aligned}$$

so that ϕ is a homomorphism. Likewise ψ is and moreover

$$\phi \circ \psi([\beta]) = [\omega \star \omega^{-1} \star \beta \star \omega \star \omega^{-1}] = [\beta].$$

The lemma follows. □

1.2. **Maps and induced homomorphisms.** Let $f\colon X \to Y$ be a map with $f(x) = y$. If γ is a loop in X based at x then $f \circ \gamma$ is a loop in Y based at Y. Define the map $f_\#\colon \pi_1(X, x) \to \pi_1(Y, y)$ as follows

$$f_\#([\gamma]) = [f \circ \gamma].$$

Theorem 3. *The map $f_\#$ is a homomorphism.*

Proof.

$$\begin{aligned} f_\#([\gamma] \cdot [\beta]) &= f_\#([\gamma \star \beta]) = [f \circ (\gamma \star \beta)] \\ &= [(f \circ \gamma) \star (f \circ \beta)] = [f \circ \gamma] \cdot [f \circ \beta] \\ &= f_\#([\gamma]) \cdot f_\#([\beta]). \end{aligned}$$

□

Theorem 4. *Homotopic maps induce the same homomorphism on π_1.*

Proof. If f_0 and f_1 are homotopic then $[f_0 \circ \gamma] = [f_1 \circ \gamma]$. □

Corollary 5. *Homotopy equivalent spaces have isomorphic fundamental groups.*

In fact there is an infinite set of homotopy groups $\pi_1(X), \pi_2(X), \dots$ such that if X and Y are spaces then X and Y are (weakly) homotopy equivalent if and only if $\pi_j(X) \approx \pi_j(Y)$ for all j.

Examples: $\pi_1(\mathbb{R}^n) = \{1\}$ and $\pi_1(D^n) = 1$. Also $\pi_1(S^n) = 1$ if $n > 1$, see Lecture 4.

Theorem 6. *The fundamental group of the circle S^1 is $\pi_1(S^1) \approx \mathbb{Z}$. The element in $\mathbb{Z}$ corresponding to $\gamma\colon I \to S^1$ with $\gamma(0) = \gamma(1) = 1$ can be computed by lifting γ to $\mathbb{R}$: find the unique continuous $\theta\colon I \to \mathbb{R}$ such that*

$$\gamma(t) = e^{2\pi i\theta(t)}, \quad \theta(0) = 0$$

and evaluate at 1, i.e. the homotopy class of γ is given by $\theta(1) \in \mathbb{Z}$.

Proof. Note that the endpoint $\theta(1)$ of θ depends continuously on γ. We thus have an integer valued continuous function on loops in S^1. Since integer valued continuous functions are locally constant we find that it gives a map $\pi_1(S^1, 1) \to \mathbb{Z}$. This map is a homomorphism by the definition of the product as concatenation. The map is obviously surjective and injective since any loop in $\mathbb{R}$ is contractible. □