GEOMETRY/TOPOLOGY – PROBLEMS 1

1. Classify regular closed curves up to regular homotopy on S^2.

2. Classify regular closed curves up to regular homotopy on T^2.

3. Classify regular closed curves up to regular homotopy on S^1.

4. Classify regular closed curves up to regular homotopy in $\mathbb{R}^n$, $n > 2$.

5. Think of S^2 as the unit sphere in $\mathbb{R}^3$ and let (x, y, z) be standard coordinates on $\mathbb{R}^3$. Let $F\colon S^1 \times [0,1] \to S^2$ be a smooth homotopy such that

$$F(t, 0) = (\cos t, \sin t, 0) \quad \text{and} \quad F(t, 1) = (\cos 2t, \sin 2t, 0).$$

Estimate the area of F,

$$\operatorname{area}(F) = \int_{S^1 \times [0,1]} \left| \frac{\partial F}{\partial t} \times \frac{\partial F}{\partial s} \right| dt\, ds,$$

where $\times$ denote the cross product in $\mathbb{R}^3$, from below. Is your estimate sharp?

6. Let $f\colon \mathbb{R} \to \mathbb{R}^n$ be Hölder continuous with exponent $\alpha \in (0, 1)$, i.e. there exists a constant C such that $|f(x) - f(y)| \le C|x - y|^\alpha$ for all $x, y \in \mathbb{R}$. Estimate the Hausdorff dimension of the image of f.

7. Let $U(n)$ denote the space of unitary complex $n \times n$-matrices and let $SU(n) \subset U(n)$ denote the subspace of the matrices of determinant 1. Compute $\pi_1(U(n))$ and $\pi_1(SU(n))$.

8. Let Q denote the space of positive definite quadratic forms on $\mathbb{R}^3$. Let $Q' \subset Q$ denote the space of forms with at least two distinct eigenvalues, and let $Q'' \subset Q'$ denote the space of forms with three distinct eigenvalues. Compute $\pi_1(Q)$, $\pi_1(Q')$, and $\pi_1(Q'')$.

9. Let X'_d denote the space of complex polynomials of degree d in one variable with at least $d-1$ distinct roots and let $X_d \subset X'_d$ denote the space of polynomials with d distinct roots. Compute $\pi_1(X'_d)$ and $\pi_1(X_d)$.

10. Consider the region $A = \{(x, y, z) \in \mathbb{R}^3 \colon 10^{-2} \le x^2 + y^2 + z^2 \le 10^2\}$. Let l_1, l_2, and l_3 be line segments in A with endpoints in ∂A as follows:

$$\begin{aligned} l_1\colon &\quad (-10, 0, 0) \text{ and } (-0.1, 0, 0),\\ l_2\colon &\quad (0, 0, -10) \text{ and } (0, 0, -0.1),\\ l_3\colon &\quad (0, 0, 0.1) \text{ and } (0, 0, 10). \end{aligned}$$

Rotating the inner boundary of A two full turns around the z-axis deforms the line segment l_1' (which is assumed completely elastic and flexible) to a curve as in the figure below.

Show the following:

- If the line segments l_2 and l_3 are made of steel (i.e. completely fixed) then we cannot untangle l_1', i.e. deform it continuously to l_1, keeping ∂A fixed.
- If the line segments l_2 and l_3 are also flexible (and allowed to move) then we can untangle l_1'.

11. Draw a step-by-step movie of a deformation of l_1' to l_1 for flexible l_2, l_3 in Problem 10. Alternatively build a physical model with three strings attached to a small ball and show how to untangle it after two full rotations of the ball around an axis.