A VERSION OF RATIONAL SFT FOR EXACT LAGRANGIAN COBORDISMS IN 1-JET SPACES

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Abstract. The following expository paper is based on two lectures given by the author at a workshop on Symplectic Field Theory in Leipzig, August 2006. First, a version of rational Symplectic Field Theory for an exact Lagrangian submanifold L of a symplectization of a 1-jet space with components subdivided into k subsets is outlined. The theory associates to L a $\mathbb{Z}$-graded chain complex of vector spaces over $\mathbb{Z}_2$, filtered with k filtration levels. The corresponding k-level spectral sequence is invariant under deformations and its functorial properties lead to Legendrian isotopy invariants. Second, the theory outlined is computed in basic examples.

1. Introduction

In this paper we give an expository description of a version of rational SFT for exact Lagrangian cobordisms of symplectizations of 1-jet spaces and compute it in basic examples. We restrict attention to 1-jet spaces because of the technical simplifications this leads to. The theory in more general settings is described in [3], which also serves as the main reference for proofs and details left out in Sections 2 – 5 of this paper. In fact, these sections follow [3] quite closely, whereas the material in Sections 6 has not appeared elsewhere.

Let M be a smooth manifold. The 1-jet space of M is the space $J^1(M) = T^*M \times \mathbb{R}$. It carries a standard contact structure given by the contact form $\lambda = dz - p\,dq$, where z is a linear coordinate on the $\mathbb{R}$-factor and where $p\,dq$ denotes the canonical 1-form on T^*M: a point P in T^*M represents a form on T_QM for some $Q \in M$ and $p\,dq$ at P is the pull-back of that form to $T_P(T^*M)$. The *Reeb vector field* R_λ of λ is simply ∂_z, the unit vector field along the $\mathbb{R}$-factor. The *symplectization* of $J^1(M)$ is the symplectic manifold $J^1(M) \times \mathbb{R}$ with symplectic form $\omega_0 = d\alpha$, where $\alpha = e^t\lambda$ and where t is a coordinate in the additional $\mathbb{R}$-factor.

An *exact Lagrangian cobordism in* $J^1(M) \times \mathbb{R}$ consists of the following data.

- A 1-form β on $J^1(M) \times \mathbb{R}$ such that $d\beta$ is a symplectic form and such that $\beta = \alpha$ for $|t| > T$, some $T > 0$.
- An exact Lagrangian submanifold $L \subset (J^1(M)\times\mathbb{R}, d\beta)$ such that, in the regions $\{t > T\}$ and $\{t < -T\}$, $L = \Lambda^+ \times \mathbb{R}_+$ and $L = \Lambda^- \times \mathbb{R}_-$, respectively, where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$, and where Λ^+ and Λ^- are Legendrian

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submanifolds. (The submanifold L being exact means that $\beta|_L = df$ for some function $f\colon L \to \mathbb{R}$.)

We will call $\Lambda^+ \times \mathbb{R}_+$ and $\Lambda^- \times \mathbb{R}_-$ *positive- and negative ends* of L, respectively, and Λ^+ and Λ^-, $(+\infty)$- *and* $(-\infty)$-*boundaries* of L, respectively.

To define SFT in this setting one should count J-holomorphic curves in $J^1(M) \times \mathbb{R}$ with boundary on L asymptotic to Reeb chord strips of L at boundary punctures, where J is an almost complex structure adjusted to the symplectic form on $J^1(M) \times \mathbb{R}$. (A Reeb chord of L is a flow segment of the Reeb vector field in $J^1(M)$ which begins and ends on $\Lambda^\pm$. No holomorphic curves in $J^1(M)\times\mathbb{R}$ have interior punctures since the Reeb vector field in $J^1(M)$ does not have any closed orbits.) Turning such curve counts into algebra is however not straightforward because of a phenomenon often called boundary bubbling. For holomorphic curves without boundary in a symplectic manifold, generic bubbling off is described by the following local model: $\{(z,w) \in \mathbb{C}^2\colon z^2 \pm w^2 = \epsilon\}$, where $\epsilon \in \mathbb{C}$, $\epsilon \to 0$. Hence it is a codimension two phenomenon and can often be disregarded when setting up homology theories. Boundary bubbling for holomorphic curves with boundary on a Lagrangian submanifold can be modeled by a restricted version of this local model as follows. The Lagrangian submanifold corresponds to $\mathbb{R}^2 \subset \mathbb{C}^2$, the deformation parameter is constrained to be real, $\epsilon \in \mathbb{R} \subset \mathbb{C}$, and the curve is half of the curve in the model for curves without boundary. Thus, bubbling off at the boundary is a codimension one phenomenon which cannot be disregarded when setting up homology theories. In Lagrangian Floer homology, techniques for dealing with boundary bubbling have been developed in [12] and in [2]. In the present paper we will construct a version of SFT counting only rational curves. Although our treatment of boundary bubbling differs from those of [12] and [2] the end result is closely related, see the discussion at the beginning of Section 6.

In order to formulate our main results we introduce the following notation. Let (L,β) be an exact Lagrangian cobordism in $J^1(M) \times \mathbb{R}$ (we often suppress the 1-form β from the notation). Let $\Lambda^\pm \times \mathbb{R}_\pm$ be the ends of L and write $(J^1(M) \times [-T,T], \bar{L})$ for the finite part obtained by cutting the infinite parts of the cylindrical ends off at $|t| = T$. We will sometimes think of Reeb chords of $\Lambda^\pm$ in the $(\pm\infty)$-boundary as lying in $J^1(M) \times \{\pm T\}$ with endpoints on $\partial\bar{L}$. A formal disk of L is a homotopy class of maps of the 2-disk D, with m marked disjoint closed subintervals in ∂D, into $J^1(M) \times [-T,T]$, where the m marked intervals are required to map in an orientation preserving (reversing) manner to Reeb chords of $\partial\bar{L}$ in the $(+\infty)$-boundary (in the $(-\infty)$-boundary) and where remaining parts of the boundary ∂D map to $\bar{L}$. Assume that the set of connected components of L has been subdivided into subsets (we call such subsets *pieces*). In Subsection 2.1, we define the notion of an *admissible* formal disk of L in this situation. A J-holomorphic disk in $J^1(M) \times \mathbb{R}$ with boundary on L determines a formal disk and if this formal disk is admissible then boundary bubbling is impossible for topological reasons.

Let $\mathbf{V}(L)$ denote the $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$ generated by admissible formal disks of L with the following grading: the degree of a formal disk is the formal dimension of the moduli space of J-holomorphic disks homotopic to the formal disk. The degree of a formal disk is thus computed in terms of the Maslov index of the boundary condition it determines and the number of Reeb chords it has, see Subsection 3.1. We use the filtration $0 \subset F^k\mathbf{V}(L) \subset \cdots \subset F^2\mathbf{V}(L) \subset F^1\mathbf{V}(L) = \mathbf{V}(L)$, where k is the number of pieces of L and where the filtration level is determined by the number of Reeb chords of a formal disk which are in the $(+\infty)$-boundary of L. (By Lemma 3.2, an admissible formal disk has at most k Reeb chords at the positive end). In Subsection 3.5, using the collections of all rigid admissible J-holomorphic disks and of all 1-parameter families of admissible J-holomorphic disks in the symplectizations of the Legendrian manifolds in the $(\pm\infty)$-boundary of L, we define a differential $d\colon \mathbf{V}(L) \to \mathbf{V}(L)$. The differential increases grading by 1 and respects the filtration. Define the *rational admissible SFT spectral sequence* $\left\{E_r^{p,q}(L)\right\}_{r=1}^{k}$ as the spectral sequence induced by the filtration preserving differential $d\colon \mathbf{V}(L) \to \mathbf{V}(L)$, see Subsection 3.6.

Theorem 1.1. *Let L be an exact cobordism in $J^1(M)\times\mathbb{R}$ with a subdivision $L = L_1 \cup \cdots \cup L_k$ into pieces. Then $\left\{E_r^{p,q}(L)\right\}$ does not depend on the choice of adjusted almost complex structure J, and is invariant under compactly supported exact deformations of L and β.*

Theorem 1.1 is proved in Section 5. The spectral sequence in Theorem 1.1 share many of the familiar properties of rational SFT in the non-relative case, see Sections 3 and 4.

If $\Lambda \subset J^1(M)$ is a Legendrian submanifold which is subdivided into pieces $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$ then $(\Lambda \times \mathbb{R}, \alpha)$ is an exact cobordism with $\Lambda \times \mathbb{R}$ subdivided into pieces. We define the *rational admissible SFT-invariant of* Λ as

$$\left\{E_r^{p,q}(\Lambda)\right\} = \left\{E_r^{p,q}(\Lambda\times\mathbb{R})\right\},$$

where the right hand side is the spectral sequence described as above.

Theorem 1.2. *If $\Lambda \subset J^1(M)$ is a Legendrian submanifold of $J^1(M)$ then $\left\{E_r^{p,q}(\Lambda)\right\}$ is invariant under Legendrian isotopies of Λ.*

Theorem 1.2 is proved in Section 5. In Section 6 we discuss how to apply it in general and illustrate this by computing the theory for three parallel copies of the 0-section in $J^1(M)$.

2. The vector space of an exact cobordism

In this section we introduce admissible formal disks. Using them, we associate a graded filtered vector space to an exact Lagrangian cobordism.

2.1. **Admissible formal disks.** Let L be an exact cobordism in $J^1(M) \times \mathbb{R}$ with $(\pm\infty)$-boundary $\Lambda^{\pm}$. Recall the subdivision $L = \bar{L} \cup (\Lambda^+ \times \mathbb{R}_+) \cup (\Lambda^- \times \mathbb{R}_-)$ into a finite part in $J^1(M) \times [-T, T]$ and ends. Make the identification $\partial\bar{L} = \Lambda^+ \cup \Lambda^-$. Assume that L comes equipped with a subdivision $L = L_1 \cup \cdots \cup L_k$ into pieces where each piece L_j is a union of connected components of L. This subdivision induces a subdivision of the ends, $\Lambda^{\pm} = \Lambda_1^{\pm} \cup \cdots \cup \Lambda_k^{\pm}$. Let $\mathcal{R}^+$ and $\mathcal{R}^-$ denote the sets of Reeb chords of Λ^+ and Λ^-, respectively. We write $\mathcal{R}^{\pm} = \mathcal{R}^{\pm}_{\mathrm{pu}} \cup \mathcal{R}^{\pm}_{\mathrm{mi}}$. Here $\mathcal{R}^{\pm}_{\mathrm{pu}}$ contains all *pure* Reeb chords with both endpoints on the same piece in the subdivision of $\Lambda^\pm$, and $\mathcal{R}^\pm_{\mathrm{mi}}$ contains all *mixed* Reeb chords with endpoints on distinct pieces. Note that a Reeb chord is oriented (by the Reeb flow).

A *formal disk map* is a map from a 2-disk D with m marked disjoint boundary segments into $J^1(M)\times[-T,T]$ with the following properties. Each marked boundary segment either maps in an orientation preserving way to a Reeb chord of $\Lambda^+\subset\partial\bar{L}$, or maps in an orientation reversing way to a Reeb chord of $\Lambda^-\subset\partial\bar{L}$. Each unmarked boundary segment maps to $\bar{L}$. We say that two formal disk maps are *homotopic* if they are homotopic through formal disk maps. In particular, two formal disk maps are homotopic only if they have the same Reeb chords and the respective induced cyclic orderings on these Reeb chords agree. Furthermore, if two formal disk maps have the same Reeb chords in the same cyclic order then their unmarked boundary arcs which connect corresponding Reeb chord endpoints form difference-loops (i.e., the path of one disk followed by the inverse path of the other) in $\bar{L}$ and the two formal disk maps are homotopic only if all these difference-loops are contractible in $\bar{L}$. Finally, in case all the difference-loops are contractible, choosing homotopies between the boundary loops, the two formal disk maps together with these homotopies give a difference-map of a 2-sphere into $J^1(M)\times[-T,T]$ and the formal disk maps are homotopic if and only if the homotopy class of this difference-map lies in the image of $\pi_2(\bar{L}')\to\pi_2(J^1(M)\times[-T,T])$, where $\bar{L}'=\bar{L}'_1\cup\dots\cup\bar{L}'_m\subset\bar{L}$ is the union of the connected components of $\bar{L}$ which contain some boundary component of the formal disk.

In conclusion, if for a fixed cyclic word of Reeb chords there is a formal disk map which realizes this word then the homotopy classes of formal disk maps is a principal homogeneous space over the product of the kernel of a map $\pi_1(\bar{L}'_1)\times\dots\times\pi_1(\bar{L}'_m)\to\pi_1(J^1(M)\times[-T,T])$ determined by the Reeb chord endpoints and the quotient $\pi_2(J^1(M)\times[-T,T])/\operatorname{im}(\pi_2(\bar{L}')\to\pi_2(J^1(M)\times[-T,T]))$. A *formal disk* is a homotopy class of formal disk maps.

We call the Reeb chords of the formal disk its *punctures*. When speaking of formal disks we will contract the marked intervals which map to the Reeb chords to points and call them punctures as well. A component of the complement of the punctures in the boundary of a formal disk will be called a *boundary component*. We say that a puncture of a formal disk is *positive* if it maps to a chord in $\mathcal{R}^+$ and that it is *negative* if it maps to a chord in $\mathcal{R}^-$. We say that a puncture of a formal disk is *mixed* if it maps to a chord in $\mathcal{R}^\pm_{\mathrm{mi}}$ and that it is *pure* if it maps to a chord in $\mathcal{R}^\pm_{\mathrm{pu}}$.

Remark 2.1. Any J-holomorphic disk in $J^1(M)\times\mathbb{R}$ with boundary on L is asymptotic to Reeb chord strips near its punctures. (See [3] Proposition 6.2, and references listed there.) It follows in particular that any holomorphic disk determines a formal disk.

Let D be the source of a formal disk and let ∂D denote the union of the boundary components of D (i.e. ∂D consists of the points on the boundary which are not punctures). A *collapsing arc* in D is an embedding $a\colon[0,1]\to D$ such that $a^{-1}(\partial D)=\{0,1\}$ and such that a is transverse to ∂D at $a(0)$ and $a(1)$. Note that a collapsing arc a subdivides D into two components $D_1(a)$ and $D_2(a)$.

Definition 2.2. A formal disk parametrized by D is *admissible* if it meets the following two conditions.

(**a1**) D has at least one puncture which is positive.

(**a2**) For every collapsing arc a in D with endpoints on boundary components mapping to the same piece of $\bar{L}$, one component $D_1(a)$ or $D_2(a)$ has either no punctures or only pure negative punctures.

Remark 2.3. Definition 2.2 is not the only possible choice. All technical results below hold true if one uses the following alternative definition instead: keep (**a2**) and change (**a1**) to

(**a1**′) D has at least one puncture which is positive or mixed.

This gives rise to a somewhat different theory briefly discussed in Subsection 6.1.

Lemma 2.4. *An admissible disk with a pure positive puncture can neither have mixed punctures nor have other positive punctures.*

Proof. If it did, a small collapsing arc cutting the pure positive puncture off from the rest of the disk contradicts (**a2**). □

Remark 2.5. Let L be an exact Lagrangian cobordism and let $\hat{L}$ be one of its pieces. Then the punctures of an admissible formal disk w which map to a pure or mixed $\hat{L}$-chord are ordered in a natural way as follows.

- If w is a pure admissible disk then the orientation of the boundary of the disk induces an ordering of the punctures of w as follows. The first puncture in the order is the first puncture in the direction of the orientation of the boundary as seen from the positive puncture, continuing in the direction of this orientation we meet all the other negative punctures in a certain order, and finally we meet the positive puncture as the last one in the order.
- If w is a mixed admissible disk with some boundary component mapping to $\hat{L}$ then the orientation of the boundary of the disk induces an ordering as follows. At exactly one puncture mapping to a mixed $\hat{L}$-chord the boundary orientation points into the boundary component mapping to $\hat{L}$, this is the first puncture. Then follow all negative punctures mapping to pure $\hat{L}$-chords in the order of the boundary orientation, and finally the second puncture mapping to a mixed $\hat{L}$-chord where the boundary orientation points out of the boundary component mapping to $\hat{L}$.

Furthermore, once the $\hat{L}$-punctures of the boundary of an admissible formal disk are ordered in this way, we order the remaining punctures in the order they appear as the boundary of the disk is traversed in the positive direction starting at the last $\hat{L}$-puncture.

2.2. **Joining exact cobordisms and gluing formal disks.** Let L^b be an exact cobordism in $J^1(M)\times\mathbb{R}$ with $(+\infty)$-boundary equal to Λ and let L^a be an exact cobordism with $(-\infty)$-boundary equal to Λ. Then we can join the exact cobordisms over Λ to an exact cobordism L^{ba} in $J^1(M)\times\mathbb{R}$. Since $\Lambda\subset\partial\bar{L}^b$ and $\Lambda\subset\partial\bar{L}^a$, we may view Λ as a submanifold of $L^{ba} \subset J^1(M) \times \mathbb{R}$. A *partial formal disk* of L^{ba} is defined as a formal disk except that it is allowed to have punctures also at Reeb chords of $\Lambda \subset \bar{L}^{ba}$.

Let $\mathcal{R}$ denote the set of Reeb chords of Λ. Let g^b be a formal disk of L^b with one of its positive punctures at a Reeb chord $c \in \mathcal{R}$. Let g^a be a formal disk of L^a with one of its negative punctures at c. Then we can attach g^b to g^a at c in an obvious way, to form a partial formal disk g_1^{ba} of L^{ba}. This construction can be repeated: at a positive- or negative puncture of g_1^{ba} which lies in $\mathcal{R}$, a formal disk of L^a or L^b, respectively, can be attached to g_1^{ba} to form a new partial formal disk g_2^{ba}, etc. We say that the formal disks of L^a and L^b used to build a partial formal disk of L^{ba} in this way are its *factors*.

Lemma 2.6. *Let w be a formal disk of L^{ba} with factors which are formal disks in L^a and L^b, all with at least one mixed- or positive puncture. If some partial formal sub-disk of w is non-admissible then so is w.*

Proof. No formal disk has only one mixed puncture. Consider a collapsing arc of the sub-disk with mixed- or positive punctures on both sides. Attaching at a mixed puncture leaves a mixed puncture and attaching at a positive puncture leaves a positive puncture. Thus, the collapsing arc of the sub-disk shows also that the final formal disk is non-admissible. □

2.3. **The vector space of formal disks.** If γ is curve in $J^1(M)$ with contact form λ, then the *action* of γ is

$$\mathcal{A}(\gamma) = \int_\gamma \lambda.$$

Let $\Lambda \subset J^1(M)$ be a Legendrian submanifold. Reeb chords of Λ are critical points for the action functional acting on curves with endpoints on Λ. If Λ is chosen sufficiently generic then the tangent spaces at the endpoints of any Reeb chord project to transverse Lagrangian subspaces in the tangent space of T^*M. In particular, if Λ is compact there are only finitely many Reeb chords.

Let L be an exact cobordism with $(\pm\infty)$-boundary $\Lambda^\pm$. If g is a formal disk of L with positive punctures at Reeb chords $a_1, \dots, a_p$ and negative punctures at Reeb chords $b_1, \dots, b_q$ then define the $(+)$-*action* of g as

$$\mathcal{A}^+(g) = \sum_{j=1}^{p} \mathcal{A}(a_j), \tag{2.1}$$

the $(-)$-*action* of g as

$$\mathcal{A}^-(g) = \sum_{j=1}^{q} \mathcal{A}(b_j), \tag{2.2}$$

and the *action* of g as

$$\mathcal{A}(g) = \mathcal{A}^+(g) - \mathcal{A}^-(g), \tag{2.3}$$

where the $(\pm)$-actions are measured with respect to the contact form $\alpha = e^{\pm T}\lambda$ in the respective cut-off slices.

Let $\tau = (\tau_1, \dots, \tau_m)$ be a vector of m formal variables. Define the module $\mathbf{V}(L;\tau)$ over the polynomial ring $\mathbb{Z}_2[\tau]$ as the free $\mathbb{Z}_2[\tau]$-module generated by admissible formal disks of L. Define $\mathbf{V}^+(L;\tau) \subset \mathbf{V}(L;\tau)$ as the submodule generated by all formal disks of positive action.

2.4. **Gluing pairing.** As in Subsection 2.2, let L^b and L^a be exact cobordisms such that Λ is the $(+\infty)$-boundary of L^b and the $(-\infty)$-boundary of L^a, and let L^{ba} be the joined cobordism. We define gluing pairings

$$\mathbf{V}(L^b;\tau^b) \times \mathbf{V}(L^a;\tau^a) \to \mathbf{V}(L^{ba};\tau^b,\tau^a),$$

as follows. If $v^b \in \mathbf{V}(L^b;\tau^b)$ and $v^a \in \mathbf{V}(L^a;\tau^a)$ then the *gluing pairing* of v^b and v^a

$$\big(v^b \big| \, v^a\big) \in \mathbf{V}(L^{ba};\tau^b,\tau^a) \tag{2.4}$$

is the vector of all admissible formal disks, with factors from v^a and v^b, weighted by the product of the weights of its factors. In order to make this definition precise we discuss properties of the gluing operation and explain how to count glued disks.

Lemma 2.7. *Let u be a mixed formal disk (i.e. a disk with at least one mixed puncture) in L^b or in L^a with at least one positive puncture. Consider a formal disk w in L^{ba} with factors which are formal disks in L^b or in L^a. Assume that all factors of w have at least one positive puncture and that w has at least two factors which both equal u. Then w is non-admissible.*

Proof. The formal disk w has a tree structure where its factors are the nodes and glued punctures are edges. Pick the shortest path in the tree of w which connects two distinct u-factors and consider the corresponding sub-disk w' of w. Lemma 2.6 implies that it is enough to show that w' is non-admissible. We write $w' = w'' \sharp\, u_1 \sharp\, u_2$, where w'' is the sub-disk of w' obtained by removing the two u-factors u_1 and u_2.

If there exists some mixed puncture p of u such that the corresponding punctures p_1 in u_1 and p_2 in u_2 are both punctures of w', then an arc in w' starting on the incoming boundary component near p_1 and ending at the incoming boundary component near p_2 gives a collapsing arc separating p_1 from p_2 and hence w' is non-admissible. If u is a formal disk of L^a then such a puncture always exists: take p equal to any of the positive punctures of u. If u is a formal disk in L^b and if there is no such puncture p, then u must have two mixed positive punctures and no other mixed punctures. Since w'' must contain some L^a-disk, w' has at least one more mixed positive puncture. (For example, the positive puncture of an L^a-factor, which must be mixed since it connects to mixed disks, see Lemmas 2.4 and 2.6.) At least one of the arcs in w' connecting matching boundary components of u_1 and u_2 is then a collapsing arc with mixed punctures on both sides showing that w' is non-admissible, see Figure 1. □

Lemma 2.8. *Let $v^a \in \mathbf{V}(L^a)$ and $v^b \in \mathbf{V}(L^b)$. Then there are only finitely many admissible formal disks g^{ba} of L^{ba} with factors from v^a and v^b. Furthermore each such admissible formal disk g^{ba} has only finitely many factorizations into factors from v^a and v^b.*

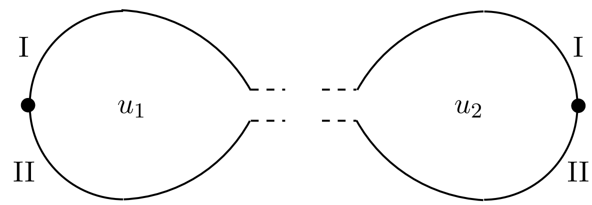

FIGURE 1. I and II label pieces of L^{ba}. If the collapsing arc connecting I to I does not have mixed punctures on both sides then the collapsing arc connecting II to II does.

Proof. Lemma 2.7 implies that any mixed formal disk in v^a or v^b can appear only once as a factor in g^{ba}. If a factor from v^a is pure then all factors of g^{ba} are pure by admissibility. Hence the total number of negative punctures of disks in v^a where pure factors from v^b can be attached is finite. The lemma follows. □

We use Lemma 2.8 to define the gluing pairing (2.4) as follows. Let $v^b \in \mathbf{V}(L^b;\tau^b)$ and $v^a \in \mathbf{V}(L^a;\tau^a)$. In order to find the coefficient of one of the finitely many formal disks g^{ba} which may contribute to $\big(v^b|\, v^a\big)$, we count disks as follows. Each formal admissible disk of L^{ba} with factors in v^a and v^b has a tree structure: vertices correspond to disk factors and edges to glued Reeb chords. Label the vertices in the tree of a formal disk of L^{ba} by the corresponding formal disk from v^a or v^b and weight the formal admissible disk of L^{ba} by the product over the vertices in the tree of the polynomials which are the weights at its vertices. Summing the weighted disks corresponding to all such trees with vertices from v^a and v^b (there are only finitely many) in $\mathbf{V}(L^{ba};\tau^b,\tau^a)$ gives $\big(v^b|\, v^a\big) \in \mathbf{V}(L^{ba};\tau^b,\tau^a)$.

We next consider compositions of gluing pairings. Let L^c be an exact cobordism with $(+\infty)$-boundary Λ^0, let L^b be an exact cobordism with $(-\infty)$-boundary Λ^0 and $(+\infty)$-boundary Λ^1, and let L^a be an exact cobordism with $(-\infty)$-boundary Λ^1. Let $v^* \in \mathbf{V}(L^*;\tau^*)$, $* \in \{c,b,a\}$. Let L^{cba} denote the exact cobordism in $J^1(M)\times\mathbb{R}$ obtained by joining these three cobordisms.

Lemma 2.9. *The following equation holds*

$$\Big(v^c|\ \big(v^b|\, v^a\big)\Big) = \Big(\big(v^c|\, v^b\big)\,|\, v^a\Big) \in \mathbf{V}(L^{cba};\tau),$$

where $\tau = (\tau^a,\tau^b,\tau^c)$.

Proof. Consider any formal disk of L^{cba} which arises after gluing according to the prescription in the left hand side. By subdividing it differently we see that it also arises after gluing as prescribed in the right hand side. Moreover, the weights are in both cases the product of the weights of all the factors. The same argument shows that any formal disk contributing to the right hand side also contributes to the left hand side. The lemma follows. □

We will often use linearized versions of the gluing pairing (2.4) defined as follows. Let L^b and L^a be as above. Fix $f^b \in \mathbf{V}(L^b)$ and consider for $v^a \in \mathbf{V}(L^a)$, $v^a \mapsto \big(f^b|\, v^a\big)$ as a function $\mathbf{V}(L^a) \to \mathbf{V}(L^{ba})$. We define the *linearization* of this function at $f^a \in \mathbf{V}(L^a)$

$$\left[\partial_{f^a}\left(f^b\right|\,\right] : \mathbf{V}(L^a) \to \mathbf{V}(L^{ba}), \tag{2.5}$$

through the following equation

$$\left(f^b\right|f^a + \epsilon w^a\big) = \left(f^b\right|f^a\big) + \epsilon\left[\partial_{f^a}\left(f^b\right|\,\right](w^a) + \mathcal{O}(\epsilon^2) \in \mathbf{V}(L^{ba};\epsilon),$$

for $w^a \in \mathbf{V}(L^a)$.

In a similar way, fixing $f^a \in \mathbf{V}(L^a)$, we consider $\left(v^b\right|f^a\big)$ as a function $\mathbf{V}(L^b) \to \mathbf{V}(L^{ba})$. Its *linearization* at $f^b \in \mathbf{V}(L^b)$

$$\left[\partial_{f^b}\left|\,f^a\right)\right] : \mathbf{V}(L^b) \to \mathbf{V}(L^{ba}), \tag{2.6}$$

is defined through

$$\left(f^b + \epsilon w^b\right|f^a\big) = \left(f^b\right|f^a\big) + \epsilon\left[\partial_{f^b}\left|\,f^a\right)\right](w^b) + \mathcal{O}(\epsilon^2) \in \mathbf{V}(L^{ba};\epsilon),$$

for $w^b \in \mathbf{V}(L^b)$.

Lemma 2.10. *Let L^*, $* \in \{c,b,a\}$, be exact cobordisms as in Lemma 2.9 and let $f^* \in \mathbf{V}(L^*)$. Then*

$$\left[\partial_{f^c}\left|\left(f^b\right|f^a\right)\right)\right](w^c) = \left[\partial_{\left(f^c\right|f^b\right)}\left|\,f^a\right)\right]\left(\left[\partial_{f^c}\right|f^b\big)\right](w^c)\Big), \quad w^c \in \mathbf{V}(L^c), \text{ and}$$
$$\left[\partial_{f^a}\left(\left(f^c\right|f^b\right)\right|\,\Big]\,(w^a) = \left[\partial_{\left(f^b\right|f^a\right)}\left(f^c\right|\,\right]\left(\left[\partial_{f^a}\left(f^b\right|\,\right](w^a)\right), \quad w^a \in \mathbf{V}(L^a).$$

Proof. Let $w^c \in \mathbf{V}(L^c)$. Lemma 2.9 implies that

$$\left(\left(f^c + \epsilon w^c\right|f^b\right)\Big|\,f^a\Big) = \left(f^c + \epsilon w^c\Big|\,\left(f^b\right|f^a\right)\right).$$

The left hand side of this equation can be rewritten as

$$\left(\left(f^c\right|f^b\right) + \epsilon\left[\partial_{f^c}\,\right|f^b\big)\right](w^c) + \mathcal{O}(\epsilon^2)\,\Big|\,f^a\Big) =$$
$$\left(\left(f^c\right|f^b\right)\Big|\,f^a\Big) + \epsilon\left[\partial_{\left(f^c\right|f^b\right)}\Big|\,f^a\Big)\right]\left(\left[\partial_{f^c}\right|f^b\big)\right](w^c)\Big) + \mathcal{O}(\epsilon^2)$$

and the right hand side as

$$\left(f^c\,\Big|\,\left(f^b\right|f^a\right)\right) + \epsilon\left[\partial_{f^c}\Big|\,\left(f^b\right|f^a\right)\right)\right](w^c) + \mathcal{O}(\epsilon).$$

The first equation follows. The second equation is proved in a similar way. □

3. Grading, filtration, and differential

In this section we introduce grading and filtration on the vector space of an exact cobordism. We also define a filtration preserving differential on this space using the gluing pairing in combination with properties of moduli spaces of J-holomorphic disks. The filtration preserving differential determines a spectral sequence and we show that if two exact cobordisms are joined at a common end then there are induced filtration preserving chain maps and hence morphisms of spectral sequences. We assume throughout this section, that every exact cobordism comes equipped with a sufficiently generic almost complex structure J which is adjusted to its symplectic form. For simplicity and to conform with the notation in more general situations (see [3], Appendix B.4 – 6) we will leave out J from the notation, writing "holomorphic disk" instead of "J-holomorphic disk". Below we will frequently use properties of moduli spaces of holomorphic curves. We refer to [3, Appendix B] and references therein for details.

3.1. **Grading.** Let L be an exact Lagrangian cobordism in $J^1(M)\times\mathbb{R}$ and let g be a formal disk of L. The moduli space $\mathcal{M}(g)$ of holomorphic disks in L which determines the formal disk g (i.e., the formal disk map associated to the holomorphic disks in $\mathcal{M}(g)$ belong to the homotopy class g, see [3, Remark B.2]) has formal dimension $\dim(\mathcal{M}(g))$, which we explain how to compute below.

Definition 3.1. The $\mathbb{Z}$-grading on the vector space $\mathbf{V}(L)$ is induced by the following grading of formal disks: the grading of a formal disk $g\in\mathbf{V}(L)$ is

$$|g| = \dim(\mathcal{M}(g)).$$

We say that an element $v\in\mathbf{V}(L)$ is *homogeneous* if it is a formal sum of admissible formal disks which all have the same degree.

If g is a formal disk then $\dim(\mathcal{M}(g))$ equals the Fredholm index of the the linearized $\bar\partial_J$-operator at a map $u\colon D_m\to J^1(M)\times\mathbb{R}$ which is asymptotic to Reeb chord strips of the Reeb chords $c_1,\dots,c_m$ at the m boundary punctures of the punctured disk D_m, and which induces a formal disk map representing g. The source space of this linearized operator splits into a sum of the tangent space of the space of conformal structures on D_m, which is $(m-3)$-dimensional, and an infinite dimensional functional analytic space U of vector fields along u which are tangent to L along the boundary. The Fredholm index of the linearized $\bar\partial_J$-operator restricted to U can be computed by relating it to the Riemann-Hilbert problem, see e.g. [8, 9], as follows. Pick a symplectic trivialization of the tangent bundle of $J^1(M)\times\mathbb{R}$ along u such that the linearized Reeb flow in this trivialization equals the identity along all Reeb chords. The tangent planes of L along $u(\partial D_m)$ give m paths $\gamma_1,\dots,\gamma_m$ of a Lagrangian subspaces of $\mathbb{C}^n$ (where $\mathbb{C}^n$ is determined by the trivialization) where γ_j connects the tangent space $T_{e^1_j}L$ of L at the Reeb chord endpoint e^1_j where the formal disk leaves the Reeb chord c_j to the tangent space $T_{e^0_{j+1}}L$ at the Reeb chord endpoint e^0_{j+1} where the formal disk enters the Reeb chord c_{j+1}. (Here we use the convention $m+1=1$). The tangent space of L at a Reeb chord endpoint e^σ_j, $\sigma=0,1$, splits as $T_{e^\sigma_j}\Lambda^\pm\times\mathbb{R}$, where $\Lambda^\pm$ is the $(\pm\infty)$-boundary of L. Let $\Pi_{\mathbb{C}}\colon J^1(M)\times\mathbb{R}\to T^*M$ denote the projection. Our genericity assumptions on $\Lambda^\pm$ imply that the linearized Reeb flow along c_j takes the projection into T^*M of the tangent space of $\Lambda^\pm$ at the Reeb chord endpoint e^σ_j, where the Reeb field points into c_j, $\Pi_{\mathbb{C}}(T_{e^\sigma_j}\Lambda^\pm)$ to a Lagrangian subspace which is transverse to the projection of the tangent space $\Pi_{\mathbb{C}}(T_{e^\tau_j}\Lambda^\pm)$ at the other endpoint e^τ_j of c_j ($\sigma\ne\tau\in\{0,1\}$). We close the paths $\gamma_1,\dots,\gamma_m$ to a loop using paths $\hat\gamma_1,\dots,\hat\gamma_m$, where $\hat\gamma_{j-1}$ connects the endpoint $T_{e^0_j}\Lambda^\pm\times\mathbb{R}$ of γ_{j-1} to the start-point $T_{e^1_j}\Lambda^\pm\times\mathbb{R}$ of γ_j, as follows. The path $\hat\gamma_{j-1}$ leaves the $\mathbb{R}$-factor fixed. If e^0_j is the endpoint of a Reeb chord in the $(+\infty)$-boundary, then rotate $\Pi_{\mathbb{C}}(T_{e_j^0}\Lambda^+)$ in the negative direction to $\Pi_{\mathbb{C}}(T_{e_j^1}\Lambda^+)$. If e_j^0 is the endpoint of a Reeb chord in the $(-\infty)$-boundary, then rotate $\Pi_{\mathbb{C}}(T_{e_j^0}\Lambda^-)$ in the negative direction to $\Pi_{\mathbb{C}}(T_{e_j^1}\Lambda^-)$. (Here we use the following notation: if W is a symplectic vector space then a negative rotation of a Lagrangian subspace $V_0 \subset W$ to a Lagrangian subspace $V_1 \subset W$ transverse to V_0, is a path of the form $e^{-s\frac{\pi}{2}I}V_0$, $0 \le s \le 1$, where I is a complex structure compatible with the symplectic structure and such that $IV_0 = V_1$.) The concatenation $\gamma = \gamma_1 * \hat{\gamma}_1 * \cdots \gamma_m * \hat{\gamma}_m$ is a a loop of Lagrangian subspaces of $\mathbb{C}^n$. The index of the restriction of the linearized $\bar{\partial}_J$-operator to U is $n + \mu(\gamma)$ and the formal dimension is consequently

$$|g| = \dim(\mathcal{M}(g)) = n - 3 + \mu(\gamma) + m,$$

where $2n = \dim(J^1(M) \times \mathbb{R})$, where μ denotes Maslov index, and where m is the number of punctures, see [7, 9].

The dimension is additive in the following sense. With notation as in Section 2.3, if $v^a \in \mathbf{V}(\Lambda^a)$ and $v^b \in \mathbf{V}(\Lambda^b)$ then the dimension of a formal disk contributing to $\left(v^b \middle| v^a\right)$ equals the sum of the dimensions of its factors. To see this one uses the following fact: at each Reeb chord where disks are glued, $(n-1)$ negative half-turns are lost (since the closing up rotations at the punctures were erased by the gluing operation) and two punctures are erased.

3.2. **Filtration.** Let L be an exact cobordism in $J^1(M) \times \mathbb{R}$ and assume that $L = L_1 \cup \cdots \cup L_k$ is subdivided into k pieces.

Lemma 3.2. *No admissible formal disk in L has more than k mixed punctures and consequently no more than k positive punctures.*

Proof. Assume that D is the source of a formal disk with more than k mixed punctures. Then there exists a pair of boundary components of D which map to the same piece L_j of L and such that both complementary arcs in ∂D of these two boundary components must contain boundary components mapping to some piece of L other than L_j. An arc connecting the two boundary components in the pair then contradicts (**a2**) of Definition 2.2. Lemma 2.4 implies that any admissible formal disk either has one pure positive puncture or all its positive punctures are mixed. The lemma follows. □

If g is a formal disk then let $\mathrm{pos}(g)$ denote the number of positive punctures of g.

Definition 3.3. For $1 \le p \le k$, let $F^p\mathbf{V}(L)$ denote the subspace of $\mathbf{V}(L)$ generated by all admissible formal disks with at least p positive punctures. The filtration of $\mathbf{V}(L)$ is

$$0 \subset F^k\mathbf{V}(L) \subset F^{k-1}\mathbf{V}(L) \subset \cdots \subset F^1\mathbf{V}(L) = \mathbf{V}(L).$$

3.3. **The potential vector.** Let L be an exact cobordism in $J^1(M) \times \mathbb{R}$. For generic almost complex structure J, the moduli space of 0-dimensional admissible holomorphic disks with boundary on L is a 0-dimensional compact manifold. In other words, it is a finite collection of points and to each point is associated a formal disk g, see Remark 2.1. Furthermore, if a formal disk g has a holomorphic representative then $\mathcal{A}(g) \ge 0$

by Stokes theorem. Define the *potential vector* $f \in \mathbf{V}^+(L)$ as

$$f = \sum_{\dim(\mathcal{M}(g))=0} |\mathcal{M}(g)|\, g,$$

where $|\mathcal{M}(g)|$ denotes the modulo 2 number of points in the 0-dimensional moduli space $\mathcal{M}(g)$. Note that $f \in \mathbf{V}^+(L)$ is a homogeneous element of grading 0.

In the special case when the cobordism is trivial, because of the translational invariance along the $\mathbb{R}$-factor, for generic adjusted almost complex structure, the only holomorphic disks of formal dimension 0 are Reeb chord strips with one positive and one negative puncture (at the same Reeb chord). Thus for a trivial concordance the potential vector f is extremely simple: if $\Lambda \subset J^1(M)$ is a Legendrian submanifold then

$$f = \sum_{c \in \mathcal{R}} g_c \in \mathbf{V}(\Lambda \times \mathbb{R}),$$

where g_c is the formal disk represented by the Reeb chord strip of the Reeb chord c.

3.4. **The Hamiltonian vector of a symplectization.** Consider the trivial cobordism $\Lambda \times \mathbb{R}$ associated to a Legendrian submanifold $\Lambda \subset J^1(M)$. Using the fact, pointed out above, that $\mathbb{R}$ acts on the moduli spaces of holomorphic disks and that for generic adjusted almost complex structure, the only 0-dimensional holomorphic disks are Reeb chord strips, we conclude that that the moduli space $\mathcal{M}$ of holomorphic disks of formal dimension 1, when divided out by this $\mathbb{R}$-action, forms a compact 0-dimensional manifold. Write $\widehat{\mathcal{M}} = \mathcal{M}/\mathbb{R}$ and call $\widehat{\mathcal{M}}$ the *reduced* moduli space. Define the *Hamiltonian vector* $h \in \mathbf{V}(\Lambda \times \mathbb{R})$ of a trivial cobordism as

$$h = \sum_{\dim(\widehat{\mathcal{M}}(g))=0} |\widehat{\mathcal{M}}(g)|\, g,$$

where the sum ranges over admissible formal disks g. Note that $h \in \mathbf{V}(\Lambda \times \mathbb{R})$ is a homogeneous element of grading 1 and that $\mathcal{A}(g) > 0$ for any g for which $\widehat{\mathcal{M}}(g)$ is non-empty and hence $h \in \mathbf{V}^+(\Lambda \times \mathbb{R})$.

3.5. **The differential.** Let L be an exact cobordism in $J^1(M) \times \mathbb{R}$ with ends $\Lambda^\pm \times \mathbb{R}_\pm$. Gluing the trivial cobordisms $\Lambda^\pm \times \mathbb{R}$ to L does not change L. In particular, if $v^\pm \in \mathbf{V}(\Lambda^\pm \times \mathbb{R}; \tau)$ and $w \in \mathbf{V}(L;\tau)$ then $(v^- |\, w) \in \mathbf{V}(L;\tau)$ and $(w|\, v^+) \in \mathbf{V}(L;\tau)$, see Subsection 2.4 for notation.

Let $f \in \mathbf{V}^+(L)$ and $f^\pm \in \mathbf{V}^+(\Lambda^\pm \times \mathbb{R})$ be the potential vectors and let $h^\pm \in \mathbf{V}^+(\Lambda^\pm \times \mathbb{R})$ be the Hamiltonian vectors. We associate operators $h^\pm \colon \mathbf{V}(L;\epsilon) \to \mathbf{V}(L;\epsilon)$ to the Hamiltonian vectors (for simplicity, we denote these operators by the same symbols as the vectors themselves) through the following equations

$$\big(v|\, f^+ + \tau h^+\big) = v + \tau\, h^+(v) + \mathcal{O}(\tau^2) \in \mathbf{V}(L;\tau,\epsilon),$$
$$\big(f^- + \tau h^- |\, v\big) = v + \tau\, h^-(v) + \mathcal{O}(\tau^2) \in \mathbf{V}(L;\tau,\epsilon),$$

where $v \in \mathbf{V}(L;\epsilon) \subset \mathbf{V}(L;\tau,\epsilon)$ and $f^\pm + \tau h^\pm \in \mathbf{V}^+(\Lambda^\pm \times \mathbb{R};\tau) \subset \mathbf{V}(\Lambda^\pm \times \mathbb{R};\tau,\epsilon)$. Define

$$h = h^+ + h^- \colon \mathbf{V}(L;\epsilon) \to \mathbf{V}(L;\epsilon).$$

Lemma 3.4. *If $f \in \mathbf{V}^+(L)$ is the potential vector then $h(f) = 0$.*

Proof. To see this we note that any admissible formal disk contributing to $h(f)$ can be viewed as a broken disk which is an endpoint of a 1-dimensional moduli space of holomorphic disks in $J^1(M) \times \mathbb{R}$ with boundary on L. By compactness and by uniqueness of gluing, the other endpoint of this 1-dimensional moduli space correspond to some other broken admissible disk with one factor from h^+ or h^- and remaining factors from f. Hence also the other endpoint contributes to $h(f)$. The lemma follows. □

We next consider the linearization of the Hamiltonian operator h at the potential vector f. This linearization is a map $d^f\colon \mathbf{V}(L) \to \mathbf{V}(L)$ defined through the following equation for $v \in \mathbf{V}(L)$,

$$h(f + \epsilon v) = \epsilon\ d^f(v) + \mathcal{O}(\epsilon^2) \in \mathbf{V}(L; \epsilon).$$

Lemma 3.5. *The map $d^f\colon \mathbf{V}(L) \to \mathbf{V}(L)$ is a filtration preserving differential of degree 1. That is, $d^f \circ d^f = 0$, $d^f(F^p\mathbf{V}(L)) \subset F^p\mathbf{V}(L)$, and if $v \in \mathbf{V}(L)$ is homogeneous of degree k then $d^f(v)$ is homogeneous of degree $k+1$. Furthermore, d^f is $(+)$-action non-decreasing: if $v \in \mathbf{V}(L)$ is a formal disk then the $(+)$-action $\mathcal{A}^+(w)$ of any formal disk w contributing to $d^f(v)$ satisfies $\mathcal{A}^+(w) \geq \mathcal{A}^+(v)$.*

Proof. Let $v \in \mathbf{V}(L)$ and let h^+ and h^- denote the Hamiltonian vectors at the positive and negative ends of L, respectively. We first show $d^f \circ d^f = 0$. By linearity of d^f and since for each fixed formal disk g of L there is only a finite number of formal disks in v, in f, and in $h^\pm$ which can contribute to the coefficient of g in $d^f \circ d^f(v)$, see Lemma 2.8, it suffices to show that $d^f \circ d^f(v) = 0$ in the case when v is a single formal disk. In other words, we must show

$$h\big(f + h(f + \epsilon v)\big) = \mathcal{O}(\epsilon^2), \tag{3.1}$$

for formal disks v.

The constant term of the left hand side in (3.1) vanishes by Lemma 3.4. Consider the linear term. Any formal disk contributing to the linear term in (3.1) has one v-factor, two $h^\pm$-factors, and all other factors f-factors. We distinguish the $h^\pm$-factor corresponding to the *outer* Hamiltonian operator h and call it *charged*. Note that the non-charged $h^\pm$-factor is connected to v. Below we will make use of slightly more general disks. We therefore define an LT-disk (LT for "linear term") as a formal disk with one v-factor, two $h^\pm$-factors one which is connected to v and the other one distinguished as charged, and remaining factors f-factors.

Consider the positive- or negative end (i.e., all positive- or negative punctures) of the charged $h^\pm$-factor of an LT-disk. We say that this end is *un-obstructed* if either there is some $h^\pm$-factor attached at the end, or if there are only f-factors attached at the end. An end which is not un-obstructed is called *obstructed*. We say that an LT-disk is *isolated*, *boundary*, or *interior* if its charged $h^\pm$-factor has two, one, or zero obstructed ends, respectively.

We first show that the contribution to (3.1) from isolated LT-disks vanishes. Both $h^\pm$-factors of an isolated LT-disk are connected to v. Changing the factorization by moving the charge from one $h^{\pm}$-factor to the other we see that the formal disk appears in two ways as an LT-disk in (3.1). Thus the contribution from isolated LT-disks vanishes.

Consider a non-isolated LT-disk with an un-obstructed end of its charged $h^{\pm}$-factor distinguished and called *active*. We define the following propagation law for such an LT-disk.

- If there are only f-factors attached at the active end of the charged $h^{\pm}$-factor, then view the broken disk consisting of the charged $h^{\pm}$-factor and all these f-factors as the boundary of a 1-dimensional moduli space of holomorphic disks with boundary on L. Moving to the other end of this moduli space we get another broken disk with f-factors and one $h^{\pm}$-factor. Charge the new $h^{\pm}$-factor and activate the end of it where the new f-factors are not attached.
- If there is an $h^{\pm}$-factor attached at the active end of the charged $h^{\pm}$-factor, then one of these two $h^{\pm}$-factors is attached to v. View the broken disk consisting of the two $h^{\pm}$-factors as the boundary of a 1-dimensional reduced moduli space in the symplectization of the $(\pm\infty)$-boundary. Moving to the other end of the reduced moduli space we get another broken disk with two $h^{\pm}$-factors. Charge the one of them which is not attached to v and activate the end of it where the other $h^{\pm}$-factor is not attached.

This propagation law associates to an LT-disk with an un-obstructed end of its charged $h^{\pm}$-factor activated a unique non-isolated LT-disk with an end of its charged $h^{\pm}$-factor activated. In particular, using this propagation law repeatedly starting at a boundary LT-disk the process stops at some other uniquely determined boundary LT-disk, see Figure 2. Noting that every non-isolated LT-disk which contributes to (3.1) is a

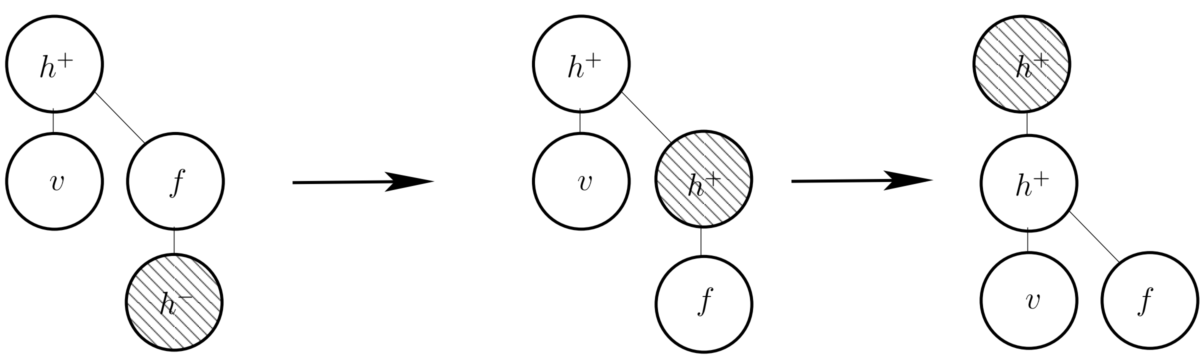

FIGURE 2. The propagation law, the charged $h^{\pm}$-factor is shaded.

boundary LT-disk and that any boundary LT-disk contributes to (3.1) it follows that also the contribution from non-isolated LT-disks to (3.1) vanishes. We conclude that $d^f \circ d^f = 0$.

The fact that d^f increases grading by 1 follows from Subsection 3.1 since f is homogeneous of degree 0 and $h^{\pm}$ are homogeneous of degree 1. To see that $d^f(F^p\mathbf{V}(L)) \subset F^p(\mathbf{V}(L))$, note that gluing operations of admissible disks never decreases the number of positive punctures: since each admissible disk has at least one positive puncture, a positive puncture erased by gluing is compensated by at least one new positive puncture.

To see that d^f is (+)-action non-decreasing we argue as follows. Formal disks contributing to $d^f(v)$ have one v-factor and remaining factors from $h^{\pm}$ and from f. Formal disks in $h^{\pm}$ and in f have holomorphic representatives and hence have non-negative action. If g is a partial formal disk and some factor is added to it from below giving a new formal disk g' then obviously $\mathcal{A}^+(g') \geq \mathcal{A}^+(g)$. If g is a partial formal disk and a disk h is added to it from above giving a new disk g' then $\mathcal{A}^+(g') \geq \mathcal{A}^+(g) + \mathcal{A}(h)$ so that $\mathcal{A}^+(g') \geq \mathcal{A}^+(g)$ if $\mathcal{A}(h) \geq 0$. It follows that d^f does not decrease (+)-action. □

Remark 3.6. In the case that $L = \Lambda \times \mathbb{R}$ is a trivial cobordism, the argument in the proof of Lemma 3.5 shows that the first order terms d^+ and d^- in $h^+(f + \epsilon g)$ and $h^-(f + \epsilon g)$, respectively are both differentials, separately. Moreover, it is immediate that $d^+ \circ d^- = d^- \circ d^+$. (In this case the *vectors* $h^+ = h$ and $h^- = h$ are identical. However, the *operators* h^+ and h^- are not: one corresponds to attaching h from above, the other to attaching h from below.)

3.6. **The rational admissible SFT spectral sequence.** Let $L \subset J^1(M) \times \mathbb{R}$ be an exact cobordism where $L = L_1 \cup \cdots \cup L_k$ is subdivided into k pieces.

Definition 3.7. The *rational admissible SFT spectral sequence* of the exact cobordism L is the cohomological spectral sequence

$$\left\{E_r^{p,q}(L)\right\}_{r=1}^{k},$$

induced by the filtration respecting differential $d^f \colon \mathbf{V}(L) \to \mathbf{V}(L)$, where f is the potential vector of L, which has E_1-term

$$E_1^{p,q}(L) = H^{p+q}\left(F^p\mathbf{V}(L)/F^{p+1}\mathbf{V}(L)\right).$$

Let $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k \subset J^1(M)$ be a Legendrian submanifold subdivided into pieces. Since the potential vector of $\Lambda \times \mathbb{R}$ acts as the identity in gluing pairings, we suppress it from notation and write $d \colon \mathbf{V}(\Lambda \times \mathbb{R}) \to \mathbf{V}(\Lambda \times \mathbb{R})$ for the differential in this case.

Definition 3.8. The *rational admissible SFT invariant* of Λ is the spectral sequence

$$\left\{E_r^{p,q}(\Lambda)\right\}_{r=1}^{k} = \left\{E_r^{p,q}(\Lambda \times \mathbb{R})\right\}_{r=1}^{k}.$$

3.7. **Joining cobordisms and filtration preserving chain maps.** Let L^b and L^a be exact cobordisms and assume that Λ^0 is the $(+\infty)$-boundary of L^b and the $(-\infty)$-boundary of L^a. Consider the exact cobordism L^{ba} obtained by joining the ends corresponding to Λ^0. Write f^a, f^b, and f^{ba} for the potential vectors in $\mathbf{V}^+(L^a)$, $\mathbf{V}^+(L^b)$, and $\mathbf{V}^+(L^{ba})$, respectively.

Lemma 3.9. *The potential vectors satisfy*

$$f^{ba} = \left(f^b \middle| \, f^a\right).$$

Proof. Stretching L^{ba} along Λ^0 it follows by compactness, that any admissible rigid holomorphic disk of L^{ba} breaks into admissible rigid disks of L^a and L^b. Conversely, any such broken configuration can be glued uniquely to an admissible rigid disk of the joined cobordism. Thus the left- and the right hand sides count the same objects. □

Consider the linearizations

$$\left[\partial_{f^a}\left(f^b|\,\right]: \mathbf{V}(L^a) \to \mathbf{V}(L^{ba}) \text{ and}$$
$$\left[\partial_{f^b}\,|\,f^a)\right]: \mathbf{V}(L^b) \to \mathbf{V}(L^{ba}),$$

introduced in (2.5) and (2.6), respectively.

Lemma 3.10. *The maps* $\left[\partial_{f^a}\left(f^b|\,\right]$ *and* $\left[\partial_{f^b}\,|\,f^a)\right]$ *are* $(+)$*-action non-decreasing, filtration preserving chain maps,*

$$\left[\partial_{f^a}\left(f^b|\,\right] \circ d^{f^a} = d^{f^{ba}} \circ \left[\partial_{f^a}\left(f^b|\,\right], \tag{3.2}$$
$$\left[\partial_{f^b}\,|\,f^a)\right] \circ d^{f^b} = d^{f^{ba}} \circ \left[\partial_{f^b}\,|\,f^a)\right]. \tag{3.3}$$

Proof. The proof is similar to the proof of Lemma 3.5: one identifies isolated broken disks and uses a propagation law to show that non-isolated broken disks cancel. □

4. Deformations and chain homotopies

In this section we show that the rational admissible SFT spectral sequence is invariant under deformations of exact cobordisms and that morphisms induced by joining cobordisms have similar invariance properties.

4.1. **Increment disks.** Let L^b and L^a be exact cobordisms in $J^1(M) \times \mathbb{R}$. Assume that Λ^0 is the $(+\infty)$-boundary of L^b and the $(-\infty)$-boundary of L^a. Let L^{ba} denote the cobordism obtained by joining L^b and L^a at Λ^0. Let $\hat\Lambda^0$ be a piece of Λ^0, let $\hat L^a$ and $\hat L_b$ be corresponding pieces of L^a and L^b, respectively, and let $\hat L^{ba}$ be the corresponding piece of L^{ba}. Let $h^a \in \mathbf{V}(L^a)$, let $k^b \in \mathbf{V}(L^b)$ be such that every formal disk in k^b has some boundary component mapping to $\hat L^b$. We define the $\left\{k^b \to h^a\right\}$ *split gluing operation*

$$\left\{k^b \to h^a\right\}: \mathbf{V}(L^b;\epsilon) \times \mathbf{V}(L^b;\epsilon) \times \mathbf{V}(L^a;\epsilon) \to \mathbf{V}(L^{ba};\epsilon)$$

in the following way. The vector $\left\{k^b \to h^a\right\}\left(v_0^b, v_1^b, v^a\right)$ is the sum of all admissible disks constructed as follows. First pick a disk in h^a and attach a disk in k^b to it. If the resulting (partial) disk is admissible then the $\hat L^{ba}$-component of its boundary induces an ordering of its punctures, see Remark 2.5. Second attach v_0^b-factors at the negative punctures at Reeb chords in Λ^0 of the partial formal disk which lie after the k^b-factor, and attach v_1^b-factors at such punctures which lie before the k^b-factor. Third attach v^a-factors at the positive punctures at Reeb chords in Λ^0 which arise from the attached v_0^b- and v_1^b-disks. Continue like this, in each step attaching v_0^b after the k^b-factor and v_1^b-factors before it at Reeb chords of Λ^0 which arose from the v^a-disks attached in the previous step. As for the gluing pairing in Subsection 2.3, we see that this is a finite process.

Similarly, if $h^b \in \mathbf{V}(L^b)$ and if $k^a \in \mathbf{V}(L^a)$ are such that every formal disk in k^a has some boundary component mapping to $\hat L^a$, then we define the $\left\{h^b \leftarrow k^a\right\}$ *split gluing operation*

$$\left\{h^b \leftarrow k^a\right\}: \mathbf{V}(L^b;\epsilon) \times \mathbf{V}(L^a;\epsilon) \times \mathbf{V}(L^a;\epsilon) \to \mathbf{V}(L^{ba};\epsilon)$$

as follows. The vector $\{h^b \leftarrow k^a\}(v^b, v_0^a, v_1^a)$ is the sum of all admissible disks constructed as follows. Pick a disk in h^b and attach a disk in k^a to it. If the resulting (partial) disk is admissible then the $\hat{L}^{ba}$-component of its boundary induces an ordering of its punctures. Attach v_0^a-factors at the positive punctures at Reeb chords in Λ^0 of the partial formal disks which lie after the k^a-factor, and attach v_1^a-factors at such punctures which lie before the k^a-factor. Then attach v^b-factors at punctures at Reeb chords in Λ^0. Continue like this, in each step attaching v_0^a after the k^b-factor and v_1^a-factors before it. Again this is a finite process.

Consider an exact cobordism L, let $\hat{L}$ be one of the pieces of L, let $u, v \in \mathbf{V}(L)$, and let $k \in \mathbf{V}(L)$ be a vector such that every formal disk in k has a boundary component mapping to $\hat{L}$. Let the $(\pm\infty)$-boundary of L be $\Lambda^\pm$, with Hamiltonian vectors $h^\pm$ and potential vectors $f^\pm$ in $\mathbf{V}^+(\Lambda^\pm \times \mathbb{R})$. Since the potential vector of a trivial cobordism acts as the identity in gluing pairings we suppress it from the notation for split gluing pairings writing simply

$$\{k \to h^+\}(u, v) := \{k \to h^+\}(u, v, f^+),$$
$$\{h^- \leftarrow k\}(u, v) := \{k \to h^+\}(f^-, u, v).$$

Define the k-*increment* of u inductively as follows. First let

$$\rho^{(1)}(u) = d^u(k) = \{k \to h^+\}(u, u) + \{h^- \leftarrow k\}(u, u)$$

(where, as usual, $d^u(k)$ is defined by $h(u) + h(u + \epsilon k) = \epsilon\, d^u(k) + \mathcal{O}(\epsilon^2)$). For $j > 1$, define inductively,

$$\begin{aligned} \rho^{(j)}(u) &= \{k \to h^+\}\left(u, u + \rho^{(j-1)}(u)\right) + \{h^- \leftarrow k\}\left(u, u + \rho^{(j-1)}(u)\right), \\ \delta^{(j)}(u, k) &= \rho^{(j)}(u) + \rho^{(j-1)}(u), \quad j > 1, \end{aligned} \tag{4.1}$$

where we take $\rho^{(0)}(u) = 0$. Finally, the k-increment of u is

$$\Delta(u, k) = \sum_{j=1}^{\infty} \delta^{(j)}(u, k), \tag{4.2}$$

where this sum is in fact finite for the following reason. Mixed disks in $u, v, k, h^\pm$ can only be used once in the construction by Lemma 2.7. A factor of $\Delta(u, k)$ which is a pure k-disk has at most one h^--disk attached at its negative punctures and a factor of $\Delta(u, k)$ which is a pure u-disk has no h^--disk attached at its negative punctures. Furthermore, if a pure factor of $\Delta(u, k)$ is a k- or a u-disk with positive puncture attached at a negative puncture of an h^+-disk then the sum of the actions of the Reeb chords at the positive punctures of that h^+-disk is larger than the action of the Reeb chord at the positive puncture of the k- or u-disk. In particular, if the h^+-disk is pure then the action at the positive puncture of the pure disk consisting of all disks attached to that h^+-disk (which is a disk that can be used in later steps of the construction) exceeds that of the pure k- or u-disk by an amount $\mathcal{A}_0 > 0$. Combining this with the fact that there are only finitely many Reeb chords at the positive end of L, finiteness follows much like in Lemma 2.8.

We next relate the above algebraic construction of increment disks to geometry. As explained in [3, Appendix B.3], in a generic 1-parameter family (L_s, β_s), $0 \le s \le 1$, of exact cobordisms there is a finite number of distinct moments $\hat{s} \in [0,1]$ for which there exists a holomorphic disk $\hat{k}$ of formal dimension -1 with boundary on $L_{\hat{s}}$ (we will call such a disk a (-1)-disk below), and the potential vector of L_s changes only when s passes one of these isolated moments. Below we describe algebraically how the passage of one such (-1)-disk moment affects the potential vector. The analytical study of moduli spaces behind this algebraic description involves solving transversality problems by introducing, so called, abstract perturbations. In particular, such perturbations give rise to new (-1)-disks near the original one. The perturbation scheme is described in [3, Appendix B.4 – 6].

Consider a 1-parameter family of exact cobordisms L_s, $0 \le s \le 1$, with a (-1)-disk $\hat{k}$ at $s = \frac{1}{2}$ which maps some boundary component to the piece $\hat{L}$ of L and with no other (-1)-disks. Let $k \in \mathbf{V}^+(L_s)$ be the following vector. If $\hat{k}$ is mixed, as in [3, Appendix B.5], then $k = \hat{k}$, if $\hat{k}$ is pure then k is the vector of (-1) disks which arises from the perturbation described in [3, Appendix B.6]. Note that all formal disks in k have some boundary component mapping to $\hat{L}$, see [3, Remark B.13].

Lemma 4.1. *The potential vectors $f_s \in \mathbf{V}^+(L_s)$, $s = 0, 1$, are related as follows*

$$f_1 = f_0 + \Delta(f_0, k) := \phi(f_0).$$

Proof. See [3, Lemmas B.9 and B.15]. □

We next consider 1-parameter families obtained by joining stationary and moving cobordisms. Let L^b_s, $0 \le s \le 1$, be a 1-parameter family with a (-1)-disk $\hat{k}$, with a $(+\infty)$-boundary Λ^0 and let L^a be an exact cobordism with a $(-\infty)$-boundary Λ^0. Let L^{ba}_s be the 1-parameter family obtained by joining the two, let f^b_s and f^a denote the potential vectors of L^b_s and L^a, respectively, and let $k \in \mathbf{V}^+(L^b_s)$ be the vector of (-1)-disks as in [3, Appendix B.5] if $\hat{k}$ is mixed or as in [3, Appendix B.6] if $\hat{k}$ is pure. Define the vector $K \in \mathbf{V}^+(L^{ba}_s)$ as

$$K = \left\{k \to f^a\right\}\left(f^b_0, f^b_1, f^a\right) = \left\{k \to f^a\right\}\left(f^b_0, f^b_0 + \Delta(f^b_0, k), f^a\right).$$

If $\hat{k}$ has a boundary component mapping to the piece $\hat{L}^b_s$ of L^b_s then every disk contributing to k has too. Consequently, if $\hat{L}^{ba}_s$ is the piece of L^{ba}_s corresponding to $\hat{L}^b$ then every disk contributing to K has some boundary component mapping to $\hat{L}^{ba}_s$.

Lemma 4.2. *The potential vectors F_s of (X^{ba}_s, L^{ba}_s), $s = 0, 1$, are related by*

$$F_1 = F_0 + \Delta(F_0, K) := \Phi(F_0).$$

Proof. See [3, Lemma 4.3]. □

Similarly, let L^b be an exact cobordism with a $(+\infty)$-boundary Λ^0 and let L^a_s, $0 \le s \le 1$, be a 1-parameter family of cobordisms with a (-1)-disk $\hat{k}$, with a $(-\infty)$-boundary Λ^0. Let L^{ba}_s be the 1-parameter family obtained by joining the two, let f^b and f^a_s denote the potential vectors of L^b and L^a_s, respectively, and let $k \in \mathbf{V}^+(L^a_s)$ be the vector of (-1)-disks constructed in as in [3, Appendix B.5] or as in [3, Appendix B.6] according to whether $\hat{k}$ is mixed or pure. Define the vector $K \in \mathbf{V}^{+}(L^{ba}_s)$ as

$$K = \{f^b \leftarrow k\}\left(f^b, f^a_0, f^a_1\right) = \{f^b \leftarrow k\}\left(f^b, f^a_0, f^a_0 + \Delta(f^a_0, k)\right).$$

If $\hat{k}$ has a boundary component mapping to the piece $\hat{L}^a_s$ of L^a_s then every disk contributing to k has too. Consequently, if $\hat{L}^{ba}_s$ is the piece of L^{ba}_s corresponding to $\hat{L}^b$ then every disk contributing to K has some boundary component mapping to $\hat{L}^{ba}_s$.

Lemma 4.3. *The potential vectors F_s of L^{ba}_s, $s = 0, 1$, are related by*

$$F_1 = F_0 + \Delta(F_0, K) := \Psi(F_0).$$

Proof. See [3, Lemma 4.4]. □

4.2. **Chain isomorphisms.** Consider a 1-parameter family L^b_s, $0 \le s \le 1$, with a $(+\infty)$-boundary Λ^0 and a stationary cobordism L^a with a $(-\infty)$-boundary Λ^0 as in Lemma 4.2. We use notation as there. Considering F_0 in the definition of $\Phi(F_0)$ as a variable we obtain a function $\Phi\colon \mathbf{V}(L^{ba}_0; \epsilon) \to \mathbf{V}(L^{ba}_1; \epsilon)$. For $F \in \mathbf{V}(L^{ba}_0)$, let $[\partial_F \Phi] : \mathbf{V}(L^{ba}_0) \to \mathbf{V}(L^{ba}_1)$ denote the linearization of this function at F defined by

$$\Phi(F + \epsilon V) = \Phi(F) + \epsilon\, [\partial_F \Phi]\,(V) + \mathcal{O}(\epsilon^2) \in \mathbf{V}(L^{ba}_1; \epsilon),$$

for $V \in \mathbf{V}(L^{ba}_0)$.

Lemma 4.4. *The map $[\partial_{F_0} \Phi] : \mathbf{V}(L^{ba}_0) \to \mathbf{V}(L^{ba}_1)$ is a $(+)$-action non-decreasing, filtration preserving chain map,*

$$d^{F_1} \circ [\partial_{F_0} \Phi] = [\partial_{F_0} \Phi] \circ d^{F_0}. \tag{4.3}$$

Furthermore $[\partial_{F_0} \Phi]$ induces an isomorphism of spectral sequences,

$$\left\{E^{p,q}_r(L^{ba}_0)\right\} \xrightarrow[\cong]{[\partial_{F_0}\Phi]_*} \left\{E^{p,q}_r(L^{ba}_1)\right\}.$$

Proof. The chain map property is proved using an argument similar to the proof of Lemma 3.5. The fact that the map is an isomorphism follows by considering the $(+)$-action. The map can be written $\mathrm{id} + B$ where B increases $(+)$-action by at least $\alpha_0 > 0$ and we have (because of $\mathbb{Z}_2$-coefficients)

$$(\mathrm{id} + B) \circ \overset{N}{\cdots} \circ (\mathrm{id} + B) = \mathrm{id} + B^N.$$

Since B^N increases $(+)$-action by at least $N\alpha_0 > \alpha$, where α is the sum of the actions of all Reeb chords at the positive end, we find that $B^N = 0$. We conclude that $[\partial_{F_0} \Phi] = \mathrm{id} + B$ satisfies $[\partial_{F_0} \Phi]^N = \mathrm{id}$. In particular, the filtration preserving chain map $[\partial_{F_0} \Phi] : \mathbf{V}(L^{ba}_0) \to \mathbf{V}(L^{ba}_1)$ has a filtration preserving inverse $[\partial_{F_0} \Phi]^{N-1} : \mathbf{V}(L^{ba}_1) \to \mathbf{V}(L^{ba}_0)$ (we use the fact that $\mathbf{V}(L^{ba}_0)$ and $\mathbf{V}(L^{ba}_1)$ are canonically isomorphic as vector spaces) which is easily seen to be a chain map. It follows that $[\partial_{F_0} \Phi]$ induces an isomorphism of spectral sequences. □

Consider a 1-parameter family L^a_s, $0 \leq s \leq 1$, with a negative end at Λ^0 and a stationary cobordism L^b with positive end at Λ^0. Considering F_0 in the definition of $\Psi(F_0)$, see Lemma 4.3, as a variable we obtain a function $\Psi\colon \mathbf{V}(L_0^{ba};\epsilon) \to \mathbf{V}(L_1^{ba};\epsilon)$. For $F \in \mathbf{V}(L_0^{ba})$, let $[\partial_F \Psi] : \mathbf{V}(L_0^{ba}) \to \mathbf{V}(L_1^{ba})$ denote the linearization of this function at F defined by

$$\Psi(F+\epsilon V) = \Psi(F) + \epsilon\, [\partial_F\Psi]\,(V) + \mathcal{O}(\epsilon^2) \in \mathbf{V}(L_1^{ba};\epsilon).$$

Lemma 4.5. *The map $[\partial_{F_0}\Psi]$ is a $(+)$-action non-decreasing, filtration preserving chain map,*

$$d^{F_1}\circ[\partial_{F_0}\Psi] = [\partial_{F_0}\Psi]\circ d^{F_0}. \tag{4.4}$$

Furthermore $[\partial_{F_0}\Psi]$ induces an isomorphism of spectral sequences,

$$\left\{E_r^{p,q}(X_0^{ba},L_0^{ba})\right\} \xrightarrow[\cong]{[\partial_{F_0}\Psi]_*} \left\{E_r^{p,q}(X_1^{ba},L_1^{ba})\right\}.$$

Proof. Analogous to Lemma 4.4. □

4.3. **Chain homotopy.** We consider the same situations as in Subsection 4.2, now focusing on how the deformation affects chain maps. We will first treat the case when a stationary cobordism is joined to a moving one from above and second the case when the stationary cobordism is joined to the moving one from below.

Let L^a be a stationary cobordism joined to a moving one L^b_s, $0 \leq s \leq 1$, from above along Λ^0 forming a new 1-parameter family L^{ba}_s. With notation as in Subsection 4.2, we have

$$F_1 = \Phi(F_0) = F_0 + \Delta(F_0,K) \tag{4.5}$$

$$= \left(f_0^b + \Delta(f_0^b,k)|\, f^a\right), \tag{4.6}$$

where $K \in \mathbf{V}^+(L^{ba}_s)$ is given by

$$K = \left\{k \to f^a\right\}\left(f_0^b, f_1^b, f^a\right) = \left\{k \to f^a\right\}\left(f_0^b, f_0^b + \Delta(f_0^b,k), f^a\right). \tag{4.7}$$

Subsections 3.7 and 4.2 give the following diagram of chain maps

$$\begin{array}{ccc}
\mathbf{V}(L^a) & \xrightarrow{\left[\partial_{f^a}\left(f_0^b|\right]\right.} & \mathbf{V}(L_0^{ba}) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle [\partial_{F_0}\Phi]} \\
\mathbf{V}(L^a) & \xrightarrow{\left[\partial_{f^a}\left(f_1^b|\right]\right.} & \mathbf{V}(L_1^{ba})
\end{array}$$

Starting at the upper left corner of this diagram, going right and then down corresponds to linearizing (4.5) with respect to f^a-factors in F_0. Starting at the upper left corner, going down and then right corresponds to linearizing (4.6) with respect to f^a. We show below that these two filtration preserving chain maps induce the same morphism of spectral sequences. In order to do so we will make use of chain homotopies which are built from the following maps constructed using the split gluing pairing, see Subsection 4.1. For $v \in \mathbf{V}(L^a)$, define

$$\theta(v) = \left\{k \to v\right\}\left(f_0^b, f_0^b + \Delta(f_0^b,k), 0\right) \in \mathbf{V}(L_s^{ba}).$$

Let $W \in \mathbf{V}(L_s^{ba};\epsilon)$ and let K be as above. Recall the inductive construction of the K-increment disks of $F \in \mathbf{V}(L_0^{ba})$. We consider a deformed version of that construction as follows. With h^+ and h^- denoting the Hamiltonian vectors at the positive and negative ends of (L_s^{ba}), define

$$\rho_W^{(1)}(F) = d^F(K) + W = \{K \to h^+\}(F,F) + \{h^- \leftarrow K\}(F,F) + W,$$

and inductively

$$\rho_W^{(j)}(F) = \{K \to h^+\}\Big(F, F + \rho_W^{(j-1)}(F)\Big) + \{h^- \leftarrow K\}\Big(F, F + \rho_W^{(j-1)}(F)\Big) + W.$$

Let

$$\delta_W^{(j)}(F) = \rho_W^{(j)}(F) + \rho_W^{(j-1)}(F), \quad j > 0,$$

where we take $\rho_W^{(0)}(F) = 0$, and finally

$$\Delta_W(F,K) = \sum_{j=1}^{\infty} \delta_W^{(j)}(F). \tag{4.8}$$

The map we will use to construct chain homotopies is $\Theta\colon \mathbf{V}(L^a) \to \mathbf{V}(L_1^{ba})$ defined through the following equation

$$\Delta_{\epsilon\,\theta(v)}(F_0,K) = \Delta(F_0,K) + \epsilon\,\Theta(v) + \mathcal{O}(\epsilon^2) \in \mathbf{V}(L_1^{ba};\epsilon). \tag{4.9}$$

Recall the notation $\hat{L}_s^{ba}$ for the piece of L_s^{ba} where all K-disks have a boundary component.

Lemma 4.6. *The $(+)$-action non-decreasing, filtration preserving chain maps $[\partial_{F_0}\Phi] \circ \big[\partial_{f^a}\big(f_0^b\,|\big]$ and $\big[\partial_{f^a}\big(f_1^b\,|\big]$ induce the same morphism of spectral sequences. In other words the following diagram commutes*

$$\begin{array}{ccc} \{E_r^{p,q}(L^a)\} & \xrightarrow{[\partial_{f^a}(f_0^b\,|]_*} & \{E_r^{p,q}(L_0^{ba})\} \\ \Big\downarrow \mathrm{id} & & \Big\downarrow [\partial_{F_0}\Phi]_* \\ \{E_r^{p,q}(L^a)\} & \xrightarrow{[\partial_{f^a}(f_1^b\,|]_*} & \{E_r^{p,q}(L_1^{ba})\} \end{array} \tag{4.10}$$

Proof. A chain homotopy is provided by the map in (4.9). It shows that the chain maps induce identical maps on the E_1-term of the spectral sequences. Since morphisms of spectral sequences which agree on the E_1-term agree everywhere the lemma follows. The proof of the chain homotopy equation is similar to the proof of Lemma 3.5. □

Let L^b be a stationary cobordism joined to a moving one L_s^a, $0 \le s \le 1$, from below along Λ^0 forming a new 1-parameter family L_s^{ba}. With notation as in Subsection 4.2, we have

$$F_1 = \Psi(F_0) = F_0 + \Delta(F_0,K) \tag{4.11}$$

$$= \big(f^b|\, f_0^a + \Delta(f_0^a,k)\big)\,, \tag{4.12}$$

where $K \in \mathbf{V}^+(L_s^{ba})$ is given by

$$K = \{f^b \leftarrow k\}\big(f^b, f_0^a, f_1^a\big) = \{f^b \leftarrow k\}\big(f^b, f_0^a, f_0^a + \Delta(f_0^a, k)\big)\,. \tag{4.13}$$

Subsections 3.7 and 4.2 give the following diagram of chain maps

$$\begin{array}{ccc}
\mathbf{V}(L^b) & \xrightarrow{[\partial_{f^b} | f_0^a)]} & \mathbf{V}(L_0^{ba}) \\
{\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle [\partial_{F_0}\Psi]} \\
\mathbf{V}(L^a) & \xrightarrow{[\partial_{f^b} | f_1^a)]} & \mathbf{V}(L_1^{ba})
\end{array}$$

Define $\Omega\colon \mathbf{V}(L^b) \to \mathbf{V}(L_1^{ba})$ as follows. For $v \in \mathbf{V}(L^b)$ let

$$\Omega(v) = \{v \leftarrow k\}\big(f^b, f_0^a, f_0^a + \Delta(f_0^a, k)\big) \in \mathbf{V}(L_1^{ba}).$$

Lemma 4.7. *The $(+)$-action non-decreasing, filtration preserving chain maps $[\partial_{F_0}\Psi] \circ \big[\partial_{f^b} \,|\, f_0^a)\big]$ and $\big[\partial_{f^b} \,|\, f_1^a)\big]$ induce the same morphism of spectral sequences. In other words the following diagram commutes*

$$\begin{array}{ccc}
\big\{E_r^{p,q}(L^b)\big\} & \xrightarrow{[\partial_{f^b} | f_0^a)]_*} & \big\{E_r^{p,q}(L_0^{ba})\big\} \\
{\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle [\partial_{F_0}\Phi]_*} \\
\big\{E_r^{p,q}(L^b)\big\} & \xrightarrow{[\partial_{f^b} | f_1^a)]_*} & \big\{E_r^{p,q}(L_1^{ba})\big\}
\end{array} \tag{4.14}$$

Proof. The proof is analogous to the proof of Lemma 4.6. A chain homotopy is given by Ω. $\square$

5. Proofs

Proof of Theorem 1.1. If L_s, $0 \le s \le 1$, is a 1-parameter family of exact cobordisms fixed outside a compact set then, after small perturbation, the differential is independent of s except when values of s for which there are (-1)-disks are passed. It follows from Lemmas 4.4 and 4.5 that the spectral sequence remains unchanged at such instances. $\square$

Proof of Theorem 1.2. Let Λ_0 and Λ_1 be Legendrian submanifolds of $J^1(M)$ which are Legendrian isotopic through Λ_s, $0 \le s \le 1$. Such an isotopy determines an exact cobordism L_{01} with $(+\infty)$-boundary Λ_1 and $(-\infty)$-boundary Λ_0, see e.g. [3, Appendix A]. Using the inverse isotopy we get a cobordism L_{10} with $(+\infty)$-boundary Λ_0 and $(-\infty)$-boundary Λ_1. Joining the negative end of L_{01} to the positive end of L_{10} we obtain a cobordism L_{11} which admits a compactly supported isotopy deforming it to $\Lambda_1 \times \mathbb{R}$. Let f^j, $j = 0, 1$, denote the potential vectors in $\mathbf{V}^+(\Lambda_j \times \mathbb{R})$ and let F^{ij}, $i, j \in \{0, 1\}$ denote the potential vectors in $\mathbf{V}^+(L_{ij})$. Then there are grading respecting chain maps as follows

$$\begin{array}{ccc} \mathbf{V}(\Lambda_1 \times \mathbb{R}) & \xrightarrow{\left[\partial_{f^1}\left(F^{01}\right|\,\right]} & \mathbf{V}(L_{01}) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \left[\partial_{F^{01}}\left(F^{11}\right|\,\right]} \\ \mathbf{V}(\Lambda_1 \times \mathbb{R}) & \xleftarrow{\;\;\Phi\;\;} & \mathbf{V}(L_{11}), \end{array}$$

where Φ is the chain map from Theorem 1.1, see also Lemma 4.4, induced by the deformation of L_{11}. Lemma 2.10 implies that the composition of the first two maps equals $[\partial_{f^1}(F^{11}|\,]$ which, after composition with Φ, by Lemma 4.6, is chain homotopic to $[\partial_{f^1}(f^1|\,] = \mathrm{id}$. It follows that the composition

$$\Phi \circ \left[\partial_{F^{01}}\left(F^{11}\right|\,\right] \circ \left[\partial_{f^1}\left(F^{01}\right|\,\right]$$

induces an isomorphism of spectral sequences. Theorem 1.1 implies that Φ induces an isomorphism. Thus, the map induced by $[\partial_{f^1}(F^{01}|\,]$ is a monomorphism and that induced by $[\partial_{F^{01}}(F^{11}|\,]$ is an epimorphism.

Gluing L_{11} at its negative end to the positive end of L_{01} we get a cobordism L_{011}, with potential vector $F^{011} \in \mathbf{V}^+(L_{011})$, which can be deformed to (L_{01}) inducing a chain map Ψ. Arguing as above using the diagram

$$\begin{array}{ccc} \mathbf{V}(L_{01}) & \xrightarrow{\left[\partial_{F^{01}}\left(F^{11}\right|\,\right]} & \mathbf{V}(L_{11}) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \left[\partial_{F^{11}}\left(F^{011}\right|\,\right]} \\ \mathbf{V}(L_{01}) & \xleftarrow{\;\;\Psi\;\;} & \mathbf{V}(L_{011}), \end{array}$$

we find that the map induced by $[\partial_{F^{01}}(F^{11}|\,]$ is also a monomorphism and it follows that

$$\left[\partial_{f^1}\left(F^{01}\right|\,\right] : \mathbf{V}(\Lambda_1 \times \mathbb{R}) \to \mathbf{V}(\Lambda_{01}) \tag{5.1}$$

induces an isomorphism of spectral sequences. We find

$$\left\{E_r^{p,q}(\Lambda_1)\right\} = \left\{E_r^{p,q}(L_{01})\right\}.$$

A similar argument, which uses Lemmas 4.5 and 4.7 instead of Lemmas 4.4 and 4.6, shows that

$$\left[\partial_{f^0}\,|\,F^{01}\right)\big] : \mathbf{V}(\Lambda_0 \times \mathbb{R}) \to \mathbf{V}(\Lambda_{01}). \tag{5.2}$$

induces an isomorphism of spectral sequences. We find that

$$\left\{E_r^{p,q}(\Lambda_0)\right\} = \left\{E_r^{p,q}(L_{01})\right\}.$$

as well. This proves the theorem. □

6. Examples

In this section, we first study consequences of different definitions of admissible disks in a simple example. Second, we compute the rational admissible SFT spectral sequence for three parallel copies of the 0-section in $J^1(M)$ where M is an n-manifold.

The second example is the first instance of a local version of the following general approach which shows that the resolution of the boundary bubbling problem presented in this paper is closely related to those of [12] and [2] in the setting of Lagrangian Floer homology. If $\Lambda \subset J^1(M)$ is a Legendrian submanifold (possibly connected), then we consider the many component Legendrian submanifold $\tilde{\Lambda}$ consisting of finitely many nearby parallel (i.e., parallel along the Reeb flow) copies of Λ. Partitioning the collection of parallel copies into pieces we can apply the rational admissible SFT invariant. In fact, much like in [5], the differential on $\mathbf{V}(\tilde{\Lambda} \times \mathbb{R})$ can in this case be computed in terms of (all) moduli spaces of holomorphic disks in $J^1(M) \times \mathbb{R}$ with boundary on $\Lambda \times \mathbb{R}$ in combination with the spaces of Morse flow trees in Λ, see [11, 3], determined by a finite collection of Morse functions on Λ. In this sense, the method for dealing with boundary bubbling in the present paper is related to [12] and to [2], though in [2], only one Morse function and flow lines appears, rather than as here and in [12], many Morse functions and flow trees.

6.1. **A comparison in a simple example.** Consider the Legendrian two component link $\Lambda_r \subset J^1(\mathbb{R}) = \mathbb{R}^3$ in Figure 3. As mentioned in Remark 2.3, there are two possible

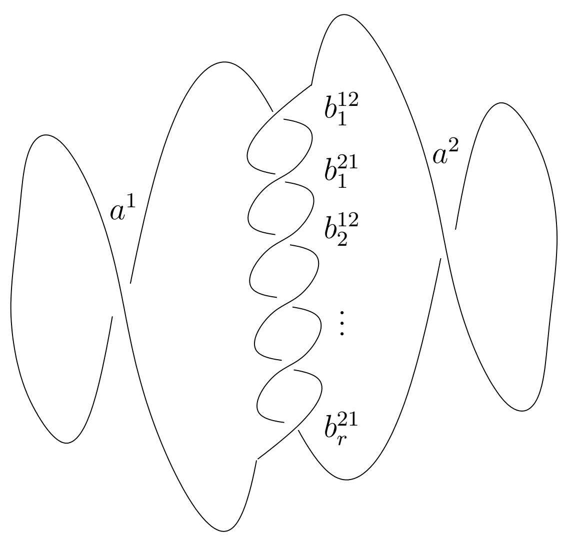

Figure 3. The Lagrangian projection of Λ_r.

definitions of admissible disks, one including formal disks with only negative punctures, as long as at least one of them is mixed, and one excluding such formal disks requiring that all disks have at least one positive puncture. The latter is the definition used in this paper. We denote vector spaces of the latter version $\mathbf{V}$ and of the former $\tilde{\mathbf{V}}$ and the corresponding differentials d and $\tilde{d}$, respectively. The corresponding spectral sequences are denoted $E_r^{p,q}$ and $\tilde{E}_r^{p,q}$, respectively. Consider the two components of the link as distinct pieces. In our treatment below we will forget the distinct homotopy classes of formal disks and simply represent them as cyclic words of Reeb chords. If c is a Reeb chord we write p_c for a positive puncture at c and q_c for a negative puncture at c.

By Lemma 2.4, the pure admissible disks in $\mathbf{V}(\Lambda_r \times \mathbb{R})$ and $\tilde{\mathbf{V}}(\Lambda_r \times \mathbb{R})$ are

$$p_{a^\sigma} q_{a^\sigma} \overset{k}{\dots} q_{a^\sigma}, \quad \sigma = 0, 1, \quad 0 \leq k.$$

The grading of such a disk is $1 - k$. The mixed admissible disks in $\mathbf{V}(\Lambda_r \times \mathbb{R})$ are

$$p_{b_j^{12}} q_{a^2} \overset{t}{\dots} q_{a^2} p_{b_k^{21}} q_{a^1} \overset{s}{\dots} q_{a^1} \quad j, k \in \{1, \dots, r\}, \tag{6.1}$$

$$p_{b_j^{12}} q_{a^2} \overset{t}{\dots} q_{a^2} q_{b_k^{12}} q_{a^1} \overset{s}{\dots} q_{a^1} \quad j, k \in \{1, \dots, r\}, \tag{6.2}$$

$$p_{b_j^{21}} q_{a^1} \overset{s}{\dots} q_{a^1} q_{b_k^{21}} q_{a^2} \overset{t}{\dots} q_{a^2} \quad j, k \in \{1, \dots, r\}. \tag{6.3}$$

The formal dimension of the disks in (6.1) is $1 - (s + t)$ and the formal dimension of the disks in (6.2) and (6.3) is $-(s + t)$. In $\tilde{\mathbf{V}}(\Lambda_r \times \mathbb{R})$ there are also the elements

$$q_{b_j^{12}} q_{a^1} \overset{t}{\dots} q_{a^1} q_{b_k^{21}} q_{a^2} \overset{s}{\dots} q_{a^2} \quad j, k \in \{1, \dots, r\}, \tag{6.4}$$

of dimension $-(s + t) - 1$.

The Hamiltonian vector is easily computed. We have

$$h = \sum_{j=1}^{r} p_{b_j^{12}} p_{b_j^{21}} + \sum_{j=1}^{r-1} p_{b_j^{21}} p_{b_{j+1}^{12}}.$$

Since the Hamiltonian does not contain any pure formal disks the differential it induces does not affect pure punctures q_{a^σ}, $\sigma = 1, 2$ in the admissible formal disks above. Consequently, the differential d on $\mathbf{V}(\Lambda_r)$ is determined by the following calculation

$$\begin{aligned} d(p_{b_j^{12}} p_{b_k^{21}}) &= 0, \\ d(p_{b_j^{12}} q_{b_k^{12}}) &= p_{b_j^{12}} p_{b_k^{21}} + p_{b_j^{12}} p_{b_{k-1}^{21}}, \\ d(p_{b_j^{21}} q_{b_k^{21}}) &= p_{b_j^{21}} p_{b_k^{12}} + p_{b_j^{21}} p_{b_{k+1}^{12}}. \end{aligned} \tag{6.5}$$

To compute $\tilde{d}$ on $\tilde{\mathbf{V}}(\Lambda_r)$, we note that it agrees with d on the generators in (6.5) and that

$$\tilde{d}(q_{b_j^{12}} q_{b_k^{21}}) = q_{b_j^{12}} p_{b_k^{21}} + p_{b_{j-1}^{21}} q_{b_k^{21}} + p_{b_k^{12}} q_{b_j^{12}} + p_{b_{k+1}^{12}} q_{b_j^{12}}. \tag{6.6}$$

In order to compare the consequences of the two different definitions of admissible formal disks we will compute $E_r^{p,q}$ and $\tilde{E}_r^{p,q}$. Since the Hamiltonian vector does not involve any pure disks the parts of $\mathbf{V}(\Lambda_r)$ and $\tilde{\mathbf{V}}(\Lambda_r)$ generated by pure disks survive in homology and give copies of the homology of a trivial two component link. In our computations below we quotient out by the piece generated by pure formal disks. The corresponding quotients of $\mathbf{V}(\Lambda_r)$ and $\tilde{\mathbf{V}}(\Lambda_r)$ are spanned by mixed admissible disks. For simplicity, we keep the same notation for these quotients as for the corresponding full spaces.

We first consider $E_r^{p,q}$. Since the differential strictly increases the number of positive punctures we find that the differential, the homology of which is the E_1-term, is trivial and that $E_1^{1,m}$ satisfies the following. If $m > -1$ then

$$\dim(E_1^{1,m}) = 0.$$

If $m \leq -1$ then

$$\dim(E_1^{1,m}) = 2(r^2 - m),$$

and $E_1^{1,m}$ is generated by the elements

$$p_{b_j^{12}} q_{a^2} \overset{t}{\dots} q_{a^2} q_{b_k^{12}} q_{a^1} \overset{s}{\dots} q_{a^1} \quad j,k = 1,\dots,r, \quad s+t = -1-m$$
$$p_{b_j^{21}} q_{a^1} \overset{s}{\dots} q_{a^1} q_{b_k^{21}} q_{a^2} \overset{t}{\dots} q_{a^2} \quad j,k = 1,\dots,r, \quad s+t = -1-m.$$

Similarly, the $E_1^{2,m}$-term satisfies the following. If $m > -1$ then

$$\dim(E_1^{2,m}) = 0.$$

If $m \leq -1$ then

$$\dim(E_1^{2,m}) = r^2 - m,$$

and $E_1^{2,m}$ is generated by the elements

$$p_{b_j^{12}} q_{a^2} \overset{t}{\dots} q_{a^2} p_{b_k^{21}} q_{a^1} \overset{s}{\dots} q_{a^1} \quad j,k = 1,\dots,r, \quad s+t = -1-m.$$

The differential on the E_1-term is trivial on $E_1^{2,m}$ since all disks in the Hamiltonian have at least two positive punctures, see Lemma 3.2. Furthermore, for $m \leq -1$, the differential

$$E_1^{1,m} \to E_1^{2,m}$$

has rank $r^2 - m$ and its kernel is generated by the elements

$$\sum_{v=1}^{k} p_{b_j^{12}} q_{a^2} \overset{s}{\dots} q_{a^2} q_{b_v^{12}} q_{a^1} \overset{t}{\dots} q_{a^1} \quad + \quad \sum_{v=j}^{r} p_{b_k^{21}} q_{a^1} \overset{t}{\dots} q_{a^1} q_{b_v^{21}} q_{a^2} \overset{s}{\dots} q_{a^2},$$

where $j,k \in \{1,\dots,r\}$ and where $s+t = -1-m$.

Consequently,

$$\dim\bigl(E_2^{p,m}(\Lambda_r)\bigr) = \begin{cases} 0, & \text{if } p \neq 1 \text{ or } m > -1, \\ r^2 - m, & \text{if } p = 1 \text{ and } m \leq -1. \end{cases}$$

In order to calculate $\tilde{E}_r^{p,q}$ we need only modify the above slightly. First, $\tilde{E}_1^{l,m}$ agrees with $E_1^{l,m}$ as described above for $l \geq 1$. However, there is in this case also the term $\tilde{E}_1^{0,m}$ which satisfies the following. If $m > -1$ then

$$\dim(\tilde{E}_1^{0,m}) = 0.$$

If $m \leq -1$ then

$$\dim(\tilde{E}_1^{0,m}) = r^2 - m$$

and $\tilde{E}_1^{0,m}$ is generated by

$$q_{b_j^{12}} q_{a^1} \overset{t}{\dots} q_{a^1} q_{b_k^{21}} q_{a^2} \overset{s}{\dots} q_{a^2} \quad j,k = 1,\dots,r, \quad s+t = -1-m$$

As in the previous case the differential on $\tilde{E}_1$ is trivial on $\tilde{E}_1^{2,m}$ and it has the same kernel on $\tilde{E}_1^{1,m}$, of dimension $r^2 - m$, as that of the differential on $E_1^{1,m}$ described above. A straightforward calculation shows that the differential

$$\tilde{E}_1^{0,m} \to \tilde{E}_1^{1,m},$$

is injective and maps onto the kernel of the differential of $\tilde{E}_1^{1,m}$ and we find that

$$\dim(\tilde{E}_2^{l,m}) = 0$$

for all l, m.

This computation shows that the choice of definition of admissible disk gives rise to quite different theories: $E_2(\Lambda_r)$ detects r whereas $\tilde{E}_2(\Lambda_r)$ does not.

6.2. **Three parallel copies of the zero-section.** Let M be an n-manifold and consider the 0-section $\Lambda_0 \subset J^1(M)$ and two parallel copies Λ_1 and Λ_2 which are the graphs of two Morse functions f_1 and f_2 on M where $0 < f_1 < f_2$. We will compute the rational SFT invariant of the Legendrian submanifold $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$, where the pieces of Λ are its connected components, in terms of the cohomology ring of M. As in Subsection 6.1, we work on the quotient of $\mathbf{V}(\Lambda \times \mathbb{R})$ obtained by forgetting the homotopy classes of the admissible formal disks. As there, we keep the notation $\mathbf{V}(\Lambda \times \mathbb{R})$ for the quotient vector space and note that the $\mathbb{Z}$-grading descends to the quotient since $c_1(T^*M) = 0$ and since the Maslov class of the 0-section vanishes.

6.2.1. *Reeb chords of* Λ. All Reeb chords of Λ are mixed. Reeb chords connecting Λ_0 and Λ_j correspond to critical points of f_j, $j = 1, 2$ and Reeb chords connecting Λ_1 and Λ_2 correspond to critical points of $f_2 - f_1$. We use the following notation: if c is a Reeb chord then we write Morse-index(c) for the index of the corresponding critical point of the positive function difference.

6.2.2. *Admissible disks.* Since we disregard homotopy classes, any formal disk is uniquely determined by its punctures. As in Subsection 6.1, we write q_c and p_c to denote a negative- respectively positive puncture at the Reeb chord c and identify formal disks with words of punctures up to cyclic permutation. It is straightforward to show that admissible formal disks of Λ are of the following types.

- (2_1) Disks with two punctures of one of the forms $p_b q_c$, where both the incoming and the outgoing components at the endpoints of b and c agree.
- (3_1) Disks with three punctures of the form $p_c q_a q_b$, where c connects Λ_0 to Λ_2, a connects Λ_1 to Λ_2, and b connects Λ_0 to Λ_1 .
- (3_2) Disks with three punctures of the form $p_a p_b q_c$, where a connects Λ_0 to Λ_1, b connects Λ_1 to Λ_2, and c connects Λ_0 to Λ_2 .

In order to compute the dimension of a formal disk we note that there is a $1-1$ correspondence between $\mathbb{R}$-families of holomorphic disks in $J^1(M) \times \mathbb{R}$ with boundary on $\Lambda \times \mathbb{R}$ and holomorphic disks in T^*M with boundary on the projection of Λ, see [8]. As shown in [4] the dimensions of the formal disks above are as follows (note however that we are considering the dimension in symplectization here whereas that in [4] corresponds to the dimension in the Lagrangian projection).

(2_1)

$$\dim(p_b q_c) = \text{Morse-index}(b) - \text{Morse-index}(c),$$

.

(3_1)

$$\dim(p_c q_a q_b) = \text{Morse-index}(c) - \text{Morse-index}(a) - \text{Morse-index}(b) + 1,$$

(3_2)

$$\dim(p_a p_b q_c) = \text{Morse-index}(a) + \text{Morse-index}(b) - \text{Morse-index}(c) - n + 1.$$

6.2.3. *The Hamiltonian vector and the spectral sequence.* Using the relation between rigid holomorphic disks and rigid flow trees lines, see [11, 4], it is straightforward to describe the Hamiltonian vector. We write

$$h = h_{2_1} + h_{3_1} + h_{3_2},$$

where h_{2_1} denotes the sum of all disks of type (2_1) etc. Then

- h_{2_1} counts rigid flow lines for the respective functions.
- h_{3_1} counts rigid flow trees in which the negative gradient flow at the vertex which is a critical of f_2 is outgoing and the negative gradient flows at the other two vertices are incoming.
- h_{3_2} counts rigid flow trees in which the negative gradient flow at the vertex which is a critical of f_2 is incoming and the negative gradient flows at the other two vertices are outgoing.

It is then straightforward to compute the E_1-term of the spectral sequence: if $f\colon M \to \mathbb{R}$ is a Morse function let $C_*(f)$ denote the corresponding Morse-Witten homology complex and let $C^*(f)$ denote the corresponding cohomology complex. From the above description of admissible disks and of the Hamiltonian vector it follows that we can identify $F^2(\mathbf{V}(\Lambda \times \mathbb{R}))$ with the chain complex

$$C^*(f_1) \otimes C^*(f_2) \otimes C_*(f_2 - f_1)$$

with the standard differential. Hence

$$E_1^{2,*}(\Lambda) = H^*(M) \otimes H^*(M) \otimes H_*(M).$$

Similarly, we identify $F^1(\mathbf{V}(\Lambda \times \mathbb{R}))$ with

$$\begin{aligned} &C^*(f_1) \otimes C_*(f_1) \oplus C^*(f_2 - f_1) \otimes C_*(f_2 - f_1) \\ &\quad \oplus C^*(f_2) \otimes C_*(f_2) \oplus C^*(f_2) \otimes C_*(f_1) \otimes C_*(f_2). \end{aligned}$$

Here the differential can be computed in two steps, filtering by the number of negative punctures. After the first step we pass to homology in each complex. In the second step there is only one non-trivial component of the differential:

$$1 \otimes D\colon H^*(f_2) \otimes H_*(f_2) \to H^*(f_2) \otimes H_*(f_1) \otimes H_*(f_2).$$

It follows from the intersection product interpretation of flow trees $\gamma_1 \otimes \gamma_2$ appears with coefficient 1 in the expansion of $D(\gamma)$ if and only if $\gamma_1 \bullet \gamma_2 = \gamma$ where $\bullet$ denotes the intersection product on homology. Hence

$$\begin{aligned} E_1^{1,*}(\Lambda) =& H^*(M) \otimes H_*(M) \oplus H^*(M) \otimes H_*(M) \\ & \oplus H^*(M) \otimes \ker(D) \oplus H^*(M) \otimes \operatorname{coker}(D). \end{aligned}$$

Here $\ker(D)$ correspond to indecomposable homology classes.

Finally, in order to compute the E_2-term (which trivially equals the E_3-term since there are no admissible disks with three positive punctures) the only non-trivial differential to be considered is

$$D' \times \iota \colon H^*(M) \otimes \ker(D) \to H^*(M) \otimes H^*(M) \otimes H_*(M),$$

where ι is the inclusion $\ker(D) \subset H_*(M)$ and where $\alpha_1 \otimes \alpha_2$ appears with coefficient 1 in the expansion of $D'(\alpha)$ if and only if $\alpha_1 \cup \alpha_2 = \alpha$ where $\cup$ denotes the cup product on cohomology. Hence

$$\begin{aligned} E_2^{1,*}(\Lambda) =& H^*(M) \otimes H_*(M) \oplus H^*(M) \otimes H_*(M) \\ & \oplus \ker(D') \otimes \ker(D) \oplus H^*(M) \otimes \operatorname{coker}(D). \end{aligned}$$

and

$$E_2^{2,*} = \operatorname{coker}(D' \otimes \iota).$$

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