DIFFERENTIAL 3-KNOTS IN 5-SPACE WITH AND WITHOUT SELF INTERSECTIONS

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ABSTRACT. Regular homotopy classes of immersions $S^3 \to \mathbb{R}^5$ constitute an infinite cyclic group. The classes containing embeddings form a subgroup. We prove that this subgroup has index 24 and that the signature of a Seifert-surface of an embedding determines its regular homotopy class. The obstruction for a generic immersion to be regularly homotopic to an embedding is described in terms of geometric invariants of its self intersection. Geometric properties of self intersections are used to construct two invariants J and St of generic immersions which are analogous to Arnold's invariants of plane curves [1]. We prove that J and St are independent first order invariants and that any first order invariant is a linear combination of these.

As by-products, some invariants of immersions $S^3 \to \mathbb{R}^4$ are obtained. Using them, we find restrictions on the topology of self intersections.

1. INTRODUCTION

Immersions and embeddings form open subspaces of the space of C^r-maps $S^k \to \mathbb{R}^{k+n}$, $r \geq 1$. Smale [16] showed that the path components of the space of immersions (or, which is the same, the regular homotopy classes of immersions) $S^k \to \mathbb{R}^{k+n}$ are in one to one correspondence with the elements of $\pi_k(V_{k+n,k})$, the k^{th} homotopy group of the Steifel manifold of k-frames in $(k+n)$-space. This far-reaching result translates problems in geometry to homotopy theory.

Indicating the way back to geometry, Smale suggested the following problems ([16], p.329): *"Find explicit representatives of regular homotopy classes ... What regular homotopy classes have an embedding for representative?"*. We solve these problems in the case $S^3 \to \mathbb{R}^5$. Representatives of regular homotopy classes are given in Section 8.3 and the somewhat surprising answer to the second problem is given by Theorem 1 below.

An answer to the second problem gives information about how the inclusion of the space of embeddings into the space of immersions is organized. We restrict attention to the second problem in a sequence of cases, interesting from the point of view of knot theory.

Knot theory is the study of path components of the space of embeddings. In codimension two (i.e. for $S^k \to \mathbb{R}^{k+2}$), where embeddings share many features of classical knots ($S^1 \to \mathbb{R}^3$), this subject is richer than in other dimensions. We restrict our consideration of Smale's problem to the case of codimension two. It turns out to be most interesting in the cases $S^{4j-1} \to \mathbb{R}^{4j+1}$. Indeed, we have:

For $k = 1$ (i.e. in the case $S^1 \to \mathbb{R}^3$), the group $\pi_1(V_{3,1})$ enumerating regular homotopy classes is trivial and all immersions are regularly homotopic. For $k = 2$ (i.e. in the case $S^2 \to \mathbb{R}^4$), $\pi_2(V_{4,2}) = \mathbb{Z}$, the regular homotopy class of an immersion is determined by the algebraic number of self intersection points and thus, only one class contains embeddings. For $k \geq 3$, $\pi_k(V_{k+2,k}) \cong \pi_k(SO(k+2)) \cong \pi_k(SO(\infty))$, where $SO(n)$ is the group of orthonormal $n \times n$-matrices of determinant one. By Bott-periodicity, $\pi_k(SO(\infty))$ is isomorphic to $\mathbb{Z}_2$ if $k = 8i$ or $k = 8i+1$, it is isomorphic to $\mathbb{Z}$ if $k = 4j-1$, and is 0 in the other cases. Thus, unless $k = 4j-1$, the space of immersions $S^k \to \mathbb{R}^{k+2}$ has either one or two components (in the latter case Smale's problem is open). We restrict attention further and consider the cases $S^{4j-1} \to \mathbb{R}^{4j+1}$, where the spaces of immersions have infinitely many components:

For $j = 1$ we get the case $S^3 \to \mathbb{R}^5$ which is the subject of this paper. We prove that exactly every 24^{th} regular homotopy class have an embedding for representative. The cases $j > 1$ will be treated in a forthcoming paper. Also here, there are embeddings that are different up to regular homotopy.

In dimensions where there are embeddings in different regular homotopy classes, it is impossible to express the regular homotopy class of a generic immersion in terms of its self intersection. This is in contrast to many other cases where this is possible: For example, in the cases $S^k \to \mathbb{R}^{2k}$, $k \geq 2$ the regular homotopy class is determined by the algebraic number of self intersection points (modulo 2 if k is odd), see [16], and in the cases $S^k \to \mathbb{R}^{2k-r}$, $r = 1, 2$ self intersection formulas for regular homotopy are given in [4] and [5].

Generic immersions have simple self intersections. For example, in the case $S^1 \to \mathbb{R}^3$ a generic immersion has empty self intersection and thus, generic immersions are embeddings. From this point of view, the analogue of classical knot theory in other dimensions is the study of path components of the space of generic immersions and we may think of generic immersions as knots with self intersections.

The space of generic immersions is dense in the corresponding space of immersions and its complement is a stratified hypersurface. Using the stratification of this complement, Vassiliev [1] introduced the notion of finite order invariants in classical knot theory. There are natural analogies of this notion for invariants of generic immersions in other dimensions (see Section 6.3). Arnold [2] found first order invariants of generic plane curves ($S^1 \to \mathbb{R}^2$). Theorem 2 below shows that, up to first order, the space of generic immersions $S^3 \to \mathbb{R}^5$ is similar to the space of generic plane curves.

The composition of an immersions $S^3 \to \mathbb{R}^4$ and the inclusion $i\colon \mathbb{R}^4 \to \mathbb{R}^5$ is an immersion into 5-space. Proposition 7.1.2 shows that two immersions of S^3 into 4-space are regularly homotopic in 5-space, after composing them with the inclusion, if and only if one of them is regularly homotopic in 4-space to the connected sum of the other one and a finite number of immersions regularly homotopic to the composition of the standard embedding and a reflection in a hyperplane in $\mathbb{R}^4$.

Theorems 3, 4, and 5 give information about the self intersections of a generic immersions $g\colon S^3 \to \mathbb{R}^4$ and its relation to the self intersection of a generic immersion $f\colon S^3 \to \mathbb{R}^5$ regularly homotopic to $i \circ g$.

2. Main results

In this section, the main theorems of the paper are formulated.

2.1. *Embeddings in the space of immersions*

Regular homotopy classes of immersions $S^3 \to \mathbb{R}^5$ are known to form an infinite cyclic group **Imm** under connected sum. The classes that contain embeddings form a subgroup **Emb** $\subset$ **Imm**. We now state Theorem 1 describing the extension

Emb $\to$ **Imm** algebraically and devote the rest of this section to identify the homomorphisms involved there in topological terms.

THEOREM 1. *The following diagram of Abelian groups has exact rows, commutes and the vertical arrows are isomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Emb} & \longrightarrow & \mathbf{Imm} & \xrightarrow{\lambda\oplus\beta} & \mathbb{Z}_3\oplus\mathbb{Z}_8 & \longrightarrow & 0 \\ \downarrow & & \sigma\downarrow & & \downarrow\Omega & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 24} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{24} & \longrightarrow & 0 \end{array}.$$

Theorem 1 is proved in Section 9.1.

The homomorphism σ: Recall that if $f\colon S^3 \to \mathbb{R}^5$ is an embedding then there exists a compact orientable 4-dimensional manifold $V^4 \subset \mathbb{R}^5$ with $\partial V^4 = f(S^3)$. We call such a manifold a *Seifert-surface* of f. Its signature $\sigma(V^4)$ is divisible by 16 and is known to depend only on f. *For $\xi \in$ **Emb**, define $\sigma(\xi) \in \mathbb{Z}$ as $\frac{\sigma(V)}{16}$, where V is a Seifert-surface of an embedding representing ξ.* Theorem 1 implies that σ induces an isomorphism **Emb** $\to \mathbb{Z}$. The idea that the signature of an embedding determines its regular homotopy class is due to Rokhlin [15].

The homomorphism Ω: Any immersion $f\colon S^3 \to \mathbb{R}^5$ is determined up to regular homotopy by its Smale invariant $\Omega(f) \in \pi_3(V_{5,3}) \cong \mathbb{Z}$ (see [16] or Definition 3.2.1). *For $\xi \in$ **Imm**, define $\Omega(\xi) \in \mathbb{Z}$ as $\Omega(g)$, where g is an immersion representing ξ.*

The homomorphism λ: If $f\colon S^3 \to \mathbb{R}^5$ is a generic immersion then its self intersection $M_f \subset \mathbb{R}^5$ is a closed 1-dimensional manifold. Orientations of S^3 and $\mathbb{R}^5$ induce an orientation of M_f. We can push M_f off the image of f (see Section 6.2). Let $\mathrm{lk}(f)$ denote the linking number of the perturbed M_f and $f(S^3)$ in $\mathbb{R}^5$. m *For $\xi \in$ **Imm**, define $\lambda(\xi) \in \mathbb{Z}_3$ as $\mathrm{lk}(g)$ modulo 3, where g is a generic immersion representing ξ.*

The homomorphism β: The normal bundle of an immersion $f_0\colon S^3 \to \mathbb{R}^5$ is 2-dimensional orientable and therefore trivial. Hence, f_0 admits a normal vector field. This implies that f_0 is regularly homotopic to a (generic) immersion $f_1\colon S^3 \to \mathbb{R}^4$, composed with the inclusion $\mathbb{R}^4 \to \mathbb{R}^5$ (see [7]). Resolving the self intersection of a generic immersion $f\colon S^3 \to \mathbb{R}^4$, we obtain a smooth surface F_f (Lemma 5.1.5) and the immersion f induces a pin (i.e. Pin^-) structure on F_f (Section 7.4). There is a one to one correspondence between pin structures on a surface F and $\mathbb{Z}_4$-quadratic functions q on its first homology $H_1(F;\mathbb{Z}_2)$. Pin structures on a surface are classified up to cobordism by the Brown invariant $\beta(q) \in \mathbb{Z}_8$ of the corresponding function. Let $\beta(f) \in \mathbb{Z}_8$ denote the Brown invariant of the quadratic function corresponding to the pin structure induced by f on F_f. *For $\xi \in$ **Imm**, define $\beta(\xi) \in \mathbb{Z}_8$ as $\beta(g)$, where $g\colon S^3 \to \mathbb{R}^4 \subset \mathbb{R}^5$ is a generic immersion representing ξ.* The author does not know how to calculate $\beta(f)$ for a generic immersion $f\colon S^3 \to \mathbb{R}^5$ in terms of the geometry of its self intersection, without pushing it down to $\mathbb{R}^4$. However, $\beta(f)$ modulo 4, can be calculated in that way (see Section 9.2).

2.2. *First order invariants of generic immersions*

In generic one-parameter families of immersions there are isolated instances of non-generic immersions. In generic one-parameter families of immersions $S^3 \to \mathbb{R}^5$ such instances are immersions with one self tangency or one triple point. The same is true for generic one-parameter families of plane curves where one can distinguish two local types of self tangencies: direct (the tangent vectors point in the same direction) and reverse (the tangent vectors point in opposite directions).

An *invariant of generic immersions* is a function which is constant on the path components of the space of generic immersions. Such a function may change when we pass through instances of non-generic immersions.

Arnold [2] found three independent first order invariants of generic plane curves: The invariant J^+ which changes at direct self tangency instances and does not change under other moves, the invariant J^- which changes under reverse self tangency moves and does not change under other moves, and the invariant Strangeness St which changes under triple point moves and does not change under other moves.

For a generic immersion $f\colon S^3 \to \mathbb{R}^5$, define $J(f)$ to be the number of self intersection components of f and define $L(f) = \frac{1}{3}(\mathrm{lk}(f) - \widetilde{\lambda}(f))$, $\widetilde{\lambda} \in \{0,1,2\} \subset \mathbb{Z}$ is a lifting of $\lambda \in \mathbb{Z}_3$ (see Section 2.1).

THEOREM 2. *The invariant J changes by ± 1 under self tangency moves and does not change under triple point moves. The invariant L changes by ± 1 under triple point moves and does not change under self tangency moves. The invariants J and L are first order invariants. Moreover, if v is any first order invariant then the restriction of $v|U$, where U is a path component of the space of immersions, is a linear combination of $J|U$ and $L|U$.*

Theorem 2 is proved in Section 6.4. Although there are two local types of selftangencies in the case $S^3 \to \mathbb{R}^5$ (see Proposition 5.3.2), in contrast to the case of plane curves, J can not be splitted into more refined first order invariants.

Arnold defined the invariant St in such a way that it is additive under connected summation of plane curves and showed that this property together with the changes of St under local moves and its is orientation independence completely characterizes the invariant (up to a constant).

The invariant L is neither additive under connected summation nor symmetric with respect to orientation. To get the analogue of Arnold's strangeness of plane curves for immersions $f\colon S^3 \to \mathbb{R}^5$ we define $\mathrm{St}(f) = \frac{1}{3}(\mathrm{lk}(f) + \Omega(f))$. Then St is additive under connected sum, changes exactly as L under local moves, changes sign under composing immersions with an orientation reversing diffeomorphism of S^3, and is completely characterized by these properties up to a constant (see Proposition 6.5.1).

The reason for stating Theorem 2 in terms of L instead of St (which works equally well) is that $L(f)$ in contrast to $\mathrm{St}(f)$ can be calculated in terms of the self intersection of f.

2.3. *Immersions into 4-space*

The self intersection of a generic immersion $f\colon S^3 \to \mathbb{R}^4$ consists of 2-dimensional sheets of double points, 1-dimensional curves of triple points and isolated quadruple points (see Definition 5.1.1). Resolving the quadruple and triple points turns the self intersection into a smooth surface F_f (Lemma 5.1.5).

The following three theorems are proved in Section 9.2.

THEOREM 3. *A generic immersion $g\colon S^3 \to \mathbb{R}^4$ has an odd number of quadruple points if and only if its (resolved) self intersection surface F_g has odd Euler characteristic.*

As mentioned (Section 2.1), any immersion $S^3 \to \mathbb{R}^5$ is regularly homotopic to a composition of an immersion $S^3 \to \mathbb{R}^4$ and the inclusion $\mathbb{R}^4 \to \mathbb{R}^5$.

THEOREM 4. *Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion and let $g\colon S^3 \to \mathbb{R}^4 \subset \mathbb{R}^5$ be a generic immersion regularly homotopic to f. Then g has an odd number of quadruple points if and only if f has an odd number of self intersection components with connected preimage.*

We define a $\mathbb{Z}_4$-valued invariant τ of generic immersions $S^3 \to \mathbb{R}^5$ (see Section 7.5). If f is a generic immersion then $\tau(f)$ is divisible by 2 if and only if f has an odd number of self intersection components with connected preimage.

THEOREM 5. *Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion and let $g\colon S^3 \to \mathbb{R}^4 \subset \mathbb{R}^5$ be a generic immersion regularly homotopic to f. If $\tau(f) \neq 0$ then the (resolved) self intersection surface F_g of g is nonorientable.*

3. The Smale invariant and Steifel manifolds

In this section, some properties of the Smale invariant and the groups where it takes values are collected fro later reference. Via the Smale invariant the set of regular homotopy classes of immersions is endowed with the structure of an Abelian group. We describe the group operations in topological terms. Most of the results presented here are known. Proofs are provided where there were hard to find references.

3.1. ***Homotopy groups of two Steifel manifolds***

The Stiefel manifold $V_{4,3}$ is homotopy equivalent to $SO(4)$. Indeed, any orthonormal 3-frame in 4-space can be uniquely completed to a positively oriented 4-frame.

Let $SO(4) \xrightarrow{p} S^3$ be the fibration with fiber $SO(3)$. Consider S^3 as the set of unit quaternions in $\mathbb{H} \cong \mathbb{R}^4$, where we identify the vectors $\partial_1, \ldots, \partial_4$ in the standard base of $\mathbb{R}^4$ with $1, i, j, k \in \mathbb{H}$. For $x, y \in \mathbb{R}^4$ let $x \cdot y \in \mathbb{R}^4$ denote quaternionic product of x and y. The map $\sigma\colon S^3 \to SO(4)$, $\sigma(x)y = x \cdot y$, is a section of $SO(4) \xrightarrow{p} S^3$. Thus, $SO(4)$ is diffeomorphic to $S^3 \times SO(3)$.

Let $\varrho\colon S^3 \to SO(3)$ be the map $\varrho(x)u = x \cdot u \cdot x^{-1}$, where u is a pure quaternion and $\mathbb{R}^3$ is identified with the set of pure quaternions. (That is, the span of the vectors $i, j, k \in \mathbb{H}$.) Let $\rho\colon S^3 \to SO(4)$ be ϱ composed with the inclusion of the fiber over $(1,0,0,0) \in S^3$. The following is immediate.

LEMMA 3.1.1.
$$\pi_3(SO(4)) \cong \mathbb{Z}[\sigma] \oplus \mathbb{Z}[\rho].$$
□

Let $SO(5) \xrightarrow{p} V_{5,3}$ be the fibration with fiber $SO(2)$ that maps an orthonormal 5×5-matrix to the 3-frame consisting of its first three column vectors. Similarly, let $SO(5) \xrightarrow{r} S^4$ be the fibration with fiber $SO(4)$.

LEMMA 3.1.2. *The homomorphism $p_*\colon \pi_3(SO(5)) \to \pi_3(V_{5,3})$ is an isomorphism. The homomorphism $r_*\colon \pi_3(SO(4)) \to \pi_3(SO(5))$ is an epimorphism with kernel N, where N is the subgroup generated by $[\sigma] - 2[\rho]$.*

Proof. The first statement follows by inspecting the homotopy sequence of the fibration. For the second, see [8], Chapter 8, Proposition 12.11. □

Using Lemma 3.1.2, we make the identifications
$$\pi_3(V_{5,3}) = \pi_3(SO(5)) = \pi_3(SO(4))/N.$$

3.2. *The Smale invariant of an immersed 3-sphere.*

We define the Smale invariant of an immersion $f\colon S^3 \to \mathbb{R}^n$: Consider S^3 as the unit sphere in $\mathbb{R}^4$ and let $s\colon S^3 \to \mathbb{R}^4 \subset \mathbb{R}^n$, denote the standard embedding. Fix a disk $D^3 \subset S^3$ containing the south pole and a framing X of $S^3 - D^3$. Using regular homotopy, deform f so that $f|D^3 = s|D^3$.

Choose a diffeomorphism $r\colon H_+ \to S^3 - D^3$ of degree $+1$, where H_+ is the hemisphere $\{x_0 \geq 0\}$ in the unit sphere $\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^4$. Let $x \mapsto x^*$ be the map $(x_0, x_1, x_2, x_3) \mapsto (-x_0, x_1, x_2, x_3)$ of the unit sphere in $\mathbb{R}^4$. Define $\phi_s^f\colon S^3 \to V_{n,3}$,

$$\phi_s^f(x) = \begin{cases} df(X(r(x))) & \text{for } x \in H_+, \\ ds(X(r(x^*))) & \text{for } x \in H_-, \end{cases}$$

where H_- is the hemisphere $\{x_0 \leq 0\}$.

Definition 3.2.1. *The Smale invariant $\Omega(f)$ of f is*

$$\Omega(f) = [\phi_s^f] \in \pi_3(V_{n,3}).$$

Smale, [16] showed that Ω gives a bijection between the regular homotopy classes of immersions $S^3 \to \mathbb{R}^n$ and the elements of $\pi_3(V_{n,3})$.

3.3. *Calculating Smale invariants in 4- and 5-space*

In computations we will not use Definition 3.2.1 literally. We use a slightly different approach: Consider

$$S^3 = \{x \in \mathbb{R}^4 = \mathbb{H} : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\},$$

where $\mathbb{H}$ denotes the quaternions. The tangent space of S^3 at $(1,0,0,0)$ is the span of the vectors i, j, k. We trivialize TS^3 using the *quaternion framing:*

$$Q(x) = (x \cdot i, x \cdot j, x \cdot k) \in T_xS^3, \text{ for } x \in S^3.$$

Let $f\colon S^3 \to \mathbb{R}^n$ be an immersion. Then there is an induced map $\Phi_n^f\colon S^3 \to V_{n,3}$, $\Phi_n^f(x) = df(Q(x))$.

Assume that $n = 4$. Then $V_{n,3} = V_{4,3} = SO(4)$. Thus, we get a map $\Phi_4^f\colon S^3 \to SO(4)$.

Lemma 3.3.1. *Let $f\colon S^3 \to \mathbb{R}^4$ be an immersion then*

$$\Omega(f) = [\Phi_4^f] - [\Phi_4^s] \in \pi_3(SO(4)),$$

and $[\Phi_4^s] = [\sigma] \in \pi_3(SO(4))$. □

Assume that $n = 5$. Consider the fibration $SO(5) \to V_{5,3}$ as in Lemma 3.1.2. Since the normal bundle of f is orientable 2-dimensional, it is trivial and we can lift Φ_5^f to $\Theta^f\colon S^3 \to SO(5)$ and $[\Theta^f] \in \pi_3(SO(5))$ is independent of this lifting.

Lemma 3.3.2. *Let $f\colon S^3 \to \mathbb{R}^5$ be an immersion then*

$$\Omega(f) = [\Theta^f] - [\Theta^s] \in \pi_3(SO(5)) = \pi_3(V_{5,3}),$$

and $[\Theta^s] = [\sigma] + N \in \pi_3(SO(5))$. □

Let $i\colon \mathbb{R}^4 \to \mathbb{R}^5$ denote the inclusion.

Lemma 3.3.3. *If $f\colon S^3 \to \mathbb{R}^4$ is an immersion then*

$$\Omega(i \circ f) = \Omega(f) + N \in \pi_3(SO(5)) = \pi_3(V_{5,3}).$$

Proof. Complete the framing of f with a vector in the fifth direction and combine Lemma 3.3.1 and Lemma 3.3.2. □

3.4. *The Smale invariant as an obstruction*

As in [16], Theorem E, we can interpret the Smale invariant as an obstruction to extending an immersion $f\colon S^3 \to \mathbb{R}^5$ to the disk D^4:

Consider S^3 as the boundary ∂D^4. Choose a trivialization Y of TD^4. We can find a normal vector field of f and use it to extend f to an immersion F of $S^3 \times I$, where I is a small interval. Considering $S^3 \times I \subset D^4$, we get an induced map $\Psi^F\colon S^3 \to V_{5,4}$. By the Smale-Hirsch h-principle (see [7]), the space of maps of the principal bundle of TD^4 into the bundle with fiber $V_{5,4}$ associated to the tangent bundle of $T\mathbb{R}^5$ is homotopy equivalent to the space of immersions of D^4 into $\mathbb{R}^5$. Thus, $[\Psi^F] \in \pi_3(V_{5,4})$ is the obstruction to extend the immersion F of $S^3 = \partial D^4$ to all of D^4. We note that $V_{5,4} \cong SO(5)$ and that $[\Psi^F] \in \pi_3(SO(5))$ is independent of the normal field chosen.

LEMMA 3.4.1. *The obstruction $[\Psi^F]$ to extend an immersion $f\colon S^3 \to \mathbb{R}^5$ to D^4 satisfies,*

$$[\Psi^F] = \Omega(f) \in \pi_3(SO(5)) = \pi_3(V_{5,3}).$$

Proof. We can find a map $\mathbf{A}\colon S^3 \to SO(4)$ such that $Y(x) \cdot \mathbf{A}(x) = (x, Q(x)) = \Phi_4^s(x)$, where $\cdot$ denotes matrix multiplication, and where we identify the point $x \in S^3$ with the unit outer normal of S^3 in $\mathbb{R}^4$ at x. Now, $[Y] = 0 \in \pi_3(SO(4))$ and $[\Phi_4^s] = [\sigma] \in SO(4)$. Hence,

$$[\mathbf{A}] = -[Y] + [\Phi_4^s] = [\sigma] \in \pi_3(SO(4)).$$

Moreover, $\Psi^F(x) \cdot \mathbf{A}(x) = \Theta^f(x)$ and thus,

$$\Omega(f) = [\Theta^f] - [\Theta^s] = [\Theta^f] - ([\mathbf{A}] + N) = [\Psi^F] \in \pi_3(SO(5)) = \pi_3(V_{5,3}).$$

$\square$

3.5. *The immersion group*

Let **Imm** denote the infinite cyclic group of regular homotopy classes of immersions $S^3 \to \mathbb{R}^5$. The Smale invariant gives an isomorphism $\Omega\colon \mathbf{Imm} \to \pi_3(V_{5,3})$.

First, we consider addition in **Imm** and other groups of regular homotopy classes of immersions $S^3 \to \mathbb{R}^n$:

Given two immersions $f, g\colon S^3 \to \mathbb{R}^n$ we define an immersion $f \star g$ as follows: Consider $S^3 \subset \mathbb{R}^4$ with coordinates $x = (x_0, x_1, x_2, x_3)$ as the subset characterized by $\sum x_i^2 = 1$. Let $a = (1,0,0,0)$ and $a' = (-1,0,0,0)$, respectively. Choose frames $u_1, \dots, u_n$ at $f(a)$ and $v_1, \dots, v_n$ at $g(a')$ that agree with the orientation of $\mathbb{R}^n$ and such that u_1, u_2, u_3 (v_1, v_2, v_3) are tangent to $f(S^3)$ (to $g(S^3)$) at $f(a)$ (at $g(a')$). We can assume, possibly after moving $g(S^3)$, that $u_i = v_i$, $i = 1,2,3$ and that $g(a') = f(a) + u_n$. Moreover, we can deform the maps so that

$$f(x) = f(a) + \sum_{i=1}^{3} x_i u_i \quad \text{for} \quad 1 - \epsilon \le x_0 \le 1,$$

$$g(x) = g(a') + \sum_{i=1}^{3} x_i v_i \quad \text{for} \quad -1 \le x_0 \le -1 + \epsilon.$$

The immersion $f \star g$ is now obtained by running a tube from $f(a)$ to $g(a')$ with axis $f(a) + tu_n$. Details can be found in Kervaire [9], Section 2, where the following is proved.

Lemma 3.5.1.

$$\Omega(f \star g) = \Omega(f) + \Omega(g).$$

□

We call the immersion $f \star g$ *the connected sum of f and g.*

Secondly, we consider inversion in **Imm**:

Given an immersion $f\colon S^3 \to \mathbb{R}^5$ we define the immersion $\widehat{f}$ as follows: Let $\widehat{f} = f \circ r$, where $r\colon S^3 \to S^3$ is the restriction of a reflection through a hyperplane in $\mathbb{R}^4$.

Lemma 3.5.2.

$$\Omega(\widehat{f}) = -\Omega(f) \in \pi_3(V_{5,3}).$$

Proof. Let the disk $U \subset S^3 \subset \mathbb{R}^4$ in the definition of the Smale invariant be the hemisphere $\{x_0 \geq 0\}$ and let $r\colon S^3 \to S^3$ be the map

$$r(x_0, x_1, x_2, x_3) = (x_0, -x_1, x_2, x_3).$$

Let $R\colon \mathbb{R}^5 \to \mathbb{R}^5$ be the map

$$R(y_1, y_2, y_3, y_4, y_5) = (y_1, -y_2, y_3, y_4, -y_5).$$

Then $R \circ s \circ r = s$ and we can use the map $\phi^{R\circ f \circ r}_{R \circ s \circ r}\colon S^3 \to V_{3,5}$ to compute $\Omega(\widehat{f})$. It is straightforward to check that this map is homotopic to $\phi^f_s \circ r$. Hence,

$$\Omega(\widehat{f}) = (\phi^{R\circ f\circ r}_{R\circ s\circ r})_*[S^3] = (\phi^f_s \circ r)_*[S^3] = (\phi^f_s)_*\left(-[S^3]\right) = -\Omega(f).$$

□

4. Embeddings considered as immersions

In this section we classify embeddings $S^3 \to \mathbb{R}^5$ up to regular homotopy and show that the regular homotopy classes that contains embeddings form a subgroup of index 24 in the group **Imm**.

4.1. *Embeddings up to regular homotopy and signature*

Let $f, g\colon S^3 \to \mathbb{R}^5$ be embeddings. Then $f \star g$ is regularly homotopic to an embedding and $\widehat{f}$ is an embedding. Thus, the regular homotopy classes that contain embeddings from a subgroup of **Imm**. We denote this subgroup **Emb**.

We get a classification of embeddings up to regular homotopy as follows:

Given an embedding $f\colon S^3 \to \mathbb{R}^5$, we can find a compact connected orientable manifold V^4 embedded into $\mathbb{R}^5$ and such that $\partial V^4 = f(S^3)$. We call such a manifold a *Seifert-surface of f*. The orientation of $f(S^3)$ induces an orientation of V^4. Filling the S^3 on the boundary of V^4 with a 4-disk we get a closed connected oriented 4-manifold W^4. The cohomology sequence of the pair (W^4, V^4) shows that the inclusion induces an isomorphism $H^2(W^4; \mathbb{Z}_2) \to H^2(V^4; \mathbb{Z}_2)$. The normal bundle of V^4 is a trivial 1-dimensional bundle. Hence, $w_2(TV) = 0$ and therefore $w_2(TW) = 0$, where w_2 is the second Steifel-Whitney class. Thus, W is a spin manifold. By Rokhlin's theorem (see Milnor and Kervaire, [13]), *the signature $\sigma(W^4)$ of W^4* is divisible by 16.

Define $\sigma(f) = \frac{\sigma(W)}{16} \in \mathbb{Z}$. (This definition agrees with that given in Section 2.1.)

PROPOSITION 4.1.1. *For $\xi \in \mathbf{Emb}$, let $\sigma(\xi) = \sigma(f)$, where f is an embedding representing ξ. Then*

$$\sigma\colon \mathbf{Emb} \to \mathbb{Z}$$

is an isomorphism.

Proof. We first show that σ is well defined: Let f_0 and f_1 be regularly homotopic embeddings with Seifert-surfaces V_0 and V_1, respectively. Closing V_i with 4-disks, as above, we get closed 4-manifolds W_i, $i = 0, 1$.

Let $f_t\colon S^3 \to \mathbb{R}^5$, $0 \le t \le 1$ be a regular homotopy. Consider $F\colon S^3 \times I \to \mathbb{R}^5 \times \mathbb{R}$, $F(x,t) = (f_t(x), t)$. Then F is an immersion of $S^3 \times I$ into $\mathbb{R}^6$. Filling the ends of this immersion with V_0 and V_1 we see that the connected sum of W_0 and $-W_1$, where $-W_1$ is W_1 with the opposite orientation, immerses into $\mathbb{R}^6$. The change of orientation of W_1 appears for the following reason: If ν_0 is the outward normal of $f_0(S^3)$ in V_0 then $\nu_0, df_0(X_p)$ gives the positive orientation of V_0, where X_p is an orienting frame of S^3 at p. Pushing this orientation along $F(p, I)$ we get the orientation $-\nu_1, df_1(X_p)$ of V_1 at $f_1(p)$, where ν_1 is the outward normal to $f_1(S^3)$ in V_1. This orientation is opposite to that induced by f_1.

The normal bundle of F is 2-dimensional. We have a section of it over V_0 and $-V_1$. By standard obstruction theory this section extends over $S^3 \times I$. Thus, the normal bundle of the connected sum of W_0 and $-W_1$ is trivial.

By the signature theorem (Theorem 19.4 in [14]), the signature of an oriented 4-manifold M is $\frac{1}{3}\langle p_1, [M]\rangle$ where p_1 is the first Pontryagin class of the tangent bundle of M and $[M]$ is the orientation class of M.

Since the normal bundle of the connected sum of W_0 and $-W_1$ is trivial, its tangent bundle is stably trivial. Thus, its first Pontryagin class vanishes and hence, the signature of the connected sum of W_0 and $-W_1$ is 0.

The signature of a connected sum is the sum of the signatures of its terms. If the orientation of a manifold is changed then its signature changes sign. We conclude that $\sigma(f_0) = \sigma(f_1)$ and the map is well defined.

Moreover, it follows that the map σ is a homomorphism: If V is a Seifert-surface of f and U one of g then the boundary connected sum of V and U is a Seifert-surface of $f \star g$.

We shall show that σ is surjective: Let $[z_0 : z_1 : z_2 : z_3]$ be homogeneous coordinates on $\mathbb{C}P^3$. The hypersurface $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$ is a K3-surface, which is a closed simply connected 4-manifold W^4 with second Betti-number 22 and signature ± 16, depending on orientation. The manifold W^4 is almost parallelizable (see [13]). Hence, by the Smale-Hirsch h-principle V^4 $(= W^4 - \mathrm{int}(D^4)$, where D^4 is a disk in $W^4)$ immerses into $\mathbb{R}^5$. Moreover, W^4 admits a handle decomposition without 1-handles and 3-handles (see [6], Theorem 1). Hence, V^4 is a regular neighborhood of its 2-skeleton C^2.

Let $G\colon V^4 \to \mathbb{R}^5$ be an immersion. By transversality, $G|C^2$ is regularly homotopic to an embedding of C^2. Extending this regular homotopy along the normal bundle of C^2 in V^4 we obtain an embedding $F\colon V^4 \to \mathbb{R}^5$. Let $h\colon S^3 \to \partial V^4$ be a diffeomorphism and let $f = F \circ h$. Then $\sigma(f) = \pm 1$ and σ is surjective. Therefore, **Emb** is a nontrivial subgroup of the infinite cyclic group **Imm**. It follows that σ is an isomorphism. $\square$

4.2. ***The index of* Emb *in* Imm**

PROPOSITION 4.2.1. *The subgroup* **Emb** $\subset$ **Imm** *has index* 24.

Proof. To see this, we calculate the Smale invariant of a generator of **Emb**. Let V and $F\colon V \to \mathbb{R}^5$ be as in the above proof. Let $h\colon S^3 \to \partial V^4$ be a diffeomorphism. Then $f' = F \circ h$ represents a generator of **Emb**.

Let $n(x) \in T_{h(x)}V$, $x \in S^3$ be an outward normal vector field of ∂V in TV and let $(x,t) \in S^3 \times (0,\delta)$ be coordinates in a collar C on ∂V in V. Straightening the embedding F close to the boundary ∂V, we may assume that it has the form $F(x,t) = f'(x) - tdF(n(x))$ for $(x,t) \in C$.

We use the coordinates (x,t) also to parameterize a collar B on ∂D^4 in D^4. The embedding F allows us to think about V as a subset of $\mathbb{R}^5$. Then

$$W = \mathrm{int}(D^4) \cup_\Phi \mathrm{int}(V), \tag{†}$$

where $\Phi\colon B \to C \subset \mathbb{R}^5$ is the map $\Phi(x,t) = f'(x) + (t-\delta)dF(n(x))$. Let $\overline{V}$ and $\overline{D}$ be the compact manifolds obtained by deleting half of the collar C and half of the collar B, respectively. Then

$$W = \overline{V} \cup_f \overline{D},$$

where $f(x, \frac{\delta}{2}) = f'(x) + \frac{\delta}{2}dF(n(x))$. Clearly, f is diffeotopic to f' and hence represents a generator of **Emb**.

Let ϵ be a trivial line bundle over W. Let ξ denote the principal $SO(5)$ bundle over W associated to $TW \oplus \epsilon$. Since W is almost parallelizable, we have the decomposition

$$\xi = \overline{V} \times SO(5) \cup_\phi \overline{D} \times SO(5),$$

where $\phi\colon \partial\overline{D} \to SO(5)$ is a gluing map.

By Lemma 2 in [13], the obstruction to trivialize ξ is $\frac{1}{2}p_1(\xi)$ times a generator of $\pi_3(SO(5))$. Hence, $[\phi] = \pm 24([\sigma] + N) \in \pi_3(SO(5))$.

The bundle $T\overline{V} \oplus \epsilon$ is naturally identified with $T\mathbb{R}^5|\overline{V}$: Let ν be a normal vector field of V in $\mathbb{R}^5$. For $p \in \overline{V}$, identify the vector $v_p + \lambda_p$ in $T_p\overline{V} \oplus \epsilon_p$ with $v_p + \nu(p) \in T_p\mathbb{R}^5$. We can now identify the homotopy class of the gluing map ϕ:

Pick a trivialization (Y, w) of $T\overline{D} \oplus \epsilon$. We need to compare it with a trivialization of $T\mathbb{R}^5|\overline{V}$ along $\partial\overline{V}$. We use the standard trivialization of $T\mathbb{R}^5$ restricted to $\overline{V}$.

Using the decomposition (†) we see that (Y, w) induces the trivialization $(d\Phi(Y(x)), \nu(h(x)))$ of $T\mathbb{R}^5$ along $\partial\overline{V}$. But $d\Phi(Y(x)) = dF(Y(x)) = \Psi^F(x)$, see Lemma 3.4.1. Hence, the induced trivialization of $T\mathbb{R}^5$, considered as a 5-tuple of vectors gives (after orthonormalization) a lifting of Ψ^F to $SO(5)$ and thus $[\phi] = [\Psi^F] \in \pi_3(SO(5))$. Applying Lemma 3.4.1, we conclude

$$\Omega(f) = [\Psi^F] = [\phi] = \pm 24([\sigma] + N) \in \pi_3(SO(5)) = \pi_3(V_{5,3}).$$

□

5. Spaces of immersions

In this section, generic immersions $S^3 \to \mathbb{R}^n$, $n = 4, 5$ and their self intersections are studied. The space of immersions $S^3 \to \mathbb{R}^n$ as described in the Introduction, will be denoted $\mathcal{F}_n$. It is an infinite dimensional manifold. The set of non-generic immersions in $\mathcal{F}_n$ is a stratified hypersurface Σ_n. We describe its strata of codimension one, for $n = 4, 5$, and of codimension two, for $n = 5$.

5.1. *Generic immersions and their self intersections*

DEFINITION 5.1.1. *An immersion $f \in \mathcal{F}_4$ is* generic *if it satisfies the following conditions:*

g1 *For any $w \in \mathbb{R}^5$, $f^{-1}(w)$ contains at most four points.*
g2 *If $f(x_1) = \cdots = f(x_j) = w$, $2 \leq j \leq 4$ for $x_i \neq x_j \in S^3$ if $i \neq j$, then $df(T_{x_i}S^3) + \bigcap_{j \neq i} df(T_{x_j}S^3) = T_w\mathbb{R}^4$.*

DEFINITION 5.1.2. *An immersion $f \in \mathcal{F}_5$ is* generic *if it satisfies the following conditions:*

G1 *For any $w \in \mathbb{R}^5$, $f^{-1}(w)$ contains at most two points.*
G2 *If $f(x) = f(y) = w$, for $x \neq y \in S^3$, then $df(T_xS^3) + df(T_yS^3) = T_w\mathbb{R}^5$.*

If $f\colon X \to Y$ is an immersion of manifolds then the *self intersection* of f is the subset of points $y \in Y$ such that $f^{-1}(y)$ contains more than one point. We denote it M_f. We denote its preimage $\widetilde{M}_f$. That is, $\widetilde{M}_f = f^{-1}(M_f) \subset X$.

LEMMA 5.1.3. *Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion. Then M_f and $\widetilde{M}_f$ are closed 1-manifolds and $f\colon \widetilde{M}_f \to M_f$ is a double cover. Moreover, there is an induced orientation on M_f.*

Proof. The first statement follows from **G2**. If we order the oriented sheets coming together along M_f then there is a standard way to assign an orientation to M_f. Since the codimension of $\widetilde{M}_f$ is even this orientation is independent of the ordering. □

LEMMA 5.1.4. *Let $g\colon S^3 \to \mathbb{R}^4$ be a generic immersion. Then M_g and $\widetilde{M}_g$ are 2-dimensional stratified spaces,*

$$M_g = M_g^0 \cup M_g^1 \cup M_g^2 \quad \text{and} \quad \widetilde{M}_g = \widetilde{M}_g^0 \cup \widetilde{M}_g^1 \cup \widetilde{M}_g^2,$$

where M_g^j and $\widetilde{M}_g^j$ are smooth manifolds of dimension j, $j = 0, 1, 2$. The strata M_g^j is the set of j-tuple points, $\widetilde{M}_g^j = g^{-1}(M_g^j)$, and $g|\widetilde{M}_g^j$ is a j-fold covering.

Proof. Immediate from **g1** and **g2**. □

The next lemma shows how to resolve the self intersection of a generic immersion $S^3 \to \mathbb{R}^4$.

LEMMA 5.1.5. *Let $g\colon S^3 \to \mathbb{R}^4$ be a generic immersion. There exist closed surfaces $\widetilde{F}_g$ and F_g (and closed 1-manifolds $\widetilde{C}_g$ and C_g), unique up to diffeomorphisms, and immersions $s\colon \widetilde{F}_g \to S^3$ and $t\colon F_g \to \mathbb{R}^4$ ($\sigma\colon \widetilde{C}_g \to S^3$ and $\tau\colon C_g \to \mathbb{R}^4$) such that*

$$\begin{array}{ccc} \widetilde{F}_g & \xrightarrow{s} & \widetilde{M}_g \subset S^3 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ F_g & \xrightarrow{t} & M_g \subset \mathbb{R}^5 \end{array}, \qquad \begin{array}{ccc} \widetilde{C}_g & \xrightarrow{\sigma} & \widetilde{M}_g^0 \cup \widetilde{M}_g^1 \subset S^3 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g} \\ C_g & \xrightarrow{\tau} & M_g^0 \cup \widetilde{M}_g^1 \subset \mathbb{R}^5 \end{array},$$

commutes. The maps s and t are surjective, have multiple points only along $\widetilde{M}_g^0 \cup \widetilde{M}_g^1$ and $M_g^0 \cup M_g^1$, respectively and p is the orientation double cover. The maps σ and τ are surjective, have multiple points only along $\widetilde{M}_g^0$ and M_g^0, respectively and π is a 3-fold cover.

Proof. This is immediate from the local pictures: Close to a j-tuple point M_g is the intersection of j 3-planes in general position in 4-space. $\square$

We call F_g the *resolved self intersection surface* of g. We use the notation $F_g^j = t^{-1}(M_g^j)$ and $\widetilde{F}_g^j = s^{-1}(\widetilde{M}_g^j)$, $j = 0, 1$.

5.2. *The discriminant hypersurface and its stratum of codimension one*

The jet transversality theorem implies that the set of generic immersions is an open dense subset of $\mathcal{F}_n$, $n = 4, 5$. Its complement $\Sigma_n \subset \mathcal{F}_n$ will be called *the discriminant hypersurface.* The discriminant hypersurface is stratified, $\Sigma_n = \Sigma_n^1 \cup \Sigma_n^2 \cup \cdots \cup \Sigma_n^\infty$, where each stratum Σ_n^k, $k < \infty$ is a smooth submanifold of codimension k of $\mathcal{F}_n$ and Σ_n^j is contained in the closure of Σ_n^1 for every j.

The following two propositions follow by applying the jet transversality theorem to 1-parameter families of immersions.

PROPOSITION 5.2.1. *The codimension one stratum $\Sigma_4^1 \subset \Sigma_4$ is the set of all immersions $f\colon S^3 \to \mathbb{R}^4$ such that*

(a) **g1** *and* **g2** *holds except at one double point $w = f(x) = f(y) \in \mathbb{R}^4$, $x \neq y \in S^3$ where*

$$\dim\left(df(T_xS^3) + df(T_yS^3)\right) = 3$$

or,

(b) **g1** *and* **g2** *holds except at one j-tuple point, $3 \leq j \leq 5$ $w = f(x_1) = \cdots = f(x_j) \in \mathbb{R}^4$, $x_i \neq x_k \in S^3$, if $i \neq k$ where*

$$\dim\left(df(T_{x_i}S^3) + \bigcap_{j\neq i\, j\neq k} df(T_{x_j}S^3)\right) = 4,$$

for $k \neq i$ but

$$\dim\left(df(T_{x_i}S^3) + \bigcap_{j\neq i} df(T_{x_j}S^3)\right) = 3.$$

$\square$

PROPOSITION 5.2.2. *The codimension one stratum $\Sigma_5^1 \subset \Sigma_5$ is the set of all immersions $f\colon S^3 \to \mathbb{R}^5$ such that either*

(a) **G1** *holds, f has double points and* **G2** *holds, except at one double point $w = f(x) = f(y) \in \mathbb{R}^5$, $x \neq y \in S^3$, where $\dim\left(df(T_xS^3) + df(T_yS^3)\right) = 4$ or,*

(b) **G2** *holds and* **G1** *holds except at one triple point $w \in \mathbb{R}^5$ with $w = f(x_1) = f(x_2) = f(x_3)$, where x_i, $1 \leq i \leq 3$ are three distinct points in S^3 and*

$$\dim\left(df(T_{x_i}S^3) + \bigcap_{j\neq i} df(T_{x_j}S^3)\right) = 4.$$

$\square$

If (a) above holds for an immersion f, we say that the exceptional double point w is a *self tangency point* of f.

5.3. *Coordinate expressions and versal deformations*

Recall that a *deformation of a map* f_0 is a 1-parameter family of maps f_λ, parameterized by $\lambda \in U$, where U is a neighborhood of $0 \in \mathbb{R}^m$. A deformation f_λ of a map f_0 is called *versal* if every deformation of f_0 is equivalent (up to left-right actions of diffeomorphisms) to one induced from f_λ.

Below, $f_0 \colon S^3 \to \mathbb{R}^n$, $n = 4, 5$ will be an immersion in Σ^1_n with its exceptional multiple point at $0 \in \mathbb{R}^n$, x, y, z, w, u will denote coordinates in small 3-balls centered at the preimages of 0 and f_t, $t \in U \subset \mathbb{R}$ will be a versal deformation of f_0. Such a deformation can be assumed to be constant in t outside of the coordinate balls. The proofs of the statements in this section and the next one are discussed in Section 5.5.

PROPOSITION 5.3.1. *Let $f_0 \colon S^3 \to \mathbb{R}^4$ be an immersion in Σ^1_4 with exceptional multiple point 0 and let f_t be a versal deformation. Locally, at 0, up to choice of coordinates around the preimages of 0 and in $\mathbb{R}^4$,*

(a) *if 0 is double point then f_t is of the form*

$$\begin{aligned} f_t(x) &= (x_1, x_2, x_3, 0), \\ f_t(y) &= (y_1, y_2, y_3, y_1^2 + y_2^2 + \epsilon y_3^2 + t), \end{aligned}$$

where $\epsilon = \pm 1$, or

(b) *if 0 is a triple point then f_t is of the form*

$$\begin{aligned} f_t(x) &= (x_1, x_2, x_3, 0), \\ f_t(y) &= (y_1, y_2, 0, y_3), \\ f_t(z) &= (z_1, z_2, z_3, z_1^2 + \epsilon z_2^2 + t - z_3), \end{aligned}$$

where $\epsilon = \pm 1$, or

(c) *if 0 is a quadruple point then f_t is of the form*

$$\begin{aligned} f_t(x) &= (x_1, x_2, x_3, 0), \\ f_t(y) &= (y_1, y_2, 0, y_3), \\ f_t(z) &= (z_1, 0, z_2, z_3), \\ f_t(w) &= (w_1, w_2, w_3, w_1^2 + t - w_2 - w_3), \end{aligned}$$

or

(d) *if 0 is a quintuple point then f_t is of the form*

$$\begin{aligned} f_t(x) &= (x_1, x_2, x_3, 0), \\ f_t(y) &= (y_1, y_2, 0, y_3), \\ f_t(z) &= (z_1, 0, z_2, z_3), \\ f_t(w) &= (0, w_1, w_2, w_3), \\ f_t(u) &= (u_1, u_2, u_3, t - u_1 - u_2 - u_3). \end{aligned}$$

□

PROPOSITION 5.3.2. *Let $f_0 \colon S^3 \to \mathbb{R}^5$ be an immersion in Σ^1_5 with exceptional multiple point 0 and let f_t be a versal deformation. Locally, at 0, up to choice of coordinates around the preimages of 0 and in $\mathbb{R}^5$,*

(a) *if* 0 *is double point then* f_t *is of the form*

$$f_t(x) = (x_1, x_2, 0, x_3, 0).$$
$$f_t(y) = (y_1, y_2, y_1^2 + \epsilon y_2^2 + t, 0, y_3).$$

where $\epsilon = \pm 1$, *or*

(b) *if* 0 *is a triple point then* f_t *is of the form*

$$f_t(x) = (x_1, x_2, x_3, 0, 0),$$
$$f_t(y) = (y_1, t, 0, y_2, y_3),$$
$$f_t(z) = (0, z_2, z_1, z_2, z_3).$$

□

If $\epsilon = +1$ ($\epsilon = -1$) in cases (a) in the above propositions we say that 0 is an *elliptic self tangency point* (a *hyperbolic self tangency point*) of f_0.

5.4. ***The stratum of codimension two***

To study first order invariants of generic immersions (see Section 6.3) we need a description of the codimension two stratum of the discriminant hypersurface. We restrict attention to immersions into 5-space.

Proposition 5.4.1. *The codimension two stratum* $\Sigma_5^2 \subset \Sigma$ *is the set of all immersions* $f_0\colon S^3 \to \mathbb{R}^5$ *such that exactly one of the following holds*

(a) f_0 *has two distinct self tangency points,*
(b) f_0 *has two distinct triple points,*
(c) f_0 *has one self tangency point and one triple point,*
(d) f_0 *has a degenerate self tangency point at* $0 = f(p) = f(q)$, *in which case its versal deformation* $f_{t,s}$ *is of the form*

$$f_{t,s}(x) = (x_1, x_2, 0, x_3, 0),$$
$$f_{t,s}(y) = (y_1, y_2, y_1^2 + y_2(y_2^2 + s) + t, 0, y_3),$$

up to choice of coordinates x, y *around* p, q, *respectively, and coordinates in* $\mathbb{R}^5$.

□

The versal deformations of (a)-(d) are evident. They are just products of 1-dimensional versal deformations.

Let E, Y and T denote the codimension one parts of Σ_5 consisting of all immersions with one elliptic self tangency point, one hyperbolic self tangency point, and one triple point, respectively. Figure 1, which is a consequence of Proposition 5.4.1, shows the possible intersections of the discriminant hypersurface Σ_5 and a small 2-disk in $\mathcal{F}_5$ which meets Σ_5^2 and is transversal to Σ_5.

5.5. ***Proofs***

The proofs of the propositions in Section 5.3 and Section 5.4 are all similar: First we need to find coordinates close to the preimages of the exceptional point such that the map is given by the expression stated. Then we show that the deformation given is infinitesimally versal. A standard theorem in singularity theory then implies that the deformation is versal. As an example we write out the proof of Proposition 5.3.1 (a):

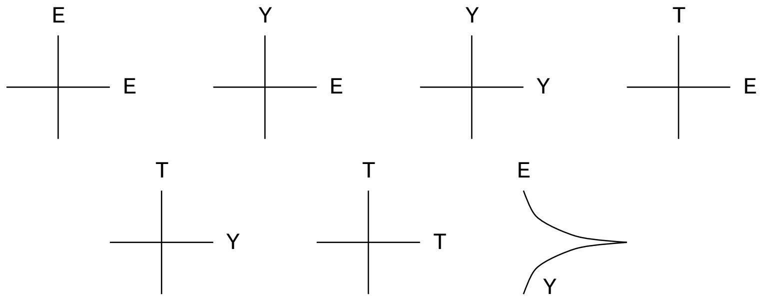

FIGURE 1. The discriminant hypersurface intersected with a generic 2-disk

Since f_0 has a tangency at 0 the tangent planes of the two sheets X and Y meeting there must agree. We may assume that this tangent plane is the plane of the first three coordinates. By the implicit function theorem we can choose coordinates so that the map of the first sheet is

$$f_0(x) = (x_1, x_2, x_3, 0).$$

We may now look upon the second sheet as the graph of a function $\phi\colon \mathbb{R}^3 \to \mathbb{R}$. The requirement that Y is tangent to X at 0 implies that $\phi(0) = 0$ and $\frac{\partial\phi}{\partial y_i}(0) = 0$, $i = 1, 2, 3$. Moreover, the second partials must be nondegenerate since the immersion is in the codimension 1 part of the discriminant hypersurface. Changing coordinates in $\mathbb{R}^4$, by adding a function of the first three coordinates that vanishes to order 3 to the fourth, we may assume that ϕ is identical to the second order terms of its Taylor polynomial. Choosing the coordinates in Y appropriately then gives f_0 the desired form.

We must check infinitesimal versality. If $(z_1, \dots, z_4)$ are coordinates on $\mathbb{R}^4$ then this amounts to showing that any smooth variations

$$\alpha(x) = \sum_{i_1}^{4} \alpha_i(x) \frac{\partial}{\partial z_i}, \quad \beta(x) = \sum_{i_1}^{4} \beta_i(y) \frac{\partial}{\partial z_i},$$

can be written as

$$\alpha(x) = \sum_{i=1}^{3} a_i(x) \frac{\partial f_0}{\partial x_i} + k(f_0(x)) + c\dot{f}_t(x)|_{t=0},$$

$$\beta(y) = \sum_{i=1}^{3} b_i(y) \frac{\partial f_0}{\partial y_i} + k(f_0(y)) + c\dot{f}_t(y)|_{t=0},$$

where k is a vector field on $\mathbb{R}^4$ and c is a constant. Writing these equations out we get the system

$$\begin{aligned}
&\alpha_i(x) = a_i(x) + k_i(x, 0), \quad i = 1, 2, 3,\\
&\alpha_4(x) = k_4(x, 0),\\
&\beta_i(y) = b_i(y) + k_i(y, y_1^2 + y_2^2 + \epsilon y_3^2), \quad i = 1, 2, 3,\\
&\beta_4(y) = 2(y_1 b_1(y) + y_2 b_2(y) + y_3 b_3(y)) + k_4(y, y_1^2 + y_2^2 + \epsilon y_3^2) + c.
\end{aligned}$$

Let $k_4(z) = \alpha_4(z_1, z_2, z_3)$. Then the fourth equation holds. Let $c = \beta_4(0) - k_4(0)$. Let $\phi(y) = \beta_4(y) - k_4(f_0(y)) - c$ and choose c so that $\phi(0) = 0$. If $y_i \neq 0$ let

$$b_1(y) = \frac{1}{6y_1}\left(\phi(y_1, y_2, y_3) - \phi(0, y_2, y_3) + \phi(y_1, 0, y_3) - \phi(0, 0, y_3) + \phi(y_1, 0, 0)\right),$$
$$b_2(y) = \frac{1}{6y_2}\left(\phi(y_1, y_2, y_3) - \phi(y_1, 0, y_3) + \phi(y_1, y_2, 0) - \phi(y_1, 0, 0) + \phi(0, y_2, 0)\right),$$
$$b_3(y) = \frac{1}{6y_3}\left(\phi(y_1, y_2, y_3) - \phi(y_1, y_2, 0) + \phi(0, y_2, y_3) - \phi(0, y_2, 0) + \phi(0, 0, y_3)\right),$$

and if $y_i = 0$ let

$$b_1(y) = \frac{1}{6}\left(\frac{\partial\phi}{\partial y_1}(0, y_2, y_3) + \frac{\partial\phi}{\partial y_1}(0, 0, y_3) + \frac{\partial\phi}{\partial y_1}(0, 0, 0)\right),$$
$$b_2(y) = \frac{1}{6}\left(\frac{\partial\phi}{\partial y_2}(y_1, 0, y_3) + \frac{\partial\phi}{\partial y_2}(y_1, 0, 0) + \frac{\partial\phi}{\partial y_2}(0, 0, 0)\right),$$
$$b_3(y) = \frac{1}{6}\left(\frac{\partial\phi}{\partial y_3}(y_1, y_2, 0) + \frac{\partial\phi}{\partial y_3}(0, y_2, 0) + \frac{\partial\phi}{\partial y_3}(0, 0, 0)\right).$$

Then the last equation holds. Choosing first $k_i(z) = k_i(z_1, z_2, z_3)$, $i = 1, 2, 3$ so that the three remaining β-equations hold and then choosing a_i, $i = 1, 2, 3$ so that the remaining α-equations hold, infinitesimal versality is proved.

6. Morse modifications, linking, and first order invariants

In this section, we study how the self intersections of generic immersions and their preimages transform as the immersions cross the discriminant hypersurface. These transformations give rise to invariants of immersions: A function on $\mathcal{F}_n$, which is constant on path components will be called an *invariant of regular homotopy*. A function on $\mathcal{F}_n - \Sigma_n$ which is constant on path components of $\mathcal{F}_n - \Sigma_n$ is an *invariant of generic immersions*. In our study of invariants of generic immersion we are interested in how they change when we pass Σ_n. This is described in terms of jumps:

Let $f_t \colon S^3 \to \mathbb{R}^n$ be a path in $\mathcal{F}_n$, $n = 4, 5$ intersecting Σ_n^1 transversally at f_0. Let $\delta > 0$ be small. Let v be an invariant of generic immersions. Then

$$\nabla v(f_0) = v(f_\delta) - v(f_{-\delta}),$$

is a locally constant function on Σ_n^1, defined up to sign. We call it *the jump of* v. In Section 6.3, we get rid of the sign ambiguity in the definition of ∇v. We prove Theorem 2 and end the section with an axiomatic characterization of strangeness St for immersions $S^3 \to \mathbb{R}^5$.

6.1. *Morse modifications*

Throughout this section, let f_t be a path in $\mathcal{F}_n$ intersecting Σ_1 transversally at f_0 and let $\delta \geq 0$ be small enough so that f_t is generic for $0 < |t| \leq \delta$.

Let $n = 5$ and let f_0 have a self tangency point. By Proposition 5.3.2, the self intersection of f_δ is obtained from the self intersection of $f_{-\delta}$ by a single Morse modification. Define

$$J(f) = \text{the number connected components of } M_f,$$

for generic immersions f.

LEMMA 6.1.1. *J is an invariant of generic immersions $S^3 \to \mathbb{R}^5$. It jumps by ± 1 when crossing the self tangency part of Σ_5^1 and remains constant when crossing the triple point part.*

Proof. Immediate from Proposition 5.3.2. □

Let $n = 4$ and let f_0 have an exceptional quadruple point. By Proposition 5.3.1 the number of quadruple points of f_δ differs from the number of quadruple points of $f_{-\delta}$ by ± 2. Define

$$Q(f) = \text{the number of quadruple points of } f, \text{ and } Q_2(f) = Q(f) \bmod 2 \in \mathbb{Z}_2,$$

for generic immersions f.

PROPOSITION 6.1.2. *Q is an invariant of generic immersions $S^3 \to \mathbb{R}^4$. It jumps by ± 2 when crossing the part of Σ_4^1 that consists of immersions with an exceptional quadruple point and does not jump when crossing any other part of Σ_4^1. Furthermore, Q_2 is an invariant of regular homotopy.*

Proof. The first part is immediate from Proposition 5.3.1. Let f and g be regularly homotopic then f and g can be joined by a path in $\mathcal{F}_4$ which intersects Σ_4^1 transversally. Since Q jumps by ± 2 or 0 on such intersections, the lemma follows. □

Let $n = 4$ and let f_0 have an exceptional triple point. By Proposition 5.3.1, C_{f_δ} is obtained from $C_{f_{-\delta}}$ by a single Morse modification (for notation, see Section 5.1). Define

$$T(f) = \text{the number of components of } C_f,$$

for generic immersions f.

PROPOSITION 6.1.3. *T is an invariant of generic immersions $S^3 \to \mathbb{R}^4$. It jumps by ± 1 when crossing the part of Σ_4^1 that consists of immersions with an exceptional triple point and does not jump when crossing any other part of Σ_4^1.*

Proof. Immediate from Proposition 5.3.1. □

Let $n = 4$ and let f_0 have an exceptional double point (a self tangency point). By Proposition 5.3.1, F_{f_δ} is obtained from $F_{f_{-\delta}}$ by a single Morse modification (for notation, see Section 5.1). Define

$$D(f) = \chi(F_f) \quad \text{and} \quad D_2(f) = D(f) \bmod 2 \in \mathbb{Z}_2,$$

for generic immersions f, where χ denotes the Euler characteristic.

PROPOSITION 6.1.4. *D is an invariant of generic immersions $S^3 \to \mathbb{R}^4$. It jumps by ± 2 when crossing the part of Σ_4^1 that consists of immersions with an exceptional double point and does not jump when crossing any other part of Σ_4^1. Furthermore, D_2 is an invariant of regular homotopy.*

Proof. See Proposition 6.1.2. □

6.2. *Linking*

We construct an invariant of generic immersions $S^3 \to \mathbb{R}^5$ which jumps under triple point moves and stays constant under self tangency moves: Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion with self intersection M_f. Consider the preimage $\widetilde{M}_f = f^{-1}(M_f)$.

Choose a normal vector field w along $\widetilde{M}_f$ satisfying the following condition: If $\widetilde{M}'_f$ denotes the result of pushing $\widetilde{M}_f$ slightly along w we require that

$$[\widetilde{M}'_f] = 0 \in H_1(S^3 - \widetilde{M}_f). \qquad (\dagger)$$

Any two vector fields satisfying this condition are homotopic. For existence, note that we can take w as the normal vector field of the boundary in a Siefert-surface of the link $\widetilde{M}_f \subset S^3$.

We define a vector field v along M_f: For $p \in M_f$, let

$$v(p) = df\, w(p_1) + df\, w(p_2), \text{ where } \{p_1, p_2\} = f^{-1}(p).$$

Let M'_f denote the result of pushing M_f slightly along v. Then $M'_f \subset \mathbb{R}^5 - f(S^3)$.

By Alexander duality $H_1(\mathbb{R}^5 - f(S^3)) \cong H^3(f(S^3))$ and $H^3(f(S^3)) \cong \mathbb{Z}$ since there is a triangulation of S^3 giving a triangulation of $f(S^3)$ after identifications in the 0 and 1 skeletons only. An orientation of S^3 gives a canonical generator of $H_1(\mathbb{R}^5 - f(S^3))$: The boundary of a small 2-disk intersecting $f(S^3)$ transversally in one point and oriented in such a way that the intersection number is positive. Note that this intersection number is independent of the ordering of the 2-disk and $f(S^3)$.

Recall that there is an induced orientation on M_f (Section 5.1) and hence on M'_f. Define

$$\mathrm{lk}(f) = [M'_f] \in H_1(\mathbb{R}^5 - f(S^3)) = \mathbb{Z} \quad \text{and} \quad \lambda(f) = \mathrm{lk}(f) \bmod 3 \in \mathbb{Z}_3.$$

Note that $\mathrm{lk}(f)$ is well defined since an homotopy between two vector fields w and w' satisfying ($\dagger$) induces a homotopy between the shifted self intersections in $\mathbb{R}^5 - f(S^3)$.

LEMMA 6.2.1. lk *is an invariant of generic immersions. It jumps by* ± 3 *when crossing the triple point part of* Σ^1_5 *and remains constant when crossing the self tangency part. Furthermore,* λ *is an invariant of regular homotopy.*

Proof. The second part follows from the first exactly as in Proposition 6.1.2. We prove the first part:

Suppose that f_t is a path in $\mathcal{F}_5$ intersecting Σ^1_5 transversally at f_0 and let $\delta > 0$ be small. We restrict attention to a small neighborhood of the exceptional multiple point of f_0 since all the immersions f_t, $|t| \leq \delta$ can be assumed to agree outside of this neighborhood.

Assume that f_0 has an elliptic self tangency point $f_0(p) = f_0(q)$. Using Proposition 5.3.2 we can write

$$f_t(x) = (x_1, x_2, t, x_3, 0).$$
$$f_t(y) = (y_1, y_2, y_1^2 + y_2^2, 0, y_3),$$

where x are coordinates around p and y are coordinates around q. The preimages of the newborn self intersection circle c of f_δ are $\{x_1^2 + x_2^2 = \delta, x_3 = 0\}$ and $\{y_1^2 + y_2^2 = \delta, y_3 = 0\}$. We may assume that the field w is given by $w(x) = \frac{\partial}{\partial x_3}$ and $w(y) = \frac{\partial}{\partial y_3}$.

Now shift c a small distance ϵ along $df_\delta w(x) + df_\delta w(y)$ to get c'. Then c' is the boundary of the 2-disk

$$(r\delta\cos(v), r\delta\sin(v), \delta, \epsilon, \epsilon), \quad 0 \le v \le 2\pi, 0 \le r \le 1,$$

which does not intersect $f_\delta(S^3)$. Hence, $\mathrm{lk}(f_\delta) = \mathrm{lk}(f_{-\delta})$.

Assume that f_0 has a hyperbolic self tangency point. As above we can write

$$f_t(x) = (x_1, x_2, t, x_3, 0).$$
$$f_t(y) = (y_1, y_2, y_1^2 - y_2^2, 0, y_3).$$

The preimages of the self intersection are $\{x_1^2 - x_2^2 = \delta, x_3 = 0\}$ and $\{y_1^2 - y_2^2 = \delta, y_3 = 0\}$. We can choose the field w so that $w(x) = \frac{\partial}{\partial x_3}$ and $w(y) = \frac{\partial}{\partial y_3}$ close to p and q, respectively. We note that with vector fields as above and equal outside neighborhoods of p and q the condition (†) holds for $f_{-\delta}$ if and only if it holds for f_δ. The rest of the argument is similar to the elliptic case.

Assume that f_0 has a triple point. According to Proposition 5.3.2 we can find coordinates x, y and z centered at p_1, p_2 and p_3, respectively such that close to the triple point $f_0(p_1) = f_0(p_2) = f_0(p_3)$ we have

$$\begin{aligned} f_t(x) &= (x_1, x_2, x_3, 0, 0), \\ f_t(y) &= (y_1, t, 0, y_2, y_3), \\ f_t(z) &= (0, z_2, z_1, z_2, z_3). \end{aligned}$$

Denote the neighborhoods of p_1, p_2 and p_3 by X, Y and Z, respectively. If we orient these by declaring the frames $(\partial_1, \partial_2, \partial_3)$ to have the positive orientation then the oriented self intersections of f_0 are the lines

$$\begin{aligned} f_t(X) \cap f_t(Y) &= (s, t, 0, 0, 0), && s \in \mathbb{R}, \\ f_t(X) \cap f_t(Z) &= (0, 0, s, 0, 0), && s \in \mathbb{R}, \\ f_t(Z) \cap f_t(Y) &= (0, t, 0, t, -s), && s \in \mathbb{R}. \end{aligned}$$

The inverse image of the self intersection of f_t, $t < 0$ is shown in Figure 2.

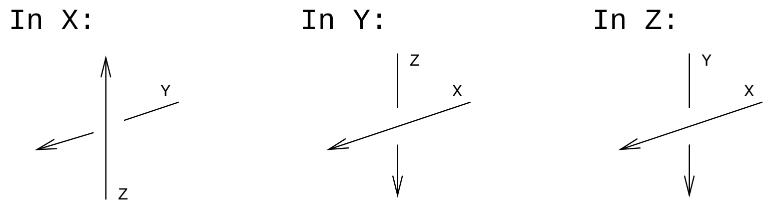

FIGURE 2. Crossings close to a triple point

We calculate $\mathrm{lk}(f_\delta)$ in terms of $\mathrm{lk}(f_{-\delta})$ and have to take two things into consideration: First the motion of Y itself. Second, the changes in the field w that this motion causes.

Assume that the shifting distance is very small in comparison to δ. Let $l_t(X, Z)$ denote the intersection $f_t(X) \cap f_t(Z)$ shifted along $v = df\, w(x) + df\, w(z)$. Let $D_t(X, Z)$ denote a part of a disc in $\mathbb{R}^5$ bounded by $l_t(X, Z)$. We may assume that $D_t(X, Z)$ is a shift along an extension of v of a disk in $f_t(X)$.

First, the intersection $D_t(X, Z) \cap f_t(Y)$ does not depend on the field w since the shift is assumed to be very small in comparision to t. We calculate the change in $D_t(X,Z) \cap f_t(Y)$ from the preimage in X. From Figure 3 it follows that the algebraic number of intersection points in $D_{-\delta}(X,Z) \cap f_{-\delta}(Y)$ differs from that corresponding to $D_\delta(X,Z) \cap f_\delta(Y)$ by $+1$.

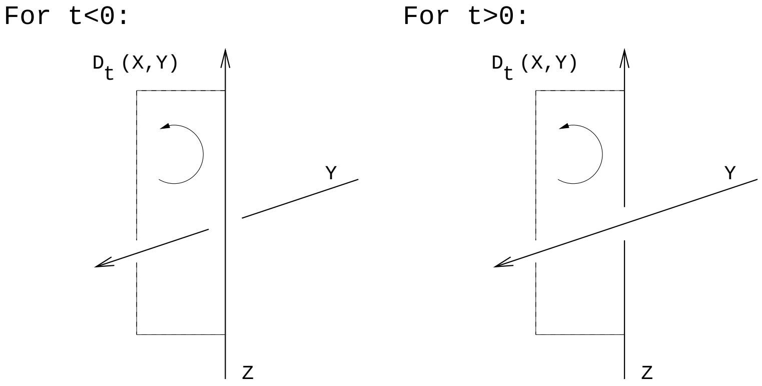

FIGURE 3. $D_t(X,Y)$ intersected with Y.

We now take the field w into consideration. In Figure 4 we see a Seifert-surface in S^3 of $\widetilde{M}_f$ close to a crossing point.

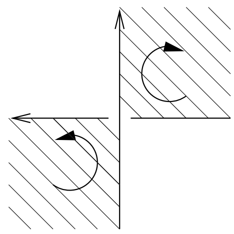

FIGURE 4. A Seifert-surface.

In Figure 5 we see w chosen as the inward normal in a Seifert-surface of $\widetilde{M}_{f_t}$. As we move through the triple point the direction of rotation of the vector field w is changed. This change in rotation gives rise to a new positive intersection point in $D_t(X,Z) \cap f_t(Z)$ for $t > 0$. As seen in Figure 6.

Similarly, we get a new positive intersection point in $D_t(X,Z) \cap f_t(X)$. Thus, in total the algebraic number of intersection points in $D_t(X,Z) \cap (f_t(X) \cup f_t(Y) \cup f_t(Z))$ increases by 1 when t is changed from $-\delta$ to δ.

Similarly, the intersection numbers corresponding to $D_t(X,Y) \cap (f_t(X) \cup f_t(Y) \cup f_t(Z))$ and $D_t(Y,Z) \cap (f_t(X) \cup f_t(Y) \cup f_t(Z))$ increase by 1 when t is changed from $-\delta$ to δ.

It follows that $\operatorname{lk}(f_\delta) - \operatorname{lk}(f_{-\delta}) = 3$. This proves the lemma. □

COROLLARY 6.2.2. *For ξ in* **Imm**, *let $\lambda(\xi) = \lambda(f)$, where f is a generic immersion representing ξ. Then*

$$\lambda\colon \mathbf{Imm} \to \mathbb{Z}_3,$$

is a homomorphism.

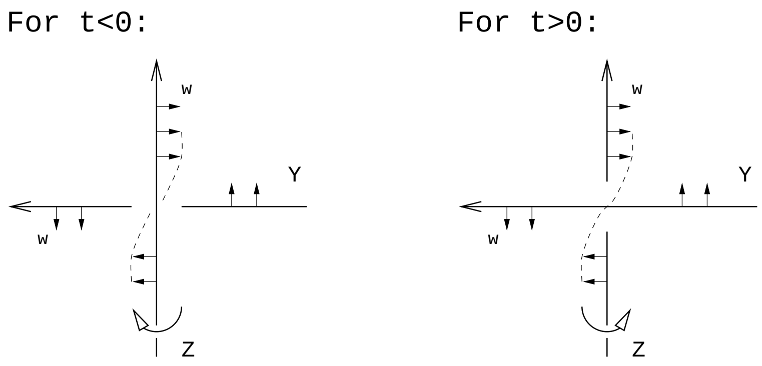

FIGURE 5. Change of w on passing a triple point.

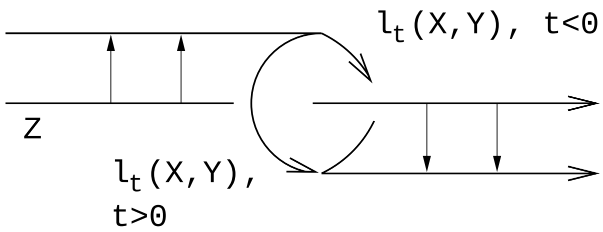

FIGURE 6. $D_t(X, Z)$ intersected with $f_t(Z)$.

Proof. As a function, λ is well defined by Lemma 6.2.1. Clearly, it is additive under connected sum and thus a homomorphism. $\square$

Remark 6.2.3. In knot theory one assigns a sign to crossings as in Figure 2 by comparing the orientation given by (v_1, v_2, v_{12}) to that of the ambient space, where v_1 is the tangent vector to the first branch, v_2 the tangent vector of the second, and v_{12} a vector from the second to the first branch. We note that all crossings appearing in Figure 2 are positive. Changing the orientation of one of the sheets changes the sign of all crossings. Hence, *all three crossings appearing close to a triple point always have the same sign.*

6.3. ***Finite order invariants***

In this section we summarize the properties of finite order invariants of generic immersions that are needed to prove Theorem 2. Recall that we have defined the jump ∇v of an invariant v of generic immersions. It was defined only up to sign. To get rid of this sign we need a coorientation of Σ_n^1:

If $f_0 \in \Sigma_n^1$ then there is a neighborhood U of f_0 in $\mathcal{F}_n$ which is cut into two parts by Σ_n^1. We make a choice of a positive part and a negative part of U. A coherent choice like that for all $f_0 \in \Sigma_n^1$ is a *coorientation of* Σ_n. A coorientation enables us to make ∇v well defined: In the definition

$$\nabla v(f_0) = v(f_\delta) - v(f_\delta),$$

we require that f_δ is on the positive and $f_{-\delta}$ on the negative side of Σ^1_n at f_0. As mentioned, ∇v is then locally constant on Σ^1_n and we may consider ∇v as an element of $H^0(\Sigma^1_n)$.

Let f_t be a path in $\mathcal{F}_5$ intersecting Σ^1_5 transversally at f_0. We coorient Σ_5 as follows: If $f_0 \in E \cup Y$ then f_δ is on the positive side of $E \cup Y$ at f_0 if M_{f_δ} has more components than $M_{f_{-\delta}}$. If $f_0 \in T$ then f_δ is on the positive side of T if $\mathrm{lk}(f_\delta) \geq \mathrm{lk}(f_{-\delta})$.

It is easy to check that this coorientation is *continuous* (see [4], Section 7.6). This enables us to iterate the above construction and define inductively

$$\nabla^{k+1} v(f_{0,0,\dots,0}) = \nabla^k(f_{\delta,0,\dots,0}) - \nabla^k(f_{-\delta,0,\dots,0}).$$

Here $f_{0,\dots,0}$ is an immersion with $k+1$ distinct degenerate points (self tangencies or triple points) and $f_{t,0,\dots,0}$ is a path in $\Sigma^k_5 \cup \Sigma^{k+1}_5$, where Σ^j_5 is the space of immersions with j distinct degenerate points, intersecting Σ^{k+1}_5 transversally. Then $\nabla^k v$ is an element in $H^0(\Sigma^k_5)$ (see [3], Remark 9.1.2). We say that *an invariant v is of order k* if $\nabla^{k+1} v \equiv 0$.

Finally, we remark that the space of invariants of generic immersions, i.e. $H^0(\mathcal{F}_n - \Sigma_n)$, splits as a direct sum over the path components of $\mathcal{F}_n$. That is $H^0(\mathcal{F}_n - \Sigma_n) = \bigoplus_U H^0(U - (\Sigma_n \cap U))$, where U runs over the path components of $\mathcal{F}_n$.

6.4. ***Proof of Theorem 2***

By Lemmas 6.1.1 and 6.2.1, J and L are invariants which changes as claimed. Clearly, L and J are independent.

Since ∇J (∇L) is 1 (0) on all self tangency parts of Σ^1_5 and is 0 (1) on all triple point parts of Σ^1_5, it follows from Proposition 5.4.1 (see Figure 1) that $\nabla^2 J \equiv 0$ and $\nabla^2 L \equiv 0$. Thus, they both have order one.

Let U be a path component of $\mathcal{F}_5$ and v an invariant, the jump of which is constant on $T \cap U$ and $(E \cup Y) \cap U$ (for notation see Section 5.3). Then the jump of $v|U$ is a linear combination of the jumps of $J|U$ and $L|U$ and it follows that $v|U$ is (up to a constant term) a linear combination of $J|U$ and $L|U$. Thus, the theorem follows once we show that any first order invariant has this property.

Let f and g be regularly homotopic immersions with one self tangency point of the same kind each or with one triple point each. Using a diffeotopy of $\mathbb{R}^5$ we can move the exceptional point of g to that of f and using a diffeotopy of S^3 we can make the maps agree in small disks around the preimages of this point. Choose unknotted paths in S^3 connecting these disks. Since there are no 1-knots in $\mathbb{R}^5$, we can after diffeotopy of $\mathbb{R}^5$ assume that the maps agree also in a neighborhood of these arcs. After this is done we have two immersions that agree on a 3-disk in S^3 and are regularly homotopic. We must show that they are regularly homotopic keeping this 3-disk fixed. However, by the Smale-Hirsch h-principle the obstruction to this regular homotopy is the difference of the Smale invariants which is zero. Choosing a generic regular homotopy we see that we can join f to g with a path in $\Sigma^1_5 \cup \Sigma^2_5$ intersecting Σ^2_5 transversally only at immersions with two distinct exceptional points.

Let v be a first order invariant. Then $\nabla^2 v \equiv 0$. This and the fact that ∇v is locally constant on Σ^1_5 implies that ∇v remains constant along a path as above. Hence, if f and g both belong to either one of $E \cap U$, $Y \cap U$, or $T \cap U$ then $\nabla v(f) = \nabla v(g)$.

It remains to show that $\nabla v|E \cap U$ equals $\nabla v|Y \cap U$: Consider the last part of Figure 1 in Section 5.4. Let f be an immersion in the E-branch and g an immersion in the Y-branch. Pick a small loop h_t, $0 \leq t \leq 1$, $h_1 = h_0$ around the cusp such that it intersects the E-branch in f and the Y-branch in g. Then $0 = v(h_1) - v(h_1) = \nabla v(f) - \nabla v(g)$. Hence,

$$\nabla v|E \cap U \equiv \nabla v(f) = \nabla v(g) \equiv \nabla v|Y \cap U.$$

The theorem follows. □

6.5. *L* ***and*** St

Arnold's invariant St for plane curves (see [2]) can be characterized axiomatically as an invariant which jumps by 1 on triple points, does not jump under self tangencies, and is additive under connected summation.

The invariant $L = \frac{1}{3}(\mathrm{lk} + \widetilde{\lambda})$ jumps by one on triple points but is not additive under connected sum. The reason for this is that the invariant $\widetilde{\lambda}$ of order zero is not additive under connected sum.

The invariant lk is additive under connected sum. It jumps by ± 3 on T and not at all on $E \cup Y$. Thus, lk and J generates the space of $\mathbb{Q}$-valued invariants. If we want an invariant which is additive under connected sum and together with J generates the space of $\mathbb{Z}$-valued invariants we may proceed as follows:

To accomplish this we note that , either $\mathrm{lk} + \Omega$ or $\mathrm{lk} - \Omega$ is divisible by3, where Ω is the Smale invariant. The sign depends on the choice of generator of

We have made a choice of a generator of $\pi_3(V_{5,3}) \cong \mathbb{Z}$ in Lemma 3.1.2. It allows us to identify this group with the integers and we consider the Smale invariant Ω to be integer valued.

In Lemma 8.3.3 we calculate $\mathrm{lk}(f)$ for an immersion with $\Omega(f) = 1$ and the result is $\mathrm{lk}(f) = 2$. Since $\lambda = \mathrm{lk} \bmod 3$ is an invariant of regular homotopy it follows that $\frac{1}{3}(\mathrm{lk}(f) + \Omega(f))$ is an integer valued invariant which jumps by ± 1 on triple points, does not jump on self tangencies, and is additive under connected sum (since both lk and Ω are). We therefore define

$$\mathrm{St}(f) = \frac{1}{3}((\mathrm{lk}(f) + \Omega(f))).$$

To complete the analogy with Arnold's St, we need to define the coorientation of the triple point stratum T of Σ^1_5 in terms of the local picture near a triple point. Recall that all crossings in the preimage close to a triple point have the same sign (Remark 6.2.3). By Lemma 6.2.1 our coorientation of T agrees with the following one defined in local terms:

We say that we pass T in the positive direction if the crossing signs in the preimage close to the triple point changes from positive to negative.

PROPOSITION 6.5.1. *There is a unique invariant* St *such that it jumps by* $+1$ (-1) *under positive (negative) triple point moves, it does not jump under self tangency moves, it is additive under connected sum, its value on an immersion changes sign if the immersion is composed with an orientation reversing diffeomorphism of* S^3*, and takes the values* 1 *on* $f_{-\frac{1}{2}}$ *(see Proposition 8.3.1).*

Proof. A generic immersion g with $\Omega(g) \neq 0$ is regularly homotopic to a connected sum of $|\Omega(g)| \in \mathbb{Z}_+$ copies of $f_{-\frac{1}{2}}$ or $f_{-\frac{1}{2}} \circ r$, where r is an orientation reversing diffeomorphism. A generic immersion g with $\Omega(g) = 0$ is regularly homotopic to $f_{\frac{1}{2}} \star f_{-\frac{1}{2}}$.

We know the values of our invariants on connected sums of $f_{\pm\frac{1}{2}}$. Thus, we know the value on any other generic immersion by adding the jumps along a path connecting it to such a connected sum. This proves uniqueness.

For existence, we need to show that St changes sign if the orientation is reversed. For Ω we already know this (Lemma 3.5.2). To see that the same is true for lk note that the induced orientation of M_f does not change if we reverse the orientation on S^3. The orientation of $f(S^3)$ does change and hence $\mathrm{lk}(f) = [M_f'] \in H_1(\mathbb{R}^5 - f(S^3))$ does change sign. □

7. Pin structures and twist framings

In this section we show that there is an induced pin structure on the self intersection surface of a generic immersion $S^3 \to \mathbb{R}^4$. The Brown invariant of this pin structure is unchanged under regular homotopy. Even stronger, if two generic immersions into $\mathbb{R}^4$ are regularly homotopic in $\mathbb{R}^5$ after composing them with the inclusion $\mathbb{R}^4 \to \mathbb{R}^5$ then the corresponding Brown invariants are equal.

Using the geometry of self intersections of generic immersions $S^3 \to \mathbb{R}^5$, we define a $\mathbb{Z}_4$-valued invariant of regular homotopy.

7.1. *Immersions into* $\mathbb{R}^4 \subset \mathbb{R}^5$

Let $g\colon S^3 \to \mathbb{R}^4$ be an immersion. Its Smale invariant $\Omega(g)$ is an element of $\pi_3(SO(4)) = \mathbb{Z}[\sigma]\oplus\mathbb{Z}[\rho]$ (see Lemma 3.1.1). The normal bundle of g is 1-dimensional and orientable. Hence, it is trivial and g admits a normal field $n\colon S^3 \to S^3$, such that the frame $(n(x), dfQ(x))$ (see Section 3.3) gives the positive orientation of $\mathbb{R}^4$. If $\Omega(g) = m[\sigma] + n[\rho]$ then, from Lemma 3.1.1, it follows that $m + 1 = \deg(n)$, the normal degree of g.

Let $s\colon S^3 \to \mathbb{R}^4$ be the standard embedding and $\widehat{s} = s \circ r$, where r is a reflection in a hyperplane as in Lemma 3.5.2.

Lemma 7.1.1.

$$\Omega(\widehat{s}) = -2[\sigma] + [\rho] \in \pi_3(V_{4,3}) = \pi_3(SO(4)).$$

Proof. Let $i\colon \mathbb{R}^4 \to \mathbb{R}^5$ be the inclusion. Then $i \circ s$ and $i \circ \widehat{s}$ are regularly homotopic in $\mathbb{R}^5$. Hence, by Lemma 3.3.3, $\Omega(\widehat{s}) \in N \subset SO(4)$.

The normal degree of $\widehat{s}$ equals the normal degree of s with opposite sign. Thus, $\Omega_4(\widehat{s}) = -2[\sigma] + n[\rho]$. But then, $\Omega_4(\widehat{s}) \in N \subset SO(4)$ implies that $n = 1$. □

Let $g, h\colon S^3 \to \mathbb{R}^4$ be immersions.

Proposition 7.1.2. *If $i \circ g$ and $i \circ h$ are regularly homotopic in $\mathbb{R}^5$ then either g and $h \star \widehat{s} \star \cdots \star \widehat{s}$ or h and $g \star \widehat{s} \star \cdots \star \widehat{s}$ are regularly homotopic in $\mathbb{R}^4$.*

Proof. If $i \circ g$ and $i \circ h$ are regularly homotopic in $\mathbb{R}^5$ then $\Omega(g) - \Omega(h) \in N$. Now, $\Omega(\widehat{s})$ is a generator of N and the Smale invariant is additive under connected sum. □

7.2. *Functionals on curves in self intersections*

The invariants we are about to define originates from functionals on curves in self intersections. Before we can construct these functionals we need some preliminaries:

Let $\mathbf{I}$ denote the identity matrix.

DEFINITION 7.2.1. *Define* $\mathbf{A}(4) \in SO(4)$ *and* $\mathbf{A}(5) \in SO(5)$ *by*

$$\mathbf{A}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}(5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The group $\{\mathbf{I}, \mathbf{A}(n)\}$ acts on $SO(n)$ from the right. Denote the corresponding quotient space $SO(n)/\mathbf{A}(n)$ and let p be the projection.

PROPOSITION 7.2.2. *For* $n = 4$ *and* $n = 5$,

$$\pi_1(SO(n)/\mathbf{A}(n)) = \mathbb{Z}_4.$$

Proof. See [3], Proposition 5.2. □

Let $u \in \pi_1(SO(n))$ denote the nontrivial element.

The functional on curves in a self intersection surface of a generic immersion into 4-space is constructed as follows (for notation, see Proposition 5.1.5):

Let $g\colon S^3 \to \mathbb{R}^4$ be a generic immersion. Let c be a closed curve in F_g meeting $F_g^0 \cup F_g^1$ transversally and let $c_S = p^{-1}(c)$. Note that c_S is either a union of two disjoint circles or one circle.

Choose a parameterization $r\colon I \to c$ of c and a normal vector $n\colon I \to TF_g$ of c in TF_g. Then r lifts to two parameterizations r_1 and r_2. If c_S is connected then the path product $r_1 * r_2$ is a parameterization of c_S.

Choose normal fields ν_1 and ν_2 of $\widetilde{M}_g$ along $s \circ r_1$ and $s \circ r_2$, respectively. Let n_1 and n_2 be such that $dp(n_i) = n$. Assume that ν_i is chosen so that $(ds\,\dot{r}_i, ds\,n_i, \nu_i)$ is a positively oriented frame, $i = 1, 2$. Then $(dt\,\dot{r}, dt\,n, dg\,\nu_1, dg\,\nu_2)$ is a framing or twist framing X of $T\mathbb{R}^4$ along $t \circ c$.

Let $\vec{c}_S$ denote the loop (or loops) in $SO(TS^3)$ (the principal $SO(3)$-bundle of the tangent bundle of S^3) represented by $s \circ c_S$ with the framing $(ds\,\dot{r}_i, ds\,n_i, \nu_i)$ along $s \circ r_i$, $i = 1, 2$ and let $[\vec{c}_S]$ denote the corresponding element in $H_1(SO(TS^3); \mathbb{Z}_2)$.

Let $[c, r, X] \in \pi_1(SO(4)/\mathbf{A}(4))$ denote the homotopy class of the loop induced by $t \circ c$ with (twist) framing X as above (which we may have to orthonormalize). Then define

DEFINITION 7.2.3.

$$\omega_g(c) = \langle \xi_S, [\vec{c}_S] \rangle \cdot p_*(u) + [c, r, X] + p_*(u) \in \pi_1(SO(4)/\mathbf{A}(4)),$$

where $\xi_S \in H^1(SO(TS^3); \mathbb{Z}_2)$ *denotes the unique spin structure on* S^3.

This definition is independent of all choices (see [3], Lemma 7.6).

We extend the functional ω_g to collections $L = \{c_1, \dots, c_m\}$ of oriented closed curves in F_g transversal to $F_g^0 \cup F_g^1$:

$$\omega_g(L) = \sum_{i=1}^{m} \omega_g(c_i).$$

Let L be a collection of oriented closed curves in F_g with m transverse intersections and L' be the collection obtained from smoothing each intersection, respecting orientation. Then

$$\omega_g(L') = \omega_g(L) + m \cdot p_*(u), \qquad (\dagger)$$

and if K and L are isotopic collections then

$$\omega_g(K) = \omega_g(L), \tag{‡}$$

see [3], Lemma 7.15.

The functional on self intersection circles of generic immersions into 5-space is constructed as follows:

Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion. Let $c \subset M_f$ be a component and $c_S = f^{-1}(c)$. Note that c_S is either a union of two disjoint circles or one circle.

Choose a parameterization r of c, agreeing with the orientation induced on c if k is odd. Then r lifts to two parameterizations r_1 and r_2. If c_S is connected then the path product $r_1 * r_2$ is a parameterization of c_S.

Choose orthonormal framings Y_1 and Y_2 of $N(c_S \subset S^3)$ (the normal bundle of c_S in S^3) along r_1 and r_2 respectively. Then $(\dot r, dfY_1, dfY_2)$ is a framing or twist framing X of $T\mathbb{R}^5$ along c.

Let $\vec c_S$ denote the loop (or loops) in $SO(TS^5)$ (the principal $SO(5)$-bundle of the tangent bundle of S^5) represented by c_S with the framing $(\dot r_i, Y_i)$ and $[\vec c_S]$ the corresponding element in $H_1(SO(TS^5);\mathbb{Z}_2)$. Let $[c,r,X] \in \pi_1(SO(5)/\mathbf{A}(5))$ denote the homotopy class of the loop induced by c with the (twist) framing X (which we may have to orthonormalize). Then define

Definition 7.2.4.

$$\omega_f(c) = \langle \xi, [\vec c_S]\rangle \cdot p_*(u) + [c,r,X] + p_*(u) \in \pi_1(SO(5)/\mathbf{A}(5)),$$

where $\xi \in H^1(SO(TS^5);\mathbb{Z}_2)$ *denotes the unique spin structure on* S^5.

This definition is independent of all choices (see [3], Lemma 6.5).

7.3. *Pin structures on surfaces*

Let V be a vector space over $\mathbb{Z}_2$ with a nonsingular symmetric bilinear form $(x,y) \mapsto x\cdot y$. A $\mathbb{Z}_4$-quadratic function on V is a function $q\colon V \to \mathbb{Z}_4$ such that $q(x+y) = q(x) + q(y) + 2(x\cdot y)$, for all $x,y \in V$.

There are four indecomposable $\mathbb{Z}_4$-quadratic spaces:

$$\begin{aligned}
&\mathcal{P}_+ = (\mathbb{Z}_2(a),\cdot,q), && a\cdot a = 1, && q(a) = 1,\\
&\mathcal{P}_- = (\mathbb{Z}_2(a),\cdot,q), && a\cdot a = 1, && q(a) = -1,\\
&\mathcal{T}_0 = (\mathbb{Z}_2(b)\oplus\mathbb{Z}_2(c),\cdot,q), && b\cdot b = c\cdot c = 0,\ b\cdot c = 1, && q(b) = q(c) = 0,\\
&\mathcal{T}_4 = (\mathbb{Z}_2(b)\oplus\mathbb{Z}_2(c),\cdot,q), && b\cdot b = c\cdot c = 0,\ b\cdot c = 1, && q(b) = q(c) = 2.
\end{aligned}$$

A $\mathbb{Z}_4$-quadratic space V is called *split* if it contains a subspace H such that $\dim H = \frac{1}{2}\dim V$, $q|H = 0$ and $H\cdot H = \{0\}$. Two $\mathbb{Z}_4$-quadratic spaces V and W belong to the same *Witt class* if there exists spilt spaces S_1 and S_2 such that $V\oplus S_1 \cong W \oplus S_2$. The Witt classes forms the *Witt group* which is generated by $[\mathcal{P}_+]$ (the Witt class of $\mathcal{P}_+$) with relations $8[\mathcal{P}_+] = 0, 4[\mathcal{P}_+] = [\mathcal{T}_4]$ and $[\mathcal{P}_+] + [\mathcal{P}_-] = 0$.

Given a $\mathbb{Z}_4$-quadratic space V with quadratic function q we define

$$\lambda(V,q) = \sum_{x\in V} e^{\frac{\pi i}{2} q(x)}.$$

Then

$$\lambda(V,q) = \sqrt{2}^{\dim V}\left(\frac{1+i}{\sqrt{2}}\right)^m.$$

Since $\frac{1+i}{\sqrt{2}}$ is an 8^{th} root of unity, m modulo 8 is well defined. This is *Brown's invariant*. It is denoted $\beta(V,q)$ and gives an isomorphism between the Witt group and $\mathbb{Z}_8$. We shall sometimes write $\beta(q)$, dropping V. More details about $\mathbb{Z}_4$-quadratic spaces can be found in [12].

Let F be a surface. By a pin structure on F we shall mean a Pin^--structure on the tangent bundle TF of F. There is a 1-1 correspondence between pin structures on F and $\mathbb{Z}_4$-quadratic functions on $H_1(F;\mathbb{Z}_2)$, see [10], Theorem 3.2.

7.4. *Invariance of the Witt class and a homomorphism* Imm $\to \mathbb{Z}_8$

Let $g\colon S^3 \to \mathbb{R}^4$ be a generic immersion. Choose, once and for all an isomorphism

$$\phi\colon \pi_1(SO(4)/\mathbf{A}(4)) \to \mathbb{Z}_4.$$

(This choice is discussed in Section 9.2.) By equations (†) and (‡) in Section 7.2 and Lemma 3.4 in [10], $\phi\circ\omega_g$ induces a $\mathbb{Z}_4$-quadratic function $q_g : H_1(F_g;\mathbb{Z}_2) \to \mathbb{Z}_4$ (and hence a pin structure on TF_g). We define

$$\beta(g) = \beta(q_g),$$

the Brown invariant of the $\mathbb{Z}_4$-quadratic function q_g.

LEMMA 7.4.1. *Let $g_0, g_1 : S^3 \to \mathbb{R}^4$ be regularly homotopic generic immersions. Then $\beta(g_0) = \beta(g_1)$*

Proof. Connect g_0 to g_1 by a path g_t in $\mathcal{F}_4$ which is transversal to Σ_4. This means that g_t intersects Σ_4 only transversally at finitely many points in Σ_4^1. Clearly, β remains constant as long as we do not cross Σ_4 and we must show that β remains unchanged when we cross Σ_4^1:

Passing Σ_4^1 at g_{t_0}, where g_{t_0} has a j-tuple point $j \geq 2$ does not change F_{g_t}. It changes the preimage in F_g of M_g^{4-j}. Since the curves representing a basis of $H_1(F_{g_t};\mathbb{Z}_2)$, used to calculate β can be chosen so that they do not meet the discs where these changes occur, β does not change at such crossings.

Passing Σ_4^1 at g_{t_0}, where g_{t_0} has a degenerate double point does change F_{g_t} by a Morse modification. If it has index 2 or 0 then F_{g_t} is changed by addition or subtraction of an S^2-component. This does not affect β. If it has index 1 then F_{g_t} is changed by addition or subtraction of a handle. In this handle there is a newborn circle c, $q_f(c) = 0$ and β remains unchanged, see [5], Lemmas 5.2.1-8. □

Let $f\colon S^3 \to \mathbb{R}^5$ be an immersion. The normal bundle of f is trivial (Section 3.3) which implies ([7], Theorem 6.4) that f is regularly homotopic to an immersion $f_1 : S^3 \to \mathbb{R}^4$ composed with the inclusion $\mathbb{R}^4 \to \mathbb{R}^5$. Moreover, we may assume that f_1 is generic. Define

$$\beta(f) = \beta(f_1).$$

PROPOSITION 7.4.2. *For $\xi \in$ **Imm**, let $\beta(\xi) = \beta(f)$, where f is an immersion representing ξ. Then*

$$\beta\colon \mathbf{Imm} \to \mathbb{Z}_8,$$

is a homomorphism.

Proof. We show first that β is well defined: If f_1 and g_1 are immersions into $\mathbb{R}^4$ which are regularly homotopic to f in $\mathbb{R}^5$ then f_1 is regularly homotopic to $g_1 \star \widehat{s} \star \cdots \star \widehat{s}$ (Proposition 7.1.2). We can perform the connected sum of two generic immersions into $\mathbb{R}^4$ in such a way that the self intersection of the immersion obtained is the union of the self intersections of the summands and at most two new S^2 self intersection components. Hence, performing connected sum with the embedding $\widehat{s}$ does not change β. Thus, $\beta\colon \mathbf{Imm} \to \mathbb{Z}_8$ is well defined by Lemma 7.4.1.

By the argument above, β is additive under connected summation. Hence, it is a homomorphism. $\square$

7.5. *A homomorphism* $\mathbf{Imm} \to \mathbb{Z}_4$

Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion. Choose, once and for all, an isomorphism

$$\psi\colon \pi_1(SO(5)/\mathbf{A}(5)) \to \mathbb{Z}_4.$$

(This choice is discussed in Section 9.2.)

Definition 7.5.1. *A component c of the self intersection M_f is* right twist framed *if $\psi(\omega_f(c)) = 1$. It is* left twist framed *if $\psi(\omega_f(c)) = -1$.*

If c_S is disconnected then there is an induced spin structure on c, see [3], Proposition 2.10. This spin structure on c is trivial (spin-cobordant to zero) if $\omega_f(c) = 0$ and nontrivial if $\omega_f(c) = p_*(u)$ (see [3], proof of Theorem 7.30).

We define $r(f)$, $l(f)$ and $n(f)$ as the number of self intersection components of a generic immersion f that are right twist framed, left twist framed and has the nontrivial spin structure, respectively. Define

$$\tau(f) = r(f) + 2n(f) - l(f) \bmod 4 \in \mathbb{Z}_4$$

Proposition 7.5.2. *For $\xi \in \mathbf{Imm}$, let $\tau(\xi) = \tau(f)$, where f is a generic immersion representing ξ. Then*

$$\tau\colon \mathbf{Imm} \to \mathbb{Z}_4,$$

is a homomorphism.

Proof. That τ is well defined follows from [4], Propositions 5.1.1-4. Clearly, τ is additive under connected sum. Hence, it is a homomorphism. $\square$

8. Special immersions

In this section, we construct immersions $S^3 \to \mathbb{R}^5$ with arbitrary Smale invariant by rotating a 2-disk with a kink i 5-space and use an immersion $\mathbb{R}P^3 \to \mathbb{R}^4$ to show that the homomorphism β (see Proposition 7.4.2) is nontrivial.

8.1. *The Whitney kink.*

As in Whitney [17] we consider the map $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by the equations

$$g_1(x,y) = x - \frac{2x}{u} \qquad g_2(x,y) = y$$
$$g_3(x,y) = \frac{1}{u} \qquad g_4(x,y) = \frac{xy}{u},$$

where $u = (1+x^2)(1+y^2)$. This is an immersion with one transversal double point:

$$(0,0,1,0) = g(1,0) = g(-1,0).$$

Moreover, if $|x|$ or $|y|$ is large then g is close to the standard embedding $\mathbb{R}^2 \to \mathbb{R}^2 \times 0 \subset \mathbb{R}^4$. Thus, we can change g slightly so that it agrees with the standard embedding outside some large disk.

The Whitney kink enjoys the following symmetry property: If $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ and $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ are the linear maps

$$L(z_1, z_2, z_3, z_4) = (-z_1, -z_2, z_3, z_4) \quad \text{and} \quad R(x, y) = (-x, -y),$$

then $L \circ g \circ R = g$.

The differential of g is

$$dg_{(x,y)} = \begin{bmatrix} 1 - \frac{2(1-x^2)}{(1+x^2)u} & \frac{4xy}{(1+y^2)u} \\ 0 & 1 \\ \frac{-2x}{(1+x^2)u} & \frac{-2y}{(1+y^2)u} \\ \frac{y(1-x^2)}{(1+x^2)u} & \frac{x(1-y^2)}{(1+y^2)u} \end{bmatrix}$$

Let ∂_x and ∂_y be unit vector fields on $\mathbb{R}^2$ in the x and y directions, respectively. Applying dg to these we get a map of $\mathbb{R}^2$ into $V_{4,2}$, which outside some large disk is constantly equal to (∂_1, ∂_2), where ∂_i is the unit vector in the z_i-direction in $\mathbb{R}^4$.

Clearly, this map is homotopic in $V_{4,2}$ to a map of the form $(v_1(x, y), \partial_2)$, where $v_1(x, y)$ is orthonormal to ∂_2. Let D be a large disk in $\mathbb{R}^2$. A straightforward calculation shows that the degree of $v_1 : (D^2, \partial D^2) \to S^2$ is one.

Let B be 2-dimensional hemisphere in $\mathbb{R}^4$ which is flat close to its north pole. To make this precise, fix a small $a > 0$ and let B be a 2-disk such that

$$\begin{aligned} &B \cap \left\{ z \in \mathbb{R}^4 : z_3 \le \sqrt{1 - 16a^2} \right\} = \\ &= \left\{ (\xi, \eta, \zeta, 0) : \xi^2 + \eta^2 + \zeta^2 = 1, 0 \le \zeta \le \sqrt{1 - 16a^2} \right\}, \\ &B \cap \left\{ z \in \mathbb{R}^4 : z_3 \ge \sqrt{1 - 9a^2} \right\} = \\ &= \left\{ (\xi, \eta, \sqrt{1 - 9a^2}, 0) : \xi^2 + \eta^2 \le 4a^2 \right\}. \end{aligned}$$

Let (ξ, η), $\xi^2 + \eta^2 \le 1$, as above be coordinates on B. Then we can define an immersion $K\colon B \to \mathbb{R}^4$ with the following properties: The map K equals the inclusion for $\xi^2 + \eta^2 \ge a^2$. On the remaining part $D_a$of B the map K is a suitably scaled Whitney kink and $L(K(R(\xi, \eta))) = K(\xi, \eta)$.

Let (r, ϑ), $0 \le r \le \pi$ and $0 \le \vartheta \le 2\pi$ be polar coordinates on the disk D_a, then the map $D_a \to V_{4,2}$ induced by the differential of K is homotopic to

$$\begin{aligned} &dK_{(r,\vartheta)}(\partial_\xi) \simeq -\cos(r)\,\partial_1 + \sin(r)\,(\cos(\vartheta)\,\partial_3 - \sin(\vartheta)\,\partial_4)\,, \\ &dK_{(r,\vartheta)}(\partial_\eta) \simeq \partial_2, \end{aligned} \tag{†}$$

since the first of these equations defines a map $(D_a, \partial D_a) \to S^2$ of degree one.

8.2. *A modified standard embedding.*

Let $(y_1, \ldots, y_5)$ be coordinates on $\mathbb{R}^5$. Let ∂_i be the unit vector in the y_i-direction. Let $0 \le \theta \le 2\pi$ and let $\mathbb{R}^4_+(\theta) \subset \mathbb{R}^5$ be the subset

$$\{(x_0, x_1, x_2 \cos\theta, x_2 \sin\theta, x_3) : x_2 \ge 0\}.$$

Let $D(\theta) \subset \mathbb{R}^4_+(\theta)$ be the disk

$$\{(x_0, x_1, x_2 \cos\theta, x_2 \sin\theta, 0) : x_2 \ge 0, x_0^2 + x_1^2 + x_2^2 = 1\}.$$

As θ varies from 0 to 2π, $D(\theta)$ sweeps the standard S^3 in $\mathbb{R}^5$. We note that if $\theta_1 \neq \theta_2$, $\theta_1 \neq 0$ then

$$D(\theta_1) \cap D(\theta_2) = \{(x_0, x_1, 0, 0, 0) : x_0^2 + x_1^2 = 1\},$$

and $D(0) = D(2\pi)$.

A straightforward calculation shows that the column vectors $S^i(\theta)$ of the matrix $S(0, 0, \cos\theta, \sin\theta, 0, 0) = \Theta^s(0, 0, \cos\theta, \sin\theta, 0, 0)$ (see Lemma 3.3.2 for notation) are

$$\begin{aligned}
S^1(\theta) &= \sin(\theta)\,\partial_3 - \cos(\theta)\,\partial_4,\\
S^2(\theta) &= -\cos(\theta)\,\partial_1 - \sin(\theta)\,\partial_2,\\
S^3(\theta) &= -\sin(\theta)\,\partial_1 + \cos(\theta)\,\partial_3,\\
S^4(\theta) &= \partial_5,\\
S^5(\theta) &= \cos(\theta)\partial_3 + \sin(\theta)\partial_4.
\end{aligned}$$

We now modify the standard embedding: Replace the disk $D(\theta)$ by $B(\theta)$, where $B(\theta)$ is obtained by rotating $B \subset \mathbb{R}^4_+(0)$ to $\mathbb{R}^4_+(\theta)$. The embedding of S^3 so obtained is clearly diffeotopic to the standard embedding. We may assume that the framing it induces on $D_a(\theta)$ depends only on θ and equals $S(\theta)$.

8.3. *Constructing immersions with arbitrary Smale invariant*

Let the linear map $L_\alpha\colon \mathbb{R}^5 \to \mathbb{R}^5$ be given by the matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & \mathbf{I}_3 \end{bmatrix}.$$

Let (ξ, η) be coordinates on $B(\theta)$, as in Section 8.1. Let $R_\alpha\colon B(\theta) \to B(\theta)$ be the map that is given by the matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

in the coordinates (ξ, η).

Consider $S^3 = \bigcup_{0\leq\theta\leq 2\pi} B(\theta)$. For every half integer $m \in \frac{1}{2}\mathbb{Z}$ we define $f_m\colon S^3 \to \mathbb{R}^5$,

$$f_m | B(\theta) = L_{m\theta} \circ K \circ R_{-m\theta}.$$

Then f_m is well defined since it agrees with the inclusion on $\partial B(\theta)$ for each θ and since K is symmetric, i.e. $L_{2m\pi} \circ K \circ R_{-2m\pi} = K$.

The self intersection of f_m is one circle. It is traced out by the double point of K under rotation. The preimage of this circle is a torus knot of type $(2m, 2)$ if m is not an integer and a two component link with unknotted components that are linked with linking number m if m is an integer.

PROPOSITION 8.3.1. *The Smale invariant $\Omega(f_m)$ satisfies,*

$$\Omega(f_m) = -2m[\sigma] + N \in \pi_3(V_{5,3}).$$

(See Lemma 3.1.2 for notation.)

Thus, as m runs through $\frac{1}{2}\mathbb{Z}$, f_m runs through the regular homotopy classes of immersions $S^3 \to \mathbb{R}^5$. The proof of Proposition 8.3.1 constitutes Section 9.3.

COROLLARY 8.3.2. *The homomorphism (see Proposition 7.5.2)*

$$\tau\colon \mathbf{Imm} \to \mathbb{Z}_4,$$

is surjective.

Proof. The immersion $f_{\frac{1}{2}}$ represents a generator of **Imm**. It has one self intersection circle with connected preimage. Thus, $\tau(f_{\frac{1}{2}})$ is a generator of $\mathbb{Z}_4$. □

LEMMA 8.3.3. *The invariant* lk *(see Lemma 6.2.1) takes the values*

$$\operatorname{lk}(f_m) = -4m.$$

We prove Lemma 8.3.3 in Section 9.4.

COROLLARY 8.3.4. *The homomorphism (see Corollary 6.2.2)*

$$\lambda\colon \mathbf{Imm} \to \mathbb{Z}_3,$$

is surjective.

Proof. By Lemma 8.3.3, $\lambda(f_{\frac{1}{2}})$ is a generator of $\mathbb{Z}_3$. □

8.4. *An immersion into 4-space*

As in Lashof and Smale [11], Theorem 3.4, we consider an immersion $k\colon S^2 \to \mathbb{R}^4$ with one transversal double point. The normal bundle of this immersion has Euler number 2 and the boundary of a small tubular neighborhood of it is therefore an immersion $h\colon US^2 \cong \mathbb{R}P^3 \to \mathbb{R}^4$, where US^2, denotes the unit tangent bundle of S^2.

Let $\pi\colon S^3 \to \mathbb{R}P^3$ be the universal double covering. Let $f = h \circ \pi\colon S^3 \to \mathbb{R}^4$. Then f is an immersion. Let $g\colon S^3 \to \mathbb{R}^4$ be a generic immersion regularly homotopic to f.

PROPOSITION 8.4.1. *The invariant $\beta(g)$ (see Proposition 7.4.1) is a generator of $\mathbb{Z}_8$, the immersion g has an odd number of quadruple points and the Euler characteristic of F_g is odd.*

Proposition 8.4.1 is proved in Section 9.5

COROLLARY 8.4.2. *The $\mathbb{Z}_2$-valued regular homotopy invariants Q_2 and D_2 defined in Section 6.1 are non-trivial.* □

COROLLARY 8.4.3. *The homomorphism (see Proposition 7.4.2)*

$$\beta\colon \mathbf{Imm} \to \mathbb{Z}_8$$

is an epimorphism. □

9. PROOFS

In this section we prove Theorems 1, 3, 4, and 5, Proposition 8.3.1, Lemma 8.3.3, and Proposition 8.4.1

9.1. ***Proof of Theorem 1***

Proposition 4.1.1 shows that σ is an isomorphism. Proposition 4.2.1 shows that **Emb** is a subgroup of the infinite cyclic group **Imm** of index 24. The Smale invariant Ω takes values in $\pi_3(V_{5,3})$. Choosing a generator, we identify this group with $\mathbb{Z}$. For one of the two possible choices, the first part of the diagram is correct. (Making the other choice, the second homomorphism in the second row would be $\times(-24)$ instead of $\times 24$).

To prove the theorem it is enough to show that $\tau \oplus \beta$ is surjective. Lemma 8.3.3 shows that $\tau(f_{\frac{1}{2}})$ is a generator of $\mathbb{Z}_3$ and if $g\colon S^3 \to \mathbb{R}^4 \subset \mathbb{R}^5$ is as in Proposition 8.4.1 then $\beta(g)$ is a generator of $\mathbb{Z}_8$. Hence, if either $\beta(f_{\frac{1}{2}})$ is a generator of $\mathbb{Z}_8$ or $\tau(g)$ is a generator of $\mathbb{Z}_3$ then $\tau \oplus \beta$ is surjective. Assume that this is not the case. Then $\beta(f_{\frac{1}{2}})$ is even and $\tau(g) = 0$ and hence

$$(\tau(f_{\frac{1}{2}} \star g), \beta(f_{\frac{1}{2}} \star g)) = (\tau(f_{\frac{1}{2}}), \beta(f_{\frac{1}{2}})) + (\tau(g), \beta(g)),$$

is a generator of $\mathbb{Z}_3 \oplus \mathbb{Z}_8$. Thus, $\tau \oplus \beta$ is surjective. (Since $f_{\frac{1}{2}}$ represents a generator of **Imm**, it actually follows that $\beta(f_{\frac{1}{2}})$ is odd.) □

9.2. ***Proofs of Theorems 3, 4, and 5***

Consider the invariants β, τ, D_2 and Q_2, defined in Section 6.1.

Since D_2 and Q_2 are both additive under connected summation (see the proof of Lemma 7.4.2), it follows exactly as for β that they are invariant under regular homotopy in $\mathbb{R}^5$. They both define homomorphisms $\mathbf{Imm} \to \mathbb{Z}_2$ and Lemma 8.4.1 implies that they are both nontrivial and therefore equal.

Proof of Theorem 3: Let $f\colon S^3 \to \mathbb{R}^4$ be a generic immersion. Consider f as an immersion into $\mathbb{R}^5$ and evaluate the $\mathbb{Z}_2$-valued invariants induced by D_2 and Q_2. By the above they are equal. The theorem follows. □

Let $\mu\colon \mathbf{Imm} \to \mathbb{Z}_2$ denote the homomorphism obtained from Q_2 and D_2. We have the following sequence:

$$\mathbf{Imm} \xrightarrow{\beta} \mathbb{Z}_8 \xrightarrow{r_4} \mathbb{Z}_4 \xrightarrow{r_2} \mathbb{Z}_2,$$

where r_2 and r_4 denote reduction modulo 2 and reduction modulo 4, respectively. The Brown invariant of a pin structure on a surface reduced modulo 2 equals the Euler characteristic of the surface reduced modulo 2. Therefore, $r_2 \circ r_4 \circ \beta = \mu$.

Moreover, if $f\colon S^3 \to \mathbb{R}^5$ represents a generator of **Imm** then $r_4(\beta(f))$ is a generator of $\mathbb{Z}_4$ and so is $\tau(f)$.

As a consequence,

$$\beta(h) \bmod 4 = \pm\tau(h) \in \mathbb{Z}_4$$

for any immersion $h\colon S^3 \to \mathbb{R}^5$. The sign in this formula depends on the choices of the isomorphisms ϕ (Section 7.4) and ψ (Section 7.5). For any one of the two possible choices of ϕ there is a unique choice (out of two possible) of ψ such that the sign in the above formula is $+$.

Proof of Theorem 4: Let $f\colon S^3 \to \mathbb{R}^5$ be a generic immersion and $g\colon S^3 \to \mathbb{R}^4$ a generic immersion regularly homotopic to f. Then $\tau(f)$ is odd if and only if $r_4(\beta(g))$ is odd. But $r_4(\beta(g))$ is odd if and only if $\beta(g)$ is odd which is equivalent to F_g having odd Euler characteristic. Apply Theorem 3. □

Proof of Theorem 5: Let f and g be as above. If F_g is orientable then $\beta(g)$ is divisible by 4. Hence, $r_4(\beta) = 0$ and therefore $\tau(f) = 0$. □

9.3. *Proof of Proposition 8.3.1*

For simplicity of notation we let $s\colon S^3 \to \mathbb{R}^5$ denote the *modified* standard embedding and $S\colon S^3 \to SO(5)$ denote the map Θ^s associated to it. Fix m and let $F\colon S^3 \to SO(5)$ denote the map Θ^{f_m} associated to the immersion $f_m\colon S^3 \to \mathbb{R}^5$. Let $\Lambda_m(x) = S^{-1}(x)F(x)$, where $S^{-1}(x)$ is the inverse of the matrix $S(x)$ and $S^{-1}(x)F(x)$ is the matrix product. To compute the Smale invariant of f_m we must compute the homotopy class (see Lemma 3.3.2)

$$[\Theta^{f_m}] - [\Theta^s] = [F] - [S] = [\Lambda_m] \in \pi_3(SO(5)) = \pi_3(V_{5,3}).$$

Since $f_m = s$ outside the solid torus $T = \bigcup_{0\le\theta\le 2\pi} D_a(\theta)$, we have

$$\Lambda_m(x) = S^{-1}(x)F(x) = \begin{bmatrix} \mathbf{I}_3 & \\ & \mathbf{J}(x) \end{bmatrix}, \text{ for } x \in S^3 - T,$$

where $\mathbf{J}(x) \in SO(2)$.

Let (θ, r, φ), $\theta, \varphi \in [0, 2\pi]$, $r \in [0, \pi]$ be coordinates on T. Here θ indicates which $D_a(\theta)$ we are in and on $D_a(\theta)$ we have

$$\xi = \tfrac{ra}{\pi}\cos(\varphi), \quad \eta = \tfrac{ra}{\pi}\sin(\varphi).$$

Using equations (†) in Section 8.1 we can compute the first columns of $F|T$. The result is the following:

$$\begin{aligned}
F^1(\theta, r, \varphi) &= \sin(\theta)\,\partial_3 - \cos(\theta)\,\partial_4,\\
F^2(\theta, r, \varphi) &= -\cos(\theta - m\theta))\,v_1(\theta, r, \varphi) - \sin(\theta - m\theta)\,v_2(\theta, r, \varphi)\\
F^3(\theta, r, \varphi) &= -\sin(\theta - m\theta)\,v_1(\theta, r, \varphi) + \cos(\theta - m\theta)\,v_2(\theta, r, \varphi),
\end{aligned}$$

where

$$\begin{aligned}
v_1(\theta, r, \phi) = &-\cos(r)\big[\cos(m\theta)\,\partial_1 + \sin(m\theta)\,\partial_2\big]\\
&+ \sin(r)\big[\cos(\varphi - m\theta)\,(\cos(\theta)\,\partial_3 + \sin(\theta)\,\partial_4) - \sin(\varphi - m\theta)\,\partial_5\big],\\
v_2(\theta, r, \varphi) = &-\sin(m\theta)\,\partial_1 + \cos(m\theta)\,\partial_2.
\end{aligned}$$

We note that $F^1(x) = S^1(x)$, for all $x \in S^3$. Hence,

$$\Lambda_m(x) = \begin{bmatrix} 1 & \\ & \mathbf{M}(x) \end{bmatrix}, \quad \text{for all } x \in S^3,$$

where $\mathbf{M}(x) \in SO(4)$. Thus, to compute $[\Lambda_m] \in \pi_3(SO(5))$ it is enough to compute the homotopy class of $\mathbf{M}\colon S^3 \to SO(4)$ which, as we shall see, can be calculated if we know the first two columns of $\mathbf{M}(x)$.

If $\mathbf{M} = (m_i^j)$ then $m_j^i = \langle S^{i-1}, F^{j-1}\rangle$, $i, j = 2, 3, 4, 5$, where $\langle,\rangle$ is the usual inner product on $\mathbb{R}^5$. Hence, we have all the information we need to determine $[\Lambda_m]$. In what follows, we first study how to calculate the homotopy class of $\mathbf{M}$ and then carry out the necessary calculations.

In terms of our standard generators of $\pi_3(SO(4))$ we have, for $\mathbf{M}\colon S^3 \to SO(4)$, $[\mathbf{M}] = u[\sigma] + v[\rho] \in \pi_3(SO(4))$, $u, v \in \mathbb{Z}$.

The integer u is simply the degree of the map $p \circ \mathbf{M} = \mathbf{M}^1\colon S^3 \to S^3$, where $SO(4) \xrightarrow{p} S^3$ is the fibration described in Section 3.1 and $\mathbf{M}^1(x)$ is the first column vector of $\mathbf{M}(x)$.

To compute $v[\rho]$, recall from Section 3.1 that $SO(4) \cong S^3 \times SO(3)$ and see that we must compare the homotopy class of the map $p_2 \circ \mathbf{M}\colon S^3 \to SO(3)$, where

$p_2\colon SO(4) \to SO(3)$ is the projection onto the second factor in the product space $SO(4) \cong S^3 \times SO(3)$, to that of $p_2\rho\colon S^3 \to SO(3)$.

The product structure on $SO(4)$ is obtained by using the section $\sigma\colon S^3 \to SO(4)$. The projection $p_2\colon SO(4) \to SO(3)$ is thus determined by the equation

$$\sigma(p(x))^{-1}x = \begin{bmatrix} 1 & \\ & p_2(x) \end{bmatrix}, \quad x \in SO(4).$$

We note that $p_2 \circ \rho = \varrho\colon S^3 \to SO(3)$. Let $\mathbf{N} = p_2 \circ \mathbf{M}$. To compare the homotopy classes of these maps we can use the fibration $SO(3) \xrightarrow{p} S^2$: Since $p_*\colon \pi_3(SO(3)) \to \pi_3(S^2)$ is an isomorphism we might as well compare the homotopy classes of the maps $p \circ \varrho$ and $p \circ \mathbf{N}$ both mapping S^3 to S^2. By the Pontryagin construction the homotopy class of any map $S^3 \to S^2$ is determined by its Hopf invariant (the linking number of two regular fibers).

The map $\varrho\colon S^3 \to SO(3)$, has Hopf invariant one if endow S^3 with the orientation that is coherent with the quaternion framing.

Thus, to determine $[\mathbf{N}] \in \pi_3(SO(3))$ we need only calculate the Hopf invariant of $p_2 \circ \mathbf{N} = \mathbf{N}^1\colon S^3 \to S^2$, where $\mathbf{N}^1(x)$ is the first column vector in the matrix $\mathbf{N}(x)$. This, in turn, means that the integer v can be computed as the Hopf invariant of the map $\mathbf{N}^1\colon S^3 \to S^2$, where S^3 has the same orientation as above.

We now turn to the actual calculations. We have $\mathbf{M}^k = \sum_{i=1}^4 \langle S^{i+1}, F^{k+1}\rangle\, \partial_i$, $k = 1, 2$ where ∂_i, $i = 1, \dots, 4$ is the standard basis in $\mathbb{R}^4$ and $\langle\,,\,\rangle$ is the standard inner product on $\mathbb{R}^5$. Computing the necessary inner products we find that if $x \in T \subset S^3$ then

$$\mathbf{M}^1(x) = \mathbf{M}^1(\theta, r, \varphi) = \begin{bmatrix} -\cos^2(m\theta-\theta)\cos(r) + \sin^2(m\theta-\theta) \\ \frac{1}{2}(\cos(r)+1)\sin(2m\theta-2\theta) \\ \cos(m\theta-\theta)\sin(r)\sin(\varphi-m\theta) \\ -\cos(m\theta-\theta)\sin(r)\cos(\varphi-m\theta) \end{bmatrix},$$

$$\mathbf{M}^2(x) = \mathbf{M}^2(\theta, r, \varphi) = \begin{bmatrix} \frac{1}{2}(\cos(r)+1)\sin(2m\theta-2\theta) \\ -\sin^2(m\theta-\theta)\cos(r) + \cos^2(m\theta-\theta) \\ -\sin(m\theta-\theta)\sin(r)\sin(\varphi-m\theta) \\ \sin(m\theta-\theta)\sin(r)\cos(\varphi-m\theta) \end{bmatrix}$$

and if $x \in S^3 - T$ then

$$\mathbf{M}^1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{M}^2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

To find the first column of $\mathbf{N}$ we must calculate $\sigma(p(\mathbf{M}))^{-1}\mathbf{M}^2$. For $y \in \mathbb{R}^4$, $\sigma(p(\mathbf{M}))y = \mathbf{M}^1 \cdot y$, (for notation see Section 3.1). For $q \in \mathbb{H}$, let $\overline{q}$ denote its conjugate then $\sigma(p(\mathbf{M}))^{-1}y = \overline{\mathbf{M}^1} \cdot y$. Multiplying gives

$$\overline{\mathbf{M}^1(x)} \cdot \mathbf{M}^2(x) = \begin{bmatrix} 0 \\ -\cos(r) \\ \sin(r)\cos(\varphi-\theta) \\ \sin(r)\sin(\varphi-\theta) \end{bmatrix} \text{ if } x \in T.$$

Hence, the the first column of $\mathbf{N}$ is given by

$$\mathbf{N}^1(x) = \begin{bmatrix} -\cos(r) \\ \sin(r)\cos(\varphi-\theta) \\ \sin(r)\sin(\varphi-\theta) \end{bmatrix} \text{ if } x \in T \text{ and } \mathbf{N}^1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ if } x \in S^3 - T.$$

To determine the degree of $\mathbf{M}^1$ we consider preimages of the regular value $(0,0,1,0) \in S^3$. For $x \in S^3 - T$, $\mathbf{M}^1(x) \neq (0,0,1,0)$ and for $x \in T$ we get the equations

$$\begin{aligned} r &= \frac{\pi}{2}, \\ \theta &= \frac{l\pi}{m-1}, \\ \varphi &= \frac{ml\pi}{m-1} + (-1)^l\,\frac{\pi}{2}, \end{aligned}$$

where $l = 0, 1, \ldots, |2m-3|$. Thus, there are $|2m-2|$ points in the preimage of $(0,0,1,0)$.

The orientation of T induced from the quaternion framing on S^3 is the same as that given by the basis $(\partial_\theta, \partial_r, \partial_\varphi)$. Thus, the orientation of S^3 at $(0,0,1,0)$ induced by $\mathbf{M}^1$ is $o((m-1)\,\partial_2, \partial_1, \partial_4)$. The orientation of S^3 at $(0,0,1,0)$ induced from the standard orientation on $\mathbb{R}^4$ is $o(\partial_1, \partial_2, \partial_4)$. So the sign of each point in the preimage is the same. It is positive if $m < 1$ and negative if $m > 1$. Thus, $\deg(\mathbf{M}^1) = u = -(2m-2) = -2m+2$.

To compute the Hopf invariant of $\mathbf{N}^1 : S^3 \to S^2$ we consider the preimage of the regular value $(0,0,1)$ and see that it is the curve c defined by the equations

$$r = \frac{\pi}{2}, \qquad \phi - \theta = \frac{\pi}{2}.$$

Moreover, the vector field $-\frac{\partial}{\partial r}$ along c, is mapped by $d\mathbf{N}$ to the vector $(1,0,0)$. Shifting the curve c along this vector field we get a curve c' and $\mathrm{lk}(c,c') = -1$ using the above orientation on T. So, the Hopf invariant of $\mathbf{N}^1$ is -1 and thus $v = -1$.

Collecting these results, we finally get

$$\begin{aligned} \Omega(f_m) = [\Lambda_m] = [\mathbf{M}]N &= ((-2m+2)[\sigma] - [\rho])N \\ &= -2m[\sigma]N \in \pi_3(SO(5)) = \pi_3(V_{5,3}), \end{aligned}$$

as claimed. □

9.4. *Proof of Lemma 8.3.3*

Let (r, θ, φ) be coordinates on the solid torus where the Whitney kinks are placed. The preimage of the self intersection c_m of f_m is $r = \epsilon$, $\varphi = m\theta$ or $\varphi = m\theta + \pi$, where ϵ is some small positive number. Consider the vector field ∂_r along this preimage. If we shift c_m along

$$v(p) = (df_m(\partial_r(p_1)) + df_m(\partial_r(p_2)),$$

where $f_m^{-1}(p) = \{p_1, p_2\}$ then we obtain a curve c_m'', which bounds a 2-disk in the x_3x_4-plane without intersections with $f_m(S^3)$. Thus, $\mathrm{lk}(c_m'', f_m(S^3)) = 0$.

Assume that m is not an integer. Then $c_S = f_m^{-1}(c_m)$ is connected. Shifting c_S along ∂_r gives a curve c_S'' with $\mathrm{lk}(c_S, c_S'') = 4m$. Thus, the framing of c_S which satisfies condition (†) in Section 6.2 is the one that differs from ∂_r by $-4m$ rotations.

Assume that m is an integer. Then $c_S = c_1 \cup c_2$ have two components and $\mathrm{lk}(c_1, c_2) = m$. Shifting c_i along ∂_r gives a curve c_i'' with $\mathrm{lk}(c_i, c_i'') = m$. Thus, the framings of the c_i that satisfy condition (†) in Section 6.2 differ from ∂_r by $-2m$ rotations each.

The rotations that we need to add to ∂_r can be made locally supported. Using a local model it is easy to see how they affect the linking number: Let X and Y be the 3-planes $X = (x_1, x_2, x_3, 0, 0)$ and $Y = (y_1, 0, 0, y_2, y_3)$. Then $X \cap Y$ is the line $l = (t, 0, 0, 0, 0)$. Shifting l along $\epsilon(\partial_2 + \partial_4)$ we get $l' = (t, \epsilon, 0, \epsilon, 0)$. Shifting l along $\epsilon(-\cos(t)\partial_2 + \sin(t)\partial_3 + \partial_4)$, $-\pi \le t \le \pi$ we get the curve $l'' = (t, -\epsilon\cos(t), \epsilon\sin(t), 0, \epsilon, 0)$. Let $D = (1-s)l' + sl''$, $0 \le s \le 1$. Then D is a 2-chain with boundary $l'' - l'$, $D \cap X = \emptyset$ and $D \cap Y = (0, 0, 0, \epsilon, 0)$ with sign -1. Thus, for each rotation in the negative direction we decrease the linking number by 1.

Adding the appropriate number of rotations to the framing ∂_r and shifting the curve c_m along the corresponding vector field we obtain a curve c_m' with

$$\mathrm{lk}(c_m', f_m(S^3)) = \mathrm{lk}(c_m'', f_m(S^3)) - 4m = -4m.$$

Thus, $\mathrm{lk}(f_m) = -4m$. $\square$

9.5. ***Proof of Proposition 8.4.1***

The self intersection of h is located close to the transverse double point $h(p) = h(q) = 0$ of S^2. Let x and y be coordinates on small disks D_1 and D_2 around p and q respectively. We may assume that

$$k(x) = (x_1, x_2, 0, 0) \quad \text{and} \quad k(y) = (0, 0, y_1, y_2).$$

Close to p and q, US^2 is of the form $D_i^2 \times S^1$, $i = 1, 2$ and the immersion h is:

$$h(x, \theta) = (x_1, x_2, \epsilon\cos(\theta), \epsilon\sin(\theta)), \qquad h(y, \varphi) = (\epsilon\cos(\varphi), \epsilon\sin(\varphi), y_1, y_2),$$

where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$ are coordinates on S^1 in the fiber close to p and q, respectively.

Thus, $M_h = T^2$ and the preimage $\widetilde{M}_h$ is $c_i \times S^1 \subset D_i \times S^1$, where c_i is a circle of radius ϵ around $0 \in D_i$.

We can find $\mathbb{R}P^2 \subset US^2$ as follows: Fix a point s in S^2 and a unit tangent vector v at that point. The space of 2-planes Π through 0 in $\mathbb{R}^3$ is $\mathbb{R}P^2$. Let r_Π be the reflection in Π. The map $\Pi \mapsto (r_\Pi(s), r_\Pi(v))$ is an embedding of $\mathbb{R}P^2$ into US^2.

Consider the double cover $p\colon S^3 \to US^2$. Observe that $K = p^{-1}(\mathbb{R}P^2) \cong S^2$ subdivides S^3 into two hemispheres D^3_+ and D^3_-.

The intersection $\widetilde{M}_h \cap \mathbb{R}P^2$ consists of two circles, isotopic to $c_i \times 1 \subset D_i \times S^1$, $i = 1, 2$. Altering the embedding of $\mathbb{R}P^2$ slightly we may assume that $\widetilde{M}_h \cap \mathbb{R}P^2 = c_1 \times 1 \cup c_2 \times 1$.

The covering $p\colon S^3 \to US^2$ is nontrivial on fibers so $p^{-1}(\widetilde{M}_h)$ consists of two tori $T_i \subset S^3$, $i = 1, 2$ and $p|T_i\colon T_i \to c_i \times S^1$ is a double cover. Thus, $K \cap T_i$ consists of two meridians in T_i, $i = 1, 2$.

Choose a normal vector field ν to h. Let $\delta > 0$ be small and let $\phi\colon S^3 \to (-\delta, \delta)$ be positive on D_+, negative on D_-, 0 on K, and have nonvanishing derivative in the normal direction of K. (Map (S^3, K) to $(S^3, \{x_4 = 0\}) \subset \mathbb{R}^4$ and use a suitably scaled height function.) We use ϕ and ν to perturb f. For $x \in S^3$, let

$$g(x) = f(x) + \phi(x)\nu(x).$$

Then g is an immersion. Clearly, g is an embedding outside a small neighborhood of $T_1 \cup T_2 \cup K$ in S^3. Moreover, inside this neighborhood it has only transverse double points if we stay away from $T_1 \cup T_2$. To determine the self intersection of g we can therefore restrict attention to neighborhoods of T_1 and T_2. Let $E_i \times S^1 = p^{-1}(D_i \times S^1)$. If $(x_1, x_2, e^{i\alpha})$, $(y_1, y_2, e^{i\omega})$ are coordinates on $E_1 \times S^1$ and $E_2 \times S^1$, respectively then $p\colon E_i \times S^1 \to D_i \times S^1$ is given by

$$\begin{aligned} p(x_1, x_2, e^{i\alpha}) &= (x_1, x_2, e^{i2\alpha}), \\ p(y_1, y_2, e^{i\omega}) &= (y_1, y_2, e^{i2\omega}). \end{aligned}$$

Normal fields along $h(D_1 \times S^1)$ and $h(D_2 \times S^1)$ are

$$\begin{aligned} \nu_1(x_1, x_2, e^{i\theta}) &= \cos(\theta)\,\partial_3 + \sin(\theta)\,\partial_4, \\ \nu_1(y_1, y_2, e^{i\varphi}) &= \cos(\theta)\,\partial_1 + \sin(\theta)\,\partial_2. \end{aligned}$$

We may assume that the function $\phi|E_i \times S^1$ is given by

$$\phi(x_1, x_2, e^{i\alpha}) = \delta\sin(\alpha) \quad \text{and} \quad \phi(y_1, y_2, e^{i\omega}) = \delta\sin(\omega).$$

Then, in $E_i \times S^1$, $g = f + \phi\nu$ is given by

$$\begin{aligned} g(x_1, x_2, e^{i\alpha}) &= (x_1, x_2, (\epsilon + \delta\sin(\alpha))\cos(2\alpha), (\epsilon + \delta\sin(\alpha))\sin(2\alpha))\,, \\ g(y_1, y_2, e^{i\omega}) &= ((\epsilon + \delta\sin(\omega))\cos(2\omega), (\epsilon + \delta\sin(\omega))\sin(2\omega), y_1, y_2)\,. \end{aligned}$$

Thus, g has one quadruple point at $(\epsilon, 0, \epsilon, 0)$, with preimages

$$(x_1, x_2, e^{i\alpha}) = (\epsilon, 0, \pm 1) = (y_1, y_2, e^{i\omega}),$$

and g has four triple lines:

$$\begin{aligned} \alpha &\mapsto (\epsilon, 0, (\epsilon + \delta\sin(\alpha))\cos(2\alpha), (\epsilon + \delta\sin(\alpha))\sin(2\alpha))\,, \\ \omega &\mapsto ((\epsilon + \delta\sin(\omega))\cos(2\omega), (\epsilon + \delta\sin(\omega))\sin(2\omega), \epsilon, 0)\,, \end{aligned}$$

where $\alpha, \omega \in (0, \pi)$ or $\alpha, \omega \in (\pi, 2\pi)$.

Constructing F_g by gluing the nine double point sheets of g together gives $F_g = T^2 \cup \mathbb{R}P^2$ as shown in Figure 7, where the six intersection points of the solid lines represents F_g^0 and the remaining twelve open arcs of the solid lines represents F_g^1. It follows that $\beta(g)$ as well as $\chi(F_g)$ are odd. Since β and D_2 are invariants of

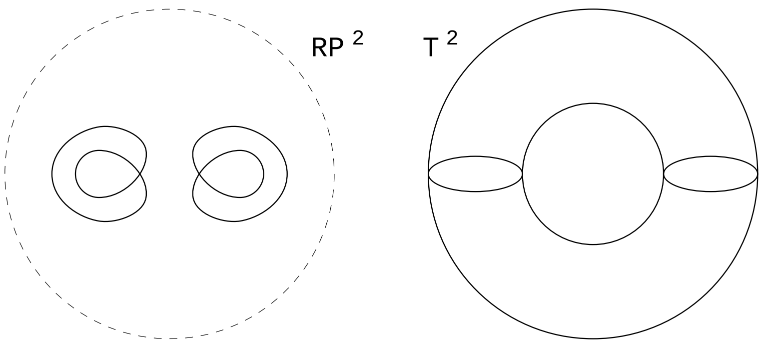

FIGURE 7. The self intersection surface F_g

regular homotopy in $\mathbb{R}^5$, the proposition follows. □

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