INVARIANTS OF GENERIC IMMERSIONS AND BERNOULLI NUMBERS

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Abstract. First order invariants of generic immersions of manifolds of dimension $nm-1$ into manifolds of dimension $n(m+1)-1$, $m, n > 1$ are constructed using the geometry of self-intersections. The range of one of these invariants is related to Bernoulli numbers.

As by-products some geometrically defined invariants of regular homotopy are found.

1. Introduction

An immersion of a smooth manifold M into a smooth manifold W is a smooth map with everywhere injective differential. Two immersions are regularly homotopic if they can be connected by a continuous 1-parameter family of immersions.

An immersion is generic if all its self-intersections are transversal. In the space $\mathcal{F}$ of immersions $M \to W$, generic immersions form an open dense subspace. Its complement is *the discriminant hypersurface*, $\Sigma \subset \mathcal{F}$. Two generic immersions belong to the same path component of $\mathcal{F} - \Sigma$ if they can be connected by a regular homotopy, which at each instance is a generic immersion. We shall consider the classification of generic immersions up to regular homotopy through generic immersions. It is similar to the classification of embeddings up to diffeotopy (knot theory). In both cases, all topological properties of equivalent maps are the same.

An *invariant of generic immersions* is a function on $\mathcal{F} - \Sigma$ which is locally constant. The value of such a function along a path in $\mathcal{F}$ jumps at intersections with Σ. Invariants may be classified according to the complexity of their jumps. The most basic invariants in this classification are called *first order invariants* (see Section 2).

In [1], Arnold studies generic regular plane curves (i.e. generic immersions $S^1 \to \mathbb{R}^2$). He finds three first order invariants J^+, J^-, and St. In [3], the author considers the case $S^3 \to \mathbb{R}^5$. Two first order invariants J and L are found. In both these cases, the only self-intersections of generic immersions are transversal double points and in generic 1-parameter families there appear isolated instances of self-tangencies and triple points. The values of the invariants $J^\pm$ and J change at instances of self-tangency and remain constant at instances of triple points. The invariants St and L change at triple points and remain constant at self-tangencies.

In this paper we shall consider high-dimensional analogs of these invariants. The most straightforward generalizations arise for immersions $M^{2n-1} \to W^{3n-1}$, where self-tangencies and triple points are the only degeneracies in generic 1-parameter families. The above range of dimensions is included in a 2-parameter family, $M^{nm-1} \to W^{n(m+1)-1}$, $m, n > 1$, where generic immersions do not have k-fold self-intersection points if $k > m$ and in generic 1-parameter families there appear isolated instances of $(m+1)$-fold self-intersection. Under these circumstances, we find first order invariants of generic immersions. (For precise statements, see Theorem 1 and Theorem 3 in Section 2, where the main results of this paper are formulated.)

In particular, if n is even and M^{nm-1} is orientable and satisfies a certain homology condition (see Theorem 1 (b)) then there exists an integer-valued invariant L of generic immersions $M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ which is an analog of Arnold's St: It changes under instances of $(m+1)$-fold self-intersection, does not change under other degeneracies which appear in generic 1-parameter families, and is additive under connected sum. The value of L at a generic immersion f is the linking number of a copy of the set of m-fold self-intersection points of f, shifted in a special way, with $f(M)$ in $\mathbb{R}^{n(m+1)-1}$. (See Definition 4.5.)

For immersions of odd-dimensional spheres in codimension two there appear restrictions on the possible values of L (see Theorem 2). This phenomenon is especially interesting in the case of immersions $S^{4j-1} \to \mathbb{R}^{4j+1}$. Using a result of Hughes and Melvin [6] we prove that the range of L is related to Bernoulli numbers: For any generic immersion $f\colon S^{4j-1} \to \mathbb{R}^{4j+1}$, $L(f)$ is divisible by p^r, where p is a prime and r is an integer such that p^{r+k} divides $2j+1$ and p^{k+1} does not divide μ_j for some integer k. Here μ_j is the denominator of $\frac{B_j}{4j}$, where B_j is the j^{th} Bernoulli number.

In general, it is not known if the above result gives all restrictions on the range of L. However, in special cases it does. For example, for immersions $S^3 \to \mathbb{R}^5$ the range of L is $\mathbb{Z}$ (see [3] and also Remark 9.3) and for immersions $S^{67} \to \mathbb{R}^{69}$ it is $35\mathbb{Z}$ (see Section 9.3).

In [4], the author gives a complete classification of generic immersions $M^k \to \mathbb{R}^{2k-r}$, $r = 0, 1, 2$, $k \geq 2r+4$ up to regular homotopy through generic immersions (under some conditions on the lower homotopy groups of M). Here, the class of a generic immersion is determined by its self-intersection with induced natural additional structures (e.g. spin structures). The existence of invariants such as L mentioned above implies that the corresponding classification in other dimensions is more involved (see Remark 9.2).

The invariants of generic immersions in Theorems 1 and 3 give rise to invariants of regular homotopy which take values in finite cyclic groups (Section 7). We construct examples showing that, depending on the source and target manifolds, the regular homotopy invariants may or may not be trivial (Section 9). It would be interesting to relate these invariants to invariants arising from the cobordism theory of immersions (see for example Eccles [2]).

2. Statements of the main results

Before stating the main results we define the notions of invariants of orders zero and one. They are similar to knot invariants of finite type introduced by Vassiliev in [10].

As in the Introduction, let $\mathcal{F}$ denote the space of immersions and let $\Sigma \subset \mathcal{F}$ denote the discriminant hypersurface. The set $\Sigma^1 \subset \Sigma$ of non-generic immersions which appear at isolated instances in generic 1-parameter families (see Lemmas 3.3 and 3.4 for descriptions of such immersions $M^{nm-1} \to W^{n(m+1)-1}$) is a smooth submanifold of $\mathcal{F}$ of codimension one. ($\mathcal{F}$ is an open subspace of the space of smooth maps, thus an infinite dimensional manifold and the notion of codimension in $\mathcal{F}$ makes sense.)

If $f_0 \in \Sigma^1$ then there is a neighborhood $U(f_0)$ of f_0 in $\mathcal{F}$ cut in two parts by Σ^1. A coherent choice of a positive and negative part of $U(f_0)$ for each $f_0 \in \Sigma^1$ is a *coorientation* of Σ. (The discriminant hypersurface in the space of immersions $M^{nm-1} \to W^{n(m+1)-1}$ will be cooriented in Section 6.1.)

Let a be an invariant of generic immersions and let $f_0 \in \Sigma^1$. Define the jump ∇a of a as

$$\nabla a(f_0) = a(f_+) - a(f_-),$$

where f_+ and f_- are generic immersions in the positive respectively negative part of $U(f_0)$. Then ∇a is a locally constant function on Σ^1.

An invariant a of generic immersions is a *zero order invariant* if $\nabla a \equiv 0$.

Let Σ^2 be the set of all immersions which appear in generic 2-parameter families but can be avoided in generic 1-parameter families (see Lemma 3.5 for the case $M^{nm-1} \to W^{n(m+1)-1}$). Then $\Sigma^2 \subset \Sigma$ is a smooth codimension two submanifold of $\mathcal{F}$.

An invariant a of generic immersions is a *first order invariant* if $\nabla a(f_0) = \nabla a(f_1)$ for any immersions $f_0, f_1 \in \Sigma^1$ which can be joined by a path in $\Sigma^1 \cup \Sigma^2$ such that at intersections with Σ^2 its tangent vector is transversal to the tangent space of Σ^2.

Theorem 1. *Let $m > 1$ and $n > 1$ be integers and let M^{nm-1} be a closed manifold.*

(a) *If m is even and $H_{n-1}(M;\mathbb{Z}_2) = 0 = H_n(M;\mathbb{Z}_2)$ then there exists a unique (up to addition of zero order invariants) first order $\mathbb{Z}_2$-valued invariant Λ of generic immersions $M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ satisfying the following conditions: It jumps by 1 on the part of Σ^1 which consists of immersions with one $(m+1)$-fold self-intersection point and does not jump on other parts of Σ^1.*

(b) *If n is even, M is orientable, and $H_{n-1}(M;\mathbb{Z}) = 0 = H_n(M;\mathbb{Z})$ then there exists a unique (up to addition of zero order invariants) first order integer-valued invariant L of generic immersions $M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ satisfying the following conditions: It jumps by $m+1$ on the part of Σ^1 which consists of immersions with one $(m+1)$-fold self-intersection point and does not jump on other parts of Σ^1.*

Theorem 1 is proved in Section 6.4. The invariants Λ and L are defined in Definition 4.4 and Definition 4.5, respectively. If appropriately normalized, Λ and L are additive under connected summation of generic immersions (see Section 5.1).

Theorem 2. *Let $f\colon S^{2m-1} \to \mathbb{R}^{2m+1}$ be a generic immersion.*

(a) *If $m = 4j+1$ then $2j+1$ divides $L(f)$.*
(b) *If $m = 4j+3$ then $4j+4$ divides $L(f)$.*
(c) *If $m = 2j$ then p^r divides $L(f)$ for every prime p and integer r such that p^{r+k} divides $2j+1$ and p^{k+1} does not divide μ_j for some integer k. Here μ_j is the denominator of $\frac{B_j}{4j}$, where B_j is the j^{th} Bernoulli number.*

We prove Theorem 2 in Section 8.2.

Theorem 3. *Let $m > 1$ and $n > 1$ be integers and let M^{nm-1} be a closed manifold and let $W^{n(m+1)-1}$ be a manifold.*

(a) *If n is odd then, for each integer $2 \le r \le m$ such that $m-r$ is even, there exist a unique (up to addition of zero order invariants) first order integer-valued invariant J_r, of generic immersions $M^{nm-1} \to W^{n(m+1)-1}$ satisfying the following conditions: The invariant jumps by 2 on the part of Σ^1 which* *consists of immersions with one degenerate r-fold self-intersection point and does not jump on other parts of Σ^1.*

(b) *If $n = 2$ then there exists a unique (up to addition of zero order invariants) first order integer-valued invariant J, of generic immersions $M^{2m-1} \to W^{2m+1}$ satisfying the following conditions: The invariant jumps by 1 on the part of Σ^1 which consists of immersions with one degenerate m-fold self-intersection point and does not jump on other parts of Σ^1.*

Theorem 3 is proved in Section 6.3. If appropriately normalized, J_r and J are additive under connected summation of generic immersions (see Section 5.1). The value of J_r on a generic immersion is the Euler characteristic of its resolved r-fold self-intersection manifold, the value of J is the number of components in its m-fold self-intersection (which is a closed 1-dimensional manifold).

3. Generic immersions and generic regular deformations

In this section we define generic immersions and describe the immersions corresponding to non-generic instances in generic 1- and 2-parameter families. We also describe the versal deformations of these non-generic immersions.

3.1. Generic immersions.

Let M and W be smooth manifolds and let $f\colon M \to W$ be an immersion. A point $q \in W$ is a *k-fold self-intersection point of f* if $f^{-1}(q)$ consists of exactly k points. Let $\Gamma_k(f) \subset W$ denote the set of k-fold self-intersection points of f and let $\widetilde{\Gamma}_k(f) = f^{-1}(\Gamma_k(f)) \subset M$ denote its preimage. Note that $f|\widetilde{\Gamma}_k(f) \to \Gamma_k(f)$ is a k-fold covering.

Definition 3.1. *Let $m > 1$ and $n > 1$. An immersion $f\colon M^{nm-1} \to W^{n(m+1)-1}$ is* generic *if*

g1 *$\Gamma_k(f)$ is empty for $k > m$, and*
g2 *if $q = f(p_1) = \dots = f(p_r) \in \Gamma_r(f)$ ($r \le m$) then for any i, $1 \le i \le r$*

$$df\,T_{p_i}M + \cap_{s \neq i} df\,T_{p_s}M = T_qW.$$

A standard application of the jet-transversality theorem shows that the set of generic immersions is open and dense in the space of all immersions $\mathcal{F}$.

We remark that the set of k-fold self-intersection points $\Gamma_k(f)$ of a generic immersion $f\colon M^{nm-1} \to W^{n(m+1)-1}$ is a smooth submanifold of W of dimension $n(m-k+1)-1$. Moreover, the deepest self-intersection $\Gamma_m(f)$ is a closed manifold.

Lemma 3.1 below gives a local coordinate description of a generic immersion close to a k-fold self-intersection point. To state it we introduce some notation: Let $x_i \in \mathbb{R}^{nm-1}$, we write $x_i = (x_i^0, x_i^1, \dots, x_i^{k-1})$, where $x_i^0 \in \mathbb{R}^{n(m-k+1)-1}$ and $x_i^r \in \mathbb{R}^n$, for $1 \le r \le k-1$. Similarly, we write $y \in \mathbb{R}^{n(m+1)-1}$ as $y = (y^0, y^1, \dots, y^k)$, where $y^0 \in \mathbb{R}^{n(m-k+1)-1}$ and $y^r \in \mathbb{R}^n$, for $1 \le r \le k$.

Lemma 3.1. *Let $f\colon M^{nm-1} \to W^{n(m+1)-1}$ be a generic immersion and let $q = f(p_1) = \dots = f(p_k)$ be a k-fold self intersection point of f. Then there are coordinates y in $V \subset W^{n(m+1)-1}$ centered at q and coordinates x_i on $U_i \subset M^{nm-1}$*

centered at p_i, $1 \le i \le k$ such that f is given by

$$\begin{aligned} f(x_1) &= (x_1^0, x_1^1, \dots, x_1^{k-2}, x_1^{k-1}, 0),\\ f(x_2) &= (x_2^0, x_2^1, \dots, x_2^{k-2}, 0, x_1^{k-1}),\\ &\vdots\\ f(x_k) &= (x_k^0, 0, x_k^1, x_k^2, \dots, x_k^{k-1}). \end{aligned}$$

(That is, if $y = f(x_i)$ then $y^0(x_i) = x_i^0$, $y^r(x_i) = x_i^r$ for $1 \le r \le k-i$, $y^{k-i+1}(x_i) = 0$, and $y^r(x_i) = x_i^{r-1}$ for $k-i+2 \le r \le k$.)

Proof. The proof is straightforward. □

When the source and target of a generic immersion are oriented and the codimension is even then there are induced orientations on the self intersection manifolds:

Proposition 3.1. *Let $n = 2j > 1$, $m > 1$, $2 \le k \le m$, and let $f\colon M^{2jm-1} \to W^{2j(m+1)-1}$ be a generic immersion. Orientations on M^{2jm-1} and $W^{2j(m+1)-1}$ induce an orientation on $\Gamma_k(f)$.*

Proof. Let N denote the normal bundle of the immersion. The decomposition

$$f^*TW^{2j(m+1)-1} = TM^{2jm-1} \oplus N$$

induces an orientation on N. If $q \in \Gamma_k(f)$, $q = f(p_1) = \dots = f(p_k)$ then

$$T_q W^{2j(m+1)-1} = T_q\Gamma_k(f) \oplus_{i=1}^k N_{p_i}.$$

The orientation on N induces an orientation on $\oplus_{i=1}^k N_{p_i}$. Since the dimension of the bundle N is $2j$ which is even, the orientation on the sum is independent on the ordering of the summands. Hence, the decomposition above induces a well-defined orientation on $T\Gamma_k(f)$. □

3.2. **The codimension one part of the discriminant hypersurface.** Our next result describes (the 1-jets of) immersions in Σ^1 (see Section 2).

Lemma 3.2. *If $f_0\colon M^{nm-1} \to W^{n(m+1)-1}$ is an immersion in Σ^1 then* **g1** *and* **g2** *of Definition 3.1 holds, except at one k-fold $(2 \le k \le m+1)$ self-intersection point $q = f_0(p_1) = \dots = f_0(p_k)$, where,*

(a) *if $k = 2$,*

$$\dim(df_0\, T_{p_1}M + df_0\, T_{p_2}M) = n(m+1) - 2,$$

or

(b) *if $2 < k \le m+1$, for $i \ne l$*

$$\dim(df_0\, T_{p_i}M + \cap_{r\ne i, r\ne l} df_0\, T_{p_r}M) = n(m+1) - 1$$

and

$$\dim(df_0\, T_{p_i}M + \cap_{r\ne i} df_0\, T_{p_r}M) = n(m+1) - 2.$$

Proof. We have to show that degeneracies as above appears at isolated parameter values in generic 1-parameter families, and that further degeneracies (of the 1-jet) may be avoided . This follows from the jet-transversality theorem applied to maps $M^{nm-1} \times [-\delta, \delta] \to W^{n(m+1)-1}$. □

A k-fold self-intersection point q of an immersion $f_0\colon M^{nm-1} \to W^{n(m+1)-1}$ where **g1** and **g2** of Definition 3.1 does not hold will be called a *degenerate self-intersection point of* f_0. If $2 \leq k \leq m$ we say that q is a *k-fold self-tangency point of* f_0 if $k = m+1$ we say that q is a *$(m+1)$-fold self-intersection point of* f_0.

Recall that a deformation F of a map f is called *versal* if any deformation of f is equivalent (up to left-right action of diffeomorphisms) to one induced from F.

Let f_0 be an immersion in Σ^1. Then its versal deformation f_t is a 1-parameter deformation. In other words, it is a path $\lambda(t) = f_t$ in $\mathcal{F}$ which intersects Σ^1 transversally at f_0 and thus, f_t are generic immersions for small $t \neq 0$.

Next, we shall describe immersions $f_0 \in \Sigma^1$ in local coordinates close to their degenerate self intersection point. Self-tangency points and $(m+1)$-fold self-intersection points are treated in Lemma 3.3 and Lemma 3.4, respectively.

To accomplish this we need a more detailed description of coordinates than that given in Lemma 3.1, we use coordinates as there with one more ingredient: We write (when necessary) $x_i^r \in \mathbb{R}^n$ and $y_i^r \in \mathbb{R}^n$, $r > 1$ as $x_i^r = (\xi_i^r, u_i^r)$ and $y^r = (\eta^r, v^r)$, where $\xi_i^r, \eta^r \in \mathbb{R}$ and $u_i^r, v_i^r \in \mathbb{R}^{n-1}$. (Greek letters for scalars and Roman for vectors).

Lemma 3.3. *Let $f_0\colon M^{nm-1} \to W^{n(m+1)-1}$ be an immersion in Σ^1 and let $q = f_0(p_1) = \cdots = f_0(p_k)$ be a point of degenerate k-fold self intersection $2 \leq k \leq m$. Then there are coordinates y on $V \subset W$ centered at q and coordinates x_i on $U_i \subset M$ centered at p_i, $1 \leq i \leq k$ such that in these coordinates the versal deformation f_t, $-\delta < t < \delta$ of f_0 is constant outside of $\cup_i U_i$ and in neighborhoods of $p_i \in U_i$ it is given by*

(a)

$$\begin{aligned} f_t(x_1) &= (x_1^0, x_1^1, \dots, x_1^{k-2}, x_1^{k-1}, 0),\\ f_t(x_2) &= (x_2^0, x_2^1, \dots, x_2^{k-2}, 0, x_1^{k-1}),\\ &\vdots\\ f_t(x_{k-1}) &= (x_{k-1}^0, x_{k-1}^1, 0, x_{k-1}^2, \dots, x_{k-1}^{k-1}), \end{aligned}$$

(That is, if $y = f_t(x_i)$ then for $1 \leq i \leq k-1$, *$y^0(x_i) = x_i^0$, $y^1(x_i) = x_i^1$, $y^r(x_i) = x_i^r$ for $1 \leq r \leq k-i$, $y^{k-i+1}(x_i) = 0$, and $y^r(x_i) = x_i^{r-1}$ for $k-i+2 \leq r \leq k$.)*

(b)

$$\begin{aligned} f_t(x_k) = \big(&x_k^0,\quad (\xi_k^1, 0),\quad (\xi_k^2, u_k^1),\quad \dots,\quad (\xi_k^{k-1}, u_k^{k-2}),\quad (-\xi_k^2 - \cdots - \xi_k^{k-1}, u_k^{k-1})\big)\\ + \big(&0,\quad (0,0),\quad (0,0),\quad \dots,\quad (0,0),\quad (Q(x_k^0, \xi_k^1) + t, 0)\big) \end{aligned}$$

where Q is a nondegenerate quadratic form in the $n(m-k+1)$ variables (x_k^0, ξ_k^1).

Proof. It is straightforward to see that we can find coordinates x_i on U_i, $i = 1, \dots, k$, so that up to first order of approximation $f_0|U_i$ is given by the expressions in (a) and the first term in (b) above.

We must consider also second order terms: Let N be a linear subspace in the coordinates y transversal to $Tf_0(U_1) \cap \cdots \cap Tf_0(U_{k-1})$. Let $\phi\colon f(U_k) \to N$ be orthogonal projection onto N. Then $\ker(d\phi) = Tf_0(U_1) \cap \cdots \cap Tf_0(U_k)$ and $\operatorname{coker}(d\phi)$

is 1-dimensional. The second derivative $d^2\phi\colon \ker(\phi) \to \operatorname{coker}(\phi)$ must be a non-degenerate quadratic form, otherwise, we can avoid f_0 in generic 1-parameter families. (This is a consequence of the jet transversality theorem.) The Morse lemma then implies that, after possibly adjusting the coordinates in U_k by adding quadratic expressions in (x_k^0, ξ_k^1) to ξ_k^j, $j > 1$, there exists coordinates for f_0 as stated.

Finally, we must prove that the deformation f_t as given above is versal. A standard result in singularity theory says that it is enough to prove that the deformation is infinitesimally versal. This is straightforward. □

In Lemma 3.4 below we will use coordinates as at generic m-fold self intersection points. That is, $x = (x^0, x^1, \dots, x^{m-1}) \in \mathbb{R}^{nm-1}$ and $y = (y^0, y^1, \dots, y^m) \in \mathbb{R}^{n(m+1)-1}$, where $x^0, y^0 \in \mathbb{R}^{n-1}$ and $x_k, y_k \in \mathbb{R}^n$ for $k \neq 0$.

Lemma 3.4. *Let $f_0\colon M^{nm-1} \to W^{n(m+1)-1}$ be an immersion in Σ^1 and let $q = f_0(p_1) = \dots = f_0(p_{m+1})$ be a point of $(m+1)$-fold self-intersection. Then there are coordinates y on $V \subset W$ centered at q and coordinates x_i on $U_i \subset M$ centered at p_i, $1 \le i \le k$ such that in these coordinates the versal deformation f_t, $-\delta < t < \delta$ of f_0 is constant outside of $\cup_i U_i$ and in neighborhoods of $p_i \in U_i$ it is given by*

(a)

$$\begin{aligned}
f_t(x_1) &= (x_1^0, x_1^1, \dots, x_1^{m-2}, x_1^{m-1}, 0),\\
f_t(x_2) &= (x_2^0, x_2^1, \dots, x_2^{m-2}, 0, x_1^{m-1}),\\
&\;\vdots\\
f_t(x_m) &= (x_m^0, 0, x_m^1, x_m^2, \dots, x_m^{m-1}),
\end{aligned}$$

(That is, if $y = f_t(x_i)$ then for $1 \le i \le m$, *$y^0(x_i) = x_i^0$, $y^r(x_i) = x_i^r$ for $1 \le r \le m-i$, $y^{m-i+1}(x_i) = 0$, and $y^r(x_i) = x_i^{r-1}$ for $m-i+2 \le r \le m$.)*

(b)

$$f_t(x_k) = \\ \big(0,\quad (\xi_k^1, u_k^0),\quad (\xi_k^2, u_k^1),\quad \dots,\quad (\xi_k^{m-1}, u_k^{m-2}),\quad (-\xi_k^1 - \dots - \xi_k^{m-1} + t, u_k^{m-1})\big)$$

Proof. The proof is similar to the proof of Lemma 3.3, but easier. □

3.3. The codimension two part of the discriminant hypersurface.

Lemma 3.5. *Let $f_{0,0}\colon M^{nm-1} \to W^{n(m+1)-1}$ be an immersion in Σ^2. Then either* (a) *or* (b) *below holds.*

(a) *$f_{0,0}$ has two distinct degenerate self-intersection points q_1 and q_2. Locally around q_i, $i = 1, 2$ $f_{0,0}$ is as in Lemma 3.3 or Lemma 3.4. The versal deformation of $f_{0,0}$ is a product of the corresponding 1-parameter versal deformations.*

(b) *$f_{0,0}$ has one degenerate k-fold $2 \le k \le m$ self-intersection point q. There are coordinates y centered at q and coordinates x_i centered at p_i, $1 \le i \le k$ such that in these coordinates the versal deformation $f_{s,t}$, $-\delta < t, s < \delta$ of $f_{0,0}$ is constant outside of $\cup_i U_i$ and in a neighborhood of $p_i \in U_i$ it is given by*

(b1)

$$
\begin{aligned}
f_{s,t}(x_1) &= (x_1^0, x_1^1, \dots, x_1^{k-2}, x_1^{k-1}, 0),\\
f_{s,t}(x_2) &= (x_2^0, x_2^1, \dots, x_2^{k-2}, 0, x_1^{k-1}),\\
&\ \ \vdots\\
f_{s,t}(x_{k-1}) &= (x_{k-1}^0, x_{k-1}^1, 0, x_{k-1}^2, \dots, x_{k-1}^{k-1}),
\end{aligned}
$$

That is, if $y = f_{s,t}(x_i)$ then for $1 \le i \le k-1$, $y^0(x_i) = x_i^0$, $y^1(x_i) = x_i^1$, $y^r(x_i) = x_i^r$ *for* $1 \le r \le k-i$, $y^{k-i+1}(x_i) = 0$, *and* $y^r(x_i) = x_i^{r-1}$ *for* $k-i+2 \le r \le k$.

(b2)

$$
\begin{aligned}
&f_{s,t}(x_k) = \\
&\big(x_k^0,\quad (\xi_k^1, 0),\quad (\xi_k^2, u_k^1),\quad \dots,\quad (\xi_k^{k-1}, u_k^{k-2}),\quad (-\xi_k^2 - \dots - \xi_k^{k-1}, u_k^{k-1})\big)\\
+&\big(0,\quad (0,0),\quad (0,0),\quad \dots,\quad (0,0),\quad (Q(x_k^0) + \xi_k^1((\xi_k^1)^2 + s) + t, 0)\big)
\end{aligned}
$$

where Q is a nondegenerate quadratic form in the $n(m-k+1)-1$ variables x_k^0.

Proof. The jet-transversality theorem applied to maps of $M^{2jm-1} \times [-\delta,\delta]^2$ into $W^{2j(m+1)-1}$ shows that immersions with points of k-fold self-intersection $k \ge m+2$, as well as immersions with points of k-fold self-intersection points $2 \le k \le m+1$ at which the 1-jet has further degenerations than the 1-jets of Lemma 3.2 can be avoided in generic 2-parameter families.

Assume that $f_{0,0}$ has a degenerate k-fold self-intersection point. If $k = m+1$ it is easy to see that $f_{0,0}$ has the same local form as the map in Lemma 3.4. So, immersions with $(m+1)$-fold self-intersection points appears along 1-parameter subfamilies in generic 2-parameter families. If $k < m+1$ we proceed as in the proof of Lemma 3.3 and construct the map ϕ. Applying the jet transversality theorem once again we see that if the rank of $d^2\phi$ is smaller than $n(m-k+1)-1$ then $f_{0,0}$ can be avoided in generic 2-parameter families. If the rank of $d^2\phi$ equals $n(m-k+1)-1$ then the third derivative of ϕ in the direction of the null-space of $d^2\phi$ must be non-zero otherwise $f_{0,0}$ can be avoided in generic 2-parameter families.

With this information at hand we can find coordinates as claimed. Finally, as in the proof of Lemma 3.3, we must check that the deformation given is infinitesimally versal. This is straightforward. □

Pictures (a) and (b) in Figure 1 correspond to cases (a) and (b) in Theorem 3.5. The codimension two parts of the discriminant are represented by points. In (a) two branches of the discriminant, which consist of immersions with one degenerate r-fold and one degenerate k-fold self-intersection point respectively, intersect in Σ^2. In (b) the smooth points of the semi-cubical cusp represents immersions with one degenerate k-fold self-intersection point. The singular point represents Σ^2. If an immersion in Σ above the singular point (Σ^2) is moved below it then the index of the quadratic form Q in local coordinates close to the degenerate self-intersection point (see Lemma 3.3) changes.

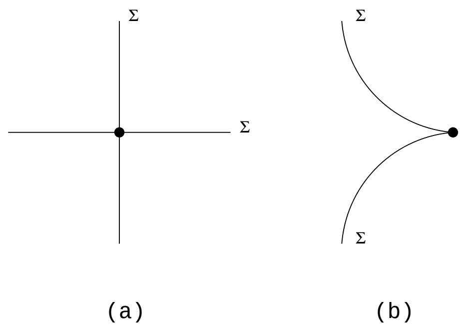

FIGURE 1. The discriminant intersected with a small generic 2-disk.

4. Definition of the invariants

In this section we define the invariants J_r, J, L and Λ. To this end, we describe resolutions of self-intersections (for the definitions of J_r and J) and we compute homology of image-complements of and of normal bundles (for the definitions of L and Λ).

4.1. **Resolution of the self intersection.** For generic immersions $f\colon M^{nm-1}\to W^{n(m+1)-1}$ let $\Gamma^j(f)=\Gamma_j(f)\cup\Gamma_{j+1}(f)\cup\cdots\cup\Gamma_m(f)$, for $2\le j\le m$ and let $\widetilde{\Gamma}^j(f)=f^{-1}(\Gamma^j(f))$. Resolving $\Gamma^j(f)$ we obtain a smooth manifold $\Delta^j(f)$:

Lemma 4.1. *Let $m>1$ and $n>1$ and $2\le j\le m$ be integers. Let $f\colon M^{nm-1}\to W^{n(m+1)-1}$ be a generic immersion. Then there exists closed $(n(m-j+1)-1)$-manifolds $\widetilde{\Delta}^j(f)$ and $\Delta^j(f)$, unique up to diffeomorphisms, and immersions $\sigma\colon\widetilde{\Delta}^j(f)\to M$ and $\tau\colon\Delta^j(f)\to W$ such that the diagram*

$$\begin{array}{ccc}\widetilde{\Delta}^j(f) & \xrightarrow{\sigma} & f^{-1}(\Gamma^j(f))\subset M\\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle f}\\ \Delta^j(f) & \xrightarrow{\tau} & \Gamma^j(f)\subset W\end{array},$$

commutes. The maps σ and τ are surjective, have multiple points only along $\widetilde{\Gamma}^{j+1}(f)$ and $\Gamma^{j+1}(f)$ respectively, and p is a j-fold cover.

Proof. This is immediate from Lemma 3.1: Close to a k-fold self intersection point $\Gamma_k(f)$ looks like the intersection of k $(nm-1)$-planes in general position in $\mathbb{R}^{n(m+1)-1}$. □

4.2. **Definition of the invariants J_r and J.**

Definition 4.1. *Let $m>1$ and $n>1$ be integers and assume that n is odd. For a generic immersion $f\colon M^{nm-1}\to W^{n(m+1)-1}$ and an integer $2\le r\le m$ such that $m-r$ is even, define*

$$J_r(f)=\chi(\Delta^r(f)),$$

where χ denotes Euler characteristic.

Definition 4.2. *Let $m>1$ be an integer. For a generic immersion $f\colon M^{2m-1}\to W^{2m+1}$, define*

$$J_r(f)=\text{The number of components of }\Gamma_m(f).$$

Lemma 4.2. *The functions J_r and J are invariants of generic immersions.*

Proof. Let f_t, $0 \leq t \leq 1$ be a regular homotopy through generic immersions. If $F\colon M \times [0,1] \to W \times [0,1]$ is the immersion $F(x,t) = (f_t(x), t)$ then $\Gamma^j(F) \cong \Gamma^j(f_0) \times I$. It follows that $\Delta^j(f_0)$ is diffeomorphic to $\Delta^j(f_1)$. $\square$

4.3. **Homology of complements of images.**

Lemma 4.3. *Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be a generic immersion. Assume that M is closed. Then*

(a)
$$H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}_2) \cong H^{nm-1}(M; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$
and

(a) *if M is oriented then*
$$H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}) \cong H^{nm-1}(M; \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. Alexander duality implies that $H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M)) \cong H^{nm-1}(f(M))$. By Lemma 3.1, we can choose triangulations of M and $f(M)$ such that the map $f\colon M \to f(M)$ is a bijection on the $(nm-k)$-skeleta for $1 \leq k \leq n$. $\square$

Remark 4.1. In case (a) of Lemma 4.3 we will use the unique $\mathbb{Z}_2$-orientation of M and the duality isomorphism to identify $H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}_2)$ with $\mathbb{Z}_2$. In case (b) of Lemma 4.3, a $\mathbb{Z}$-orientation of M determines an isomorphism $H^{nm-1}(M; \mathbb{Z}) \to \mathbb{Z}$. We shall use this and the duality isomorphism to identify $H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z})$ with $\mathbb{Z}$.

A small $(n-1)$-dimensional sphere going around a fiber in the normal bundle of $f(M) - \Gamma^2(f)$ generates these groups.

4.4. **Homology of normal bundles of immersions.** Consider an immersion $f\colon M^{nm-1} \to W^{n(m+1)-1}$. Let N denote its normal bundle. Then N is a vector bundle of dimension n over M. Choose a Riemannian metric on N and consider the associated bundle ∂N of unit vectors in N. This is an $(n-1)$-sphere bundle over M. Let $\partial F \cong S^{n-1}$ denote a fiber of ∂N.

Lemma 4.4. *Let $f\colon M^{nm-1} \to W^{n(m+1)-1}$ be an immersion. Let $i\colon \partial F \to \partial N$ denote the inclusion of the fiber.*

(a) *If $H_{n-1}(M; \mathbb{Z}_2) = 0 = H_n(M; \mathbb{Z}_2)$ then*
$$i_*\colon H_{n-1}(\partial F; \mathbb{Z}_2) \to H_{n-1}(\partial N; \mathbb{Z}_2),$$
is an isomorphism.

(b) *If M and W are oriented and $H_{n-1}(M; \mathbb{Z}) = 0 = H_n(M; \mathbb{Z})$ then*
$$i_*\colon H_{n-1}(\partial F; \mathbb{Z}) \to H_{n-1}(\partial N; \mathbb{Z}),$$
is an isomorphism.

Proof. This follows from the Leray-Serre spectral sequence. $\square$

Remark 4.2. In case (a) of Lemma 4.4 we use the isomorphism i_* and a canonical generator of $H_{n-1}(\partial F; \mathbb{Z}_2)$ to identify $H_{n-1}(\partial N; \mathbb{Z}_2)$ with $\mathbb{Z}_2$. In case (b) of Lemma 4.4, orientations of M and W induce an orientation of the fiber sphere ∂F. We use this orientation and i_* in Lemma 4.4 to identify $H_{n-1}(\partial N; \mathbb{Z})$ with $\mathbb{Z}$.

4.5. **Shifting the m-fold self-intersection.** Let $f\colon M^{nm-1} \to W^{n(m+1)-1}$ be a generic immersion. Recall that the set of m-fold self-intersection points of f is a closed submanifold $\Gamma_m(f)$ of $W^{n(m+1)-1}$ of dimension $n-1$ and its preimage $\widetilde{\Gamma}_m(f)$ is a closed submanifold of the same dimension in M^{nm-1}.

Lemma 4.5. *If $f\colon M^{nm-1} \to W^{n(m+1)-1}$ is an immersion then there exists a smooth section $s\colon \widetilde{\Gamma}_m(f) \to \partial N$, where ∂N is the unit sphere bundle of the normal bundle of f.*

Proof. This is immediate: The base space of the n-dimensional vector bundle $N|\widetilde{\Gamma}_m(f)$ has dimension $n-1$. □

Let ϕ denote the canonical bundle map over $f\colon M^{nm-1} \to W^{n(m+1)-1}$,

$$\phi\colon N \to TW^{n(m+1)-1},$$

and let s be as in Lemma 4.5. We define a vector field $v\colon \Gamma_m(f) \to TW^{n(m+1)-1}$ along $\Gamma_m(f)$: For $q = f(p_1) = \cdots = f(p_m)$ let

$$v(q) = \phi(s(p_1)) + \cdots + \phi(s(p_m))).$$

It is easy to see that v is well-defined and smooth.

We now restrict attention to the case where the target manifold is Euclidean space.

Definition 4.3. *For a generic immersion $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$, let $\Gamma'_m(f, v, \epsilon)$ be the submanifold of $\mathbb{R}^{n(m+1)-1}$ obtained by shifting $\Gamma_m(f)$ a small distance $\epsilon > 0$ along v.*

Remark 4.3. Note that for $\epsilon > 0$ small enough $\Gamma'_m(f, v, \epsilon) \cap f(M) = \emptyset$. Moreover, if $a > 0$ is such that for all $0 < \epsilon < a$ the corresponding $\Gamma'_m(f, v, \epsilon)$ is disjoint from $f(M)$ then $\Gamma'_m(f, v, \epsilon)$ and $\Gamma'_m(f, v, a)$ are homotopic in $\mathbb{R}^{n(m+1)-1} - f(M)$.

4.6. **Definition of the invariants L and Λ.**

Definition 4.4. *Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be a generic immersion. Assume that M satisfies*

$$H_{n-1}(M; \mathbb{Z}_2) = 0 = H_n(M; \mathbb{Z}_2). \tag{CΛ}$$

Define $\Lambda(f) \in \mathbb{Z}_2$ as

$$\Lambda(f) = [\Gamma'_m(f, v, \epsilon)] - s_*[\widetilde{\Gamma}_m(f)] \in \mathbb{Z}_2,$$

where $\epsilon > 0$ is very small and $[\Gamma'_m(f, v, \epsilon)] \in H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}) \cong \mathbb{Z}_2$ (see Remark 4.1) and $s_*[\widetilde{\Gamma}_m] \in H_{n-1}(\partial N; \mathbb{Z}) \cong \mathbb{Z}_2$ (see Remark 4.2).

Definition 4.5. *Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be a generic immersion. Assume that n is even and that M is oriented and satisfies*

$$H_{n-1}(M; \mathbb{Z}) = 0 = H_n(M; \mathbb{Z}). \tag{CL}$$

Define $L(f) \in \mathbb{Z}$ as

$$L(f) = [\Gamma'_m(f, v, \epsilon)] - s_*[\widetilde{\Gamma}_m(f)] \in \mathbb{Z},$$

where $\epsilon > 0$ is very small and $[\Gamma'_m(f, v, \epsilon)] \in H_{n-1}(\mathbb{R}^{2m(j+1)-1} - f(M); \mathbb{Z}) \cong \mathbb{Z}$ (see Remark 4.1) and $s_*[\widetilde{\Gamma}_m] \in H_{n-1}(\partial N; \mathbb{Z}) \cong \mathbb{Z}$ (see Remark 4.2).

Remark 4.4. If $\mathrm{Tor}(H_{n-2}(M; Z), \mathbb{Z}_2) = 0$ and M satisfies condition (CL) in Definition 4.5 then by the universal coefficient theorem M also satisfies condition (CΛ) in Definition 4.4. It follows from Remarks 4.1, 4.2 that in this case $\Lambda = L \bmod 2$.

Lemma 4.6. *Λ and L are well-defined. That is, they do neither depend on the choice of s, nor on the choice of $\epsilon > 0$.*

Remark 4.5. We shall often drop the awkward notation $\Gamma'_m(f, v, \epsilon)$ and simply write $\Gamma'(f)$. This is justified by Lemma 4.6.

Proof. The independence of $\epsilon > 0$ follows immediately from Remark 4.3. We will therefore not write all ϵ's out in the sequel of this proof.

Let s_0 and s_1 be two homotopic sections of $\partial N|\widetilde{\Gamma}_m(f)$. Let v_0 and v_1 be the corresponding normal vector fields along $\Gamma_m(f)$.

A homotopy s_t between s_0 and s_1 induces a homotopy v_t between v_0 and v_1 and hence between $\Gamma'_m(f, v_0)$ and $\Gamma'_m(f, v_1)$ in $\mathbb{R}^{n(m+1)-1} - f(M)$. This shows that Λ and L only depend on the homotopy class of s.

To see that Λ and L are independent of the vector field we introduce the notion of adding a local twist: Let $s\colon \widetilde{\Gamma}_m(f) \to \partial N$ be a section and let $p \in \widetilde{\Gamma}_m(f)$. Choose a neighborhood U of p in $\widetilde{\Gamma}_m(f)$ and a trivialization $\partial N|U \cong U \times S^{n-1}$ such that $s(u) \equiv w$, where w is a point in S^{n-1}. Let D^{n-1} be disk inside U and let $\sigma\colon D \to S^{n-1}$ be a smooth map of degree ± 1 such that $\sigma \equiv w$ in a neighborhood of ∂D. Let s^{tw} be the section which is s outside of D and σ in D. We say that s^{tw} is the result of adding a local twist to s. (In the case when M is oriented and n is even, the local twist is said to be positive if the degree of σ is $+1$ and negative if the degree of σ is -1.)

Let s^{tw} be the vector field obtained by adding a twist to s. Let v^{tw} and v be the vector fields along $\Gamma_m(f)$ obtained from s^{tw} and s respectively. Clearly,

$$s^{\mathrm{tw}}{}_*[\Gamma_m(f)] = s_*[\Gamma_m(f)] \pm 1 \quad \text{and} \quad [\Gamma'_m(f, v^{\mathrm{tw}})] = [\Gamma'_m(f, v)] \pm 1.$$

Hence, Λ and L are invariant under adding local twists.

A standard obstruction theory argument shows that if s and s' are two sections of $\partial N|\Gamma_m$ then by adding local twists to s' we can obtain a section s'' which is homotopic to s. Hence, Λ and L are independent of the choice of section. $\square$

Lemma 4.7. *Λ and L are invariants of generic immersions.*

Proof. Let f_t, $0 \le t \le 1$ be a regular homotopy through generic immersions. Consider the induced map

$$F\colon M \times I \to \mathbb{R}^{n(m+1)-1} \times I, \quad F(x, t) = (f_t(x), t),$$

where I is the unit interval $[0, 1]$. Shifting $\Gamma_m(F) \cong \Gamma_m(f_0) \times I$ off $F(M \times I)$ using a suitable vector field in $N|\widetilde{\Gamma}_m(F)$ it is easy to see that $L(f_0) = L(f_1)$. $\square$

5. Additivity properties, connected sum, and reversing orientation

In this section we study how our invariants behave under two natural operations on generic immersions: connected sum and reversing orientation.

5.1. **Connected summation of generic immersions.** For (oriented) manifolds M and V of the same dimension, let $M \sharp V$ denote the (oriented) connected sum of M and V.

Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ and $g\colon V^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be two generic immersions. We shall define the connected sum $f \sharp g$ of these. It will be a generic immersion $M \sharp V \to \mathbb{R}^{n(m+1)-1}$.

Let (u, x), $u \in \mathbb{R}$ and $x \in \mathbb{R}^{n(m+1)-2}$ be coordinates on $\mathbb{R}^{n(m+1)-1}$. Composing the immersions f and g with translations we may assume that $f(M) \subset \{u \leq -1\}$ and $g(V) \subset \{u \geq 1\}$. Choose a point $p \in M$ and a point $q \in V$ such that there is only one point in $f^{-1}(f(p))$ and in $g^{-1}(g(q))$. Pick an arc α in $\mathbb{R}^{n(m+1)-1}$ connecting $f(p)$ to $g(q)$ and such that $\alpha \cap (f(M) \cup g(V)) = \{f(p), g(q)\}$. Moreover, assume that α meets $f(M)$ and $g(V)$ transversally at its endpoints. Let N be the normal bundle of α. Pick a (oriented) basis of $T_{f(p)}f(M)$ and an (anti-oriented) one of $T_{g(q)}(g(V))$. These give rise to $nm-1$ vectors over ∂a in N. Extend these vectors to $nm-1$ independent normal vector fields along α. Using a suitable map of N into a tubular neighborhood of α these vector fields give rise to an embedding $\phi\colon \alpha \times D \to \mathbb{R}^{n(m+1)-1}$, where D denotes a disk of dimension $nm-1$, such that $\phi | f(p) \times D$ is an (orientation preserving) embedding into $f(M)$ and $\phi | g(q) \times D$ is an (orientation reversing) embedding into $g(V)$. The tube $\phi(\alpha \times \partial D)$ now joins $f(M) - \phi(f(p) \times \operatorname{int}(D))$ to $g(V) - \phi(g(q) \times \operatorname{int}(D))$. Smoothing the corners we get a generic immersion $f \sharp g\colon M \sharp V \to \mathbb{R}^{n(m+1)-1}$.

Lemma 5.1. *Let $f, g\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be generic immersions. The connected sum $f \sharp g$ is independent up to regular homotopy through generic immersions of both the choices of points $f(p)$ and $g(q)$ and the choice of the path α used to connect them.*

Proof. This is straightforward. (Note that the preimages of self-intersections has codimension $n > 1$.)

□

Lemma 5.2. *If $f, f'\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ and $g, g'\colon V^{nm-1} \to \mathbb{R}^{n(m+1)-1}$, $m, n > 1$ are regularly homotopic through generic immersions then $f \sharp g$ and $f' \sharp g'$ are regularly homotopic through generic immersions.*

Proof. This is straightforward □

Proposition 5.1. *The invariants J_r and J are additive under connected summation.*

Proof. $\Delta^j(f \sharp g) = \Delta^j(f) \sqcup \Delta^j(g)$. □

Note that if M^{nm-1} and V^{nm-1} are manifolds which both satisfy condition (CΛ) in Definition 4.4 or condition (CL) in Definition 4.5 then so does $M \sharp V$.

Proposition 5.2. *Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ and $g\colon V^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be generic immersions.*

(a) *If M and V both satisfy condition* (CΛ) *then*
$$\Lambda(f \sharp g) = \Lambda(f) + \Lambda(g).$$

(b) *If n is even and M and V are both oriented and both satisfy condition* (CL) *then*
$$L(f \sharp g) = L(f) + L(g).$$

Proof. Note that $\Gamma_m(f \sharp g) = \Gamma_m(f) \sqcup \Gamma_m(g)$. Consider case (b). Choose $2j$-chains D and E in $\{u < -1\}$ and $\{u > 1\}$ bounding $\Gamma_m(f)$ and $\Gamma_m(g)$ respectively and disjoint from the arc α, used in the construction of $f \sharp g$. Then

$$L(f \sharp g) = (D \cup E) \cdot f \sharp g(M \sharp V) = D \cdot f(M) + E \cdot g(V) = L(f) + L(g).$$

Case (a) is proved in exactly the same way. □

5.2. **Changing orientation.** The invariants J_r, J, and Λ are clearly orientation independent. In contrast to this, the invariant L is orientation sensitive.

To have L defined, let $n = 2j$ and consider an oriented closed manifold M^{2jm-1} which satisfies condition (CL).

Proposition 5.3. *Assume that there exists an orientation reversing diffeomorphism $r\colon M \to M$. Let $f\colon M^{2jm-1} \to \mathbb{R}^{2j(m+1)-1}$ be a generic immersion. Then $f \circ r$ is a generic immersion and*

$$L(f \circ r) = (-1)^{m+1} L(f)$$

Proof. Note that $\Gamma_m(f \circ r) = \Gamma_m(f) = \Gamma_m$. The orientation of Γ_m is induced from the decomposition

$$T_q\mathbb{R}^{2j(m+1)-1} = T_q\Gamma_m(f) \oplus N_1 \oplus \cdots \oplus N_m.$$

The immersions f and $f \circ r$ induces opposite orientations on each N_i hence the orientations induced on Γ_m agrees if m is even and does not agree if m is odd. Let D be a $2j$-chain bounding Γ_m, with its orientation induced from f. If m is even then

$$L(f \circ r) = D \cdot f(r(M)) = D \cdot -f(M) = -L(f).$$

If m is odd then

$$L(f \circ r) = -D \cdot f(r(M)) = -D \cdot -f(M) = L(f).$$

□

Proposition 5.4. *Let $R\colon \mathbb{R}^{2j(m+1)-1} \to \mathbb{R}^{2j(m+1)-1}$ be reflection in a hyperplane. Let $f\colon M^{2jm-1} \to \mathbb{R}^{2j(m+1)-1}$ be a generic immersion. Then $R \circ f$ is a generic immersion and*

$$L(R \circ f) = (-1)^m L(f)$$

Proof. Note that the oriented normal bundle of Rf is $-RN$.

So the correctly oriented $2j$-chain bounding $\Gamma_m(R \circ f)$ is $(-1)^{m+1}RD$. Thus,

$$L(R \circ f) = (-1)^{m+1}(RD \cdot Rf(M)) = (-1)^m(D \cdot f(M)) = (-1)^m L(f).$$

□

6. Coorientations and proofs of Theorems 1 and 3

In this section we prove Theorems 1 and 3. To do that we need a coorientation of the discriminant hypersurface in the space of immersions.

6.1. **Coorienting the discriminant.**

Remark 6.1. Let a be an invariant of generic immersions. If a is $\mathbb{Z}_2$-valued then ∇a (see Section 2) is well-defined without reference to any coorientation of Σ. Moreover, if a is integer-valued and Δ is a union of path components of Σ^1 then the notion $\nabla a|\Delta \equiv 0$ is well-defined without reference to a coorientation of Σ.

We shall coorient the relevant parts (see Remark 6.1) of the discriminant hypersurface. That is, we shall find coorientations of the parts of the discriminant hypersurface in the space of generic immersions where our invariants have non-zero jumps. These coorientations will be *continuous* (see [5], Section 7). That is, the intersection number of any generic small loop in $\mathcal{F}$ and Σ^1 vanishes and the coorientation extends continuously over Σ^2.

Definition 6.1. *Let $m > 1$ and $n > 1$. Assume that n is odd. Let $2 \leq r \leq m$ and assume that $m-r$ is even. Let $f_0\colon M^{nm-1} \to W^{n(m+1)-1}$ be a generic immersion in Σ^1 with one degenerate r-fold self-intersection point. Let f_t be a versal deformation of f_0. We say that f_δ is on the positive side of Σ^1 and $f_{-\delta}$ on the negative side if $\chi(\Delta^r(f_\delta)) > \chi(\Delta^r(f_{-\delta}))$.*

Remark 6.2. Note that by Lemma 3.3 $\chi(\Delta^r(f_\delta))$ is obtained from $\chi(\Delta^r(f_{-\delta}))$ by a Morse modification. Since the dimension of $\chi(\Delta^r(f_\delta))$ is $n(m-r+1)-1$ is even the Euler characteristic changes under such modifications.

Definition 6.2. *Let $m > 1$. Let $f_0\colon M^{2m-1} \to W^{2m+1}$ be a generic immersion in Σ^1 with one degenerate m-fold self-intersection point. Let f_t be a versal deformation of f_0. We say that f_δ is on the positive side of Σ^1 and $f_{-\delta}$ on the negative side if the number of components in $\Gamma_m(f_\delta)$ is larger than the number of components in $\Gamma_m(f_{-\delta})$.*

Remark 6.3. Note that by Lemma 3.3 $\Gamma_m(f_\delta)$ is obtained from $\Gamma_m(f_{-\delta})$ by a Morse modification. Since these are 1-manifolds the number of components change.

The construction of the relevant coorientation for L is more involved. It is based on a high-dimensional counterpart of the notion of over- and under-crossings in classical knot theory.

Let $f_0\colon M^{2j(m+1)-1} \to W^{2j(m+1)-1}$ be an immersion of oriented manifolds. Assume that f_0 has an $(m+1)$-fold self-intersection point, $q = f_0(p_1) = \cdots = f_0(p_{m+1})$. Let f_t be a versal deformation of f_0. Let U_i be small neighborhoods of p_i and let S_i^t denote the oriented sheet $f_t(U_i)$. Note that $S_i^0 \cap \cdots \cap S_{m+1}^0 = q$ and that $S_1^t \cap \cdots \cap S_{m+1}^t = \emptyset$ if $t \neq 0$.

Let $D_i^t = \cap_{j\neq i} S_j^t$. Let w_i be a line transversal to $TS_i^0 + TD_i^0$ at q. For small $t \neq 0$ both S_i^t and D_i^t intersects w_i. Orienting the line from S_i^t to D_i^t gives a local orientation $(D_i^t, S_i^t, \vec{w}_i)$ of $\mathbb{R}^{2j(m+1)-1}$. Comparing it with the standard orientation of $\mathbb{R}^{2j(m+1)-1}$ we get a sign $\sigma_t(i) = \mathrm{Or}(D_i^t, S_i^t, \vec{w}_i)$, where Or denotes the sign of the orientation. Note that, if we orient the line from D_i^t to S_i^t we get the opposite orientation $-\vec{w}_i$ of w and

$$\mathrm{Or}(D_i^t, S_i^t, \vec{w}_i) = \mathrm{Or}(S_i^t, D_i^t, -\vec{w}_i),$$

since D_i^t and S_i^t are both odd-dimensional. Note also that $\sigma_t(i) = -\sigma_{-t}(i)$ (see Lemma 3.4).

We next demonstrate that $\sigma_t(i) = \sigma_t(j)$ for all i, j: Let N_i^t be the oriented normal bundle of S_i^t. Let w be a vector from D_1^t to D_2^t, transversal to both $TS_1^0 + TD_1^0$ and $TS_2^0 + TD_2^0$. Orient it from D_1^t to D_2^t. Then

$$\sigma_t(1) = \mathrm{Or}(S_1^t, D_1^t, \vec{w}),$$

and hence $TD_1^t + w$ gives a normal bundle N_1^t and by the convention used to orient the normal bundle $\mathrm{Or}(N_1^t) = \sigma_t(1)\,\mathrm{Or}(D_1^t, \vec{w})$. In a similar way it follows that $\mathrm{Or}(N_2^t) = \sigma_t(2)\,\mathrm{Or}(D_2^t, -\vec{w})$.

Now, by definition

$$\begin{aligned}1 = \mathrm{Or}(D_1^t, N_2^t, \dots, N_m^t) &= \sigma_t(2)\,\mathrm{Or}(D_1^t, (D_2^t, -\vec{w}), N_3^t, \dots, N_m^t) = \\ =&[\dim(D_2^t) \text{ is odd}] = -\sigma_t(2)\,\mathrm{Or}(D_1^t, -\vec{w}, D_2^t, N_3^t, \dots, N_m^t) = \\ =&\sigma_t(2)\sigma_t(1)\,\mathrm{Or}(N_1^t, D_2^t, N_3^t, \dots, N_m^t) = \sigma_t(1)\sigma_t(2).\end{aligned}$$

Hence, $\sigma_t(1)\sigma_t(2) = 1$ as claimed.

Definition 6.3. *Let $f_0\colon M^{2j(m+1)-1} \to W^{2j(m+1)-1}$ be an immersion of oriented manifolds. Assume that f_0 has an $(m+1)$-fold self-intersection point. Let f_t be a versal deformation of f_0. We say that f_δ is on the positive side of Σ^1 at f_0 if*

$$\sigma_\delta(1) = \dots = \sigma_\delta(m+1) = +1.$$

6.2. **First order invariants.** The following obvious lemma will be used below.

Lemma 6.1. *Any first order invariant of generic immersions is uniquely (up to addition of zero order invariants) determined by its jump.* □

6.3. **Proof of Theorem 3.** We know from Lemma 4.2 that J and J_r are invariants of generic immersions. We must calculate their jumps.

We start in case (a): Assume that n is odd. Let $f_t\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$, $t \in [-\delta, \delta]$ be a versal deformation of $f_0 \in \Sigma^1$ and fix r, $2 \le r \le m$ such that $m - r$ is even.

If f_0 has a degenerate k-fold intersection point $2 \le k \le m+1$ and $k \ne r$ then $\Delta^r(f_{-\delta})$ is diffeomorphic to $\Delta^r(f_\delta)$: If $k < r$ then $\Delta^r(f_{-\delta})$ is not affected at all by the versal deformation. If $k > r$ then the immersed submanifold $\tau^{-1}(\Gamma_k(f_{-\delta}))$ of $\Delta^r(f_{-\delta})$ is changed by surgery (or is deformed by regular homotopy if $k = m+1$) under the versal deformation. This does not affect the diffeomorphism class of $\Delta^r(f_{-\delta})$.

If f_0 has a degenerate r-fold self-intersection point then $\Delta^r f_\delta$ is obtained from $\Delta^r f_{-\delta}$ by a surgery (see Lemma 3.3). Since the dimension of $\Delta^r(f_{-\delta})$ is $n(m-r+1)-1$ which is even this changes the Euler characteristic by ± 2.

According to our coorientation conventions $\nabla J_r(f_0) = 2$ if f_0 has a degenerate r-fold self-intersection point. Also, $\nabla J_r(f_0) = 0$ if f_0 has any other degeneracy.

The jump of J in case (b) is 1 by the same argument.

It is evident from Lemma 3.5 (see Figure 1) that J and J^r are first order invariants. The theorem now follows from Lemma 6.1. □

6.4. **Proof of Theorem 1.** We start with (b): Let $f_t\colon M^{2jm-1} \to \mathbb{R}^{2j(m+1)-1}$, $t \in [-\delta, \delta]$ be a generic one-parameter family intersecting Σ^1 in f_0. If the degenerate intersection point of f_0 is a k-fold intersection point with $2 \le k \le m-1$ then clearly $L(f_{-\delta}) = L(f_\delta)$, since the m-fold self-intersection is not affected under such deformations.

Assume that f_0 has a degenerate m-fold self-intersection point. Without loss of generality we may assume that f_t is a deformation of the form in Lemma 3.3 (we use coordinates as there):

Let $F(x,t) = (f_t(x), t)$. Shifting $((\{Q(y^0, v^1) = t\}, 0 \ldots, 0), t)$ we obtain a $2j$-chain bounded by $\Gamma'_m(f_\delta)\times\delta - \Gamma'_m(f_{-\delta})\times -\delta$ in $\mathbb{R}^{2m(j+1)-1}\times[-\delta,\delta] - F(M\times[-\delta,\delta])$. It follows that $L(f_\delta) = L(f_{-\delta})$.

If f_0 has an $(m+1)$-fold self-intersection then the discussion preceding Definition 6.3 shows that $L(f_\delta) = L(f_{-\delta}) + (m+1)$: At an $(m+1)$-fold self intersection point $m+1$ crossings are turned into crossings of opposite sign.

Hence, $\nabla L(f_0) = m+1$ if f_0 has an $(m+1)$-fold self-intersection point and $\nabla L(f_0) = 0$ if f_0 has any other degeneracy.

The calculation of $\nabla\Lambda$ in (a) is analogous. Let us just make a remark about the parity of m: At an $(m+1)$-fold self-intersection point $m+1$ crossings are changed. Hence, at instances of $(m+1)$-fold self-intersection the invariant Λ changes by $1 \in \mathbb{Z}_2$ if $m+1$ is odd and does not change if $m+1$ is even.

It is immediate from Lemma 3.5, see Figure 1 that Λ and L are first order invariants. The theorem now follows from Lemma 6.1. □

7. Invariants of regular homotopy

A function of immersions $M \to W$ which is constant on path components of $\mathcal{F}$ will be called an *invariant of regular homotopy*. Our geometrically defined invariants of generic immersion give rise to torsion invariants of regular homotopy.

Definition 7.1. *Let n be odd. Let $f\colon M^{nm-1} \to W^{n(m+1)-1}$ be an immersion. Let f' be a generic immersion regularly homotopic to f. For $2 \le r \le m$ such that $m-r$ is even, define $j_r(f) \in \mathbb{Z}_2$ as*

$$j_r(f) = J_r(f') \bmod 2.$$

Proposition 7.1. *The function j_r is an invariant of regular homotopy.*

Proof. Clearly it is enough to show that for any two regularly homotopic generic immersions f_0 and f_1, $j_r(f_0) = j_r(f_1)$. Let f_t be a generic regular homotopy from f_0 to f_1. Then f_t intersects Σ transversally in a finite number of points in Σ^1. It follows from Theorem 3 that j_r remains unchanged at such intersections. Hence, $j_r(f_0) = j_r(f_1)$. □

Definition 7.2. *Let n be even. Assume that M^{nm-1} is a manifold which satisfy condition* (CL)*. Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be an immersion. Let f' be a generic immersion regularly homotopic to f. Define $l(f) \in \mathbb{Z}_{m+1}$ as*

$$l(f) = L(f') \bmod (m+1).$$

Proposition 7.2. *The function l is an invariant of regular homotopy.*

Proof. The proof is identical to the proof of Proposition 7.1. □

Definition 7.3. *Let $m > 1$ be odd. Assume that M^{nm-1} is a manifold which satisfy condition* (CΛ)*. Let $f\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be an immersion. Let f' be a generic immersion regularly homotopic to f. Define $\lambda(f) \in \mathbb{Z}_2$ as*

$$\lambda(f) = \Lambda(f').$$

Proposition 7.3. *The function λ is an invariant of regular homotopy.*

Proof. This follows from the proof of Theorem 1, where it is noted that when m is odd Λ remains constant when a regular homotopy intersects Σ^1. $\square$

8. Sphere-immersions in codimension two

In this section Theorem 2 will be proved. To do that we will first discuss the classifications of sphere-immersions and sphere-embeddings up to regular homotopy.

8.1. **The Smale invariant.** In Smale's classical work [9] it is proved that there is a bijection between the set of regular homotopy classes $\mathbf{Imm}(k,n)$ of immersions $S^k \to \mathbb{R}^{k+n}$ and the elements of the group $\pi_k(V_{k+n,k})$, the k^{th} homotopy group of the Stiefel manifold of k-frames in $(k+n)$-space. If $f\colon S^k \to \mathbb{R}^{k+n}$ is an immersion we let $\Omega(f) \in \pi_k(V_{k+n,k})$ denote its Smale invariant. Via Ω we can view $\mathbf{Imm}(k,n)$ as an Abelian group.

If the codimension is two the groups appearing in Smale's classification are easily computed: The exact homotopy sequence of the fibration

$$SO(2) \hookrightarrow SO(2m+1) \to V_{2m+1,2m-1}$$

implies that $\pi_{2m-1}(V_{2m+1,2m-1}) \cong \pi_{2m-1}(SO(2m+1))$. Bott-periodicity then gives:

$$\pi_{2m-1}(V_{2m+1,2m-1}) = \begin{cases} \mathbb{Z} & \text{if } m = 2j, \\ \mathbb{Z}_2 & \text{if } m = 4j+1, \\ 0 & \text{if } m = 4j+3. \end{cases}$$

Remark 8.1. It is possible to identify the group operations in $\mathbf{Imm}(k,2)$ geometrically. Kervaire [8] proves that the Smale invariant Ω is additive under connected sum of immersions. This gives the geometric counterpart of addition. If $f\colon S^n \to \mathbb{R}^{n+2}$ is an immersion, $r\colon S^n \to S^n$ is an orientation reversing diffeomorphism, and $n \neq 2$ then $\Omega(f) = -\Omega(f \circ r)$. (See [3] for the case $S^3 \to \mathbb{R}^5$, the other cases are analogous). This gives the geometric counterpart of the inverse operation in $\mathbf{Imm}(n,2)$ for $n \neq 2$.

It is interesting to note that for immersions $f\colon S^2 \to \mathbb{R}^4$, $\Omega(f) = \Omega(f \circ r)$: The Smale invariant can in this case be computed as the algebraic number of self-intersection points. This number is clearly invariant under reversing orientation. (The same is true also for immersions $S^{2k} \to \mathbb{R}^{4k}$, $k \geq 1$.)

Lemma 8.1. *The regular homotopy invariant l induces a homomorphism*

$$\mathbf{Imm}(2m-1,2) \to \mathbb{Z}_{m+1}.$$

Proof. Note that the dimensions are such that L is defined and spheres certainly satisfy condition (CL). The invariant L is additive under connected summation and changes sign if an immersion is composed on the left with an orientation reversing diffeomorphism. Hence, l induces a homomorphism. $\square$

Let $\mathbf{Emb}(n,2) \subset \mathbf{Imm}(n,2)$ be the set of regular homotopy classes which contain embeddings. By Remark 8.1 $\mathbf{Emb}(n,2)$ is a subgroup of $\mathbf{Imm}(n,2)$. A result of Hughes and Melvin [6] states that $\mathbf{Emb}(4j-1,2) \subset \mathbf{Imm}(4j-1,2) \cong \mathbb{Z}$ is a subgroup of index μ_j. Here $\frac{B_j}{4j} = \frac{\nu_j}{\mu_j}$, where B_j is the j^{th} Bernoulli number and ν_j and μ_j are coprime integers.

8.2. **Proof of Theorem 2.** By Lemma 8.1, $l\colon \mathbf{Imm}(2m-1,2) \to \mathbb{Z}/(m+1)\mathbb{Z}$ is a homomorphism and clearly, $\mathbf{Emb}(2m-1,2) \subset \ker(l)$.

In case (b) $\mathbf{Imm}(8j+5,2) = 0$ and hence the image of l is zero, which proves that $L(f)$ is always divisible by $m+1 = 4j+4$.

In case (a) $\mathbf{Imm}(8j+1,2) \cong \mathbb{Z}/2\mathbb{Z}$. Hence, for any immersion f, $l(f \sharp f) = 0$. Thus, $L(f \sharp f) = 2L(f)$ is divisible by $m+1 = 4j+2$, which implies that $L(f)$ is divisible by $2j+1$.

In case (c) $\mathbf{Emb}(4j-1,2) \cong \mu_j\mathbb{Z}$ as a subgroup of $\mathbf{Imm}(4j-1,2) \cong \mathbb{Z}$. Hence, for any immersion f, $m+1 = 2j+1$ divides

$$L(f \sharp \dots [\mu_j \text{ summands}] \dots \sharp f) = \mu_j L(f).$$

Thus, if if p is a prime and r, k are integers such that p^{r+k} divides $2j+1$ and p^{k+1} does not divide μ_j then p^r divides $L(f)$. $\square$

9. Examples and problems

In this section we construct examples of 1-parameter families of immersions which shows that all the first order invariants we have defined are non-trivial. We also discuss some problems in connection with the regular homotopy invariants defined in Section 7.

9.1. **Examples.** Using our local coordinate description of immersions with one degenerate self-intersection point we can construct examples showing that the invariants J, J_r, Λ and L are non-trivial. We start with Λ and L:

Choose m standard spheres $S_1, \dots, S_m$ of dimension $(nm-1)$ intersecting in general position in $\mathbb{R}^{n(m+1)-1}$ so that $S_1 \cap \dots \cap S_m \cong S^{n-1}$. Pick a point $p \in S_1 \cap \dots \cap S_m$ and let S_{m+1} be another standard $(nm-1)$-sphere intersecting $S_1 \cap \dots \cap S_m$ at p so that in a neighborhood of p the embeddings are given by the expressions in Lemma 3.4 and so that p is the only degenerate intersection point. Let $f_0\colon S^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be the immersions which is the connected sum of $S_1, \dots, S_{m+1}$. Let f_t be a versal deformation of f. Then, after possibly reversing the direction of the versal deformation we have $\Lambda(f_\delta) = 1 \in \mathbb{Z}_2$, if n is odd or $L(f_\delta) = \pm(m+1)$ if n is even.

In the latter case we would get the other sign of $L(f_\delta)$ if the orientation of S_{m+1} in the construction above is reversed. To distinguish these two immersions denote one by h^+ and the other by h^-, so that $L(h^+) = m+1 = -L(h^-)$. Then it is an immediate consequence of Theorem 1 that:

Any generic regular homotopy from h^+ to h^- has at least two instances of $(m+1)$-*fold self-intersection.*

Theorem 1 has many corollaries as the one just mentioned.

To see that L and Λ are nontrivial for other source manifolds. We use connected sum with the immersions $S^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ just constructed and Lemma 5.2.

The proof that the invariants J and J_r are nontrivial are along the same lines: Construct a sphere-immersion into Euclidean space with one degenerate self- intersection point as in the local models in Lemma 3.3. Then embed this Euclidean space with immersed sphere into any target manifold and use connected sum together with Lemma 5.1.

Remark 9.1. Applying connected sum to the sphere-immersion constructed above in the case when it is an immersion $S^{2m-1} \to \mathbb{R}^{2m+1}$ shows that for any integer k there exists a generic immersions $f\colon S^{2m-1} \to \mathbb{R}^{2m+1}$ such that $L(f) = (m+1)k$.

Remark 9.2. Let $f_0\colon M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be an immersion in Σ^1 with an $(m+1)$-fold self-intersection point and let f_t, $-\delta < t < \delta$ be a versal deformation of f_0.

Let $\Gamma_\pm = \cup_{i=1}^m \Gamma_i(f_{\pm\delta})$ and $\widetilde{\Gamma}_\pm = f_{\pm\delta}^{-1}(\Gamma_\pm)$. Then if $U_\pm$ are small regular neighborhoods of $\Gamma_\pm$ and $\widetilde{U}_\pm = f_{\pm\delta}^{-1}(U_\pm)$ then there exist diffeomorphisms ϕ and ψ such that the following diagram commutes

$$\begin{tikzcd}
(\widetilde{U}_-, \widetilde{\Gamma}_-) \arrow[r, "\psi"] \arrow[d, "f_{-\delta}"'] & (\widetilde{U}_+, \widetilde{\Gamma}_+) \arrow[d, "f_{+\delta}"] \\
(U_-, \Gamma_-) \arrow[r, "\phi"] & (U_+, \Gamma_+)
\end{tikzcd}.$$

That is, the local properties of a generic immersion close to its self-intersection does not change at instances of $(m+1)$-fold self-intersection points in generic 1-parameter families.

Assume that M fulfills the requirements of Theorem 1. Then Λ or L is defined and $\Lambda(f_{-\delta}) \neq \Lambda(f_\delta)$ or $L(f_{-\delta}) \neq L(f_\delta)$, respectively. This implies that $f_{-\delta}$ and f_δ are not regularly homotopic through generic immersions. Hence, knowledge of the local properties of a generic immersion $M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ close to its self-intersection is not enough to determine it up to regular homotopy through generic immersions.

9.2. **Problems.** *Are the regular homotopy invariants l, λ and j_r nontrivial?*

A negative answer to this question implies restrictions on self-intersection manifold. A positive answer gives non-trivial geometrically defined regular homotopy invariants.

9.3. **Remarks on the invariant l.** Theorem 2 gives information on the possible range of l for sphere-immersions in codimension two. We consider the most interesting cases of immersions $S^{4j-1} \to \mathbb{R}^{4j+1}$, $j \geq 1$ (Theorem 2 (c)). In the first case $S^3 \to \mathbb{R}^5$, l is non-trivial. This was shown in [3]. Hence, for any integer b there are generic immersions $f\colon S^3 \to \mathbb{R}^5$ such that $L(f) = b$.

Remark 9.3. The invariant l is called λ in [3] and is the mod 3 reduction of an integer-valued invariant called lk. The definition of lk given in [3] differs slightly from the definition of L given here. There is an easy indirect way to see that, nonetheless, the two invariants are the same: Due to Theorem 1 and Theorem 2 in [3], $L - \mathrm{lk}$ is an invariant of regular homotopy which is 0 on embeddings and additive under connected summation. Assume that $L(f) - L'(f) = a$ for some immersion. Then, since $\mu_1 = 24$, the connected sum of 24 copies of any immersion is regularly homotopic to an embedding. Hence, $24a = 0$ and therefore $a = 0$. Thus, $L \equiv \mathrm{lk}$.

If $2j+1$ is prime then Theorem 2 does not impose any restrictions on l since, in this case, $2j+1$ divides μ_j (see Milnor and Stasheff [7]).

On the other hand, if none of the prime factors of $2j+1$ divides μ_j, Theorem 2 implies that l is trivial and in such cases Remark 9.1 allows us to determine the range of L which is $(2j+1)\mathbb{Z}$. The first two cases where this happens are $S^{67} \to \mathbb{R}^{69}$ and $S^{107} \to \mathbb{R}^{109}$, where $2j+1$ equals $35 = 5 \cdot 7$ and $55 = 5 \cdot 11$ and μ_j equals $24 = 2^3 \cdot 3$ and $86184 = 2^3 \cdot 3^4 \cdot 7 \cdot 19$, respectively.

9.4. **Remarks on the invariant j_2.** The first case in which to consider the invariant j_2 are immersions of a 5-manifold into an 8-manifold.

Theorem 7.30 in [4] shows that the self-intersection surface of a generic immersions of $S^5 \to \mathbb{R}^8$ must have *even* Euler characteristic (even though it may be non-orientable). Thus, in this case j_2 is trivial. (Note that $\pi_5(V_{8,5}) \cong \mathbb{Z}_2$ so that there are two regular homotopy classes of immersions $S^5 \to \mathbb{R}^8$. Hence, it is non-trivial to see that l is trivial in this case.)

In contrast, the invariant j_2 is non-trivial for immersions $S^5 \to \mathbb{R}P^8$: Clearly, S^5 embeds in $\mathbb{R}P^8$. Thus, there is an immersion f with $j_2(f) = 0$.

Consider three hyperplanes H_1, H_2, H_3 in general position in $\mathbb{R}P^8$. Fix a point $q \in \mathbb{R}P^8 - (H_1 \cup H_2 \cup H_3)$. Note that $X = \mathbb{R}P^8 - \{p\}$ is a non-orientable line bundle over H_i, $i = 1, 2, 3$. Choose sections v_i of X over H_i such that $\{v_i = 0\} \cong \mathbb{R}P^6$ meets $H_1 \cap H_2 \cap H_3$ transversally in H_i for $i = 1, 2, 3$. Now, $H_1 \cap H_2 \cap H_3 \cong \mathbb{R}P^5$. Let $g\colon \mathbb{R}P^5 \to \mathbb{R}P^8$ be the corresponding embedding. Restricting v_i to $\mathbb{R}P^5$ we get three normal vector fields v_1, v_2, v_3 along $\mathbb{R}P^5 \subset \mathbb{R}P^8$ and $\{v_1 = v_2 = v_3 = 0\} \cong \mathbb{R}P^2$.

Let $p\colon S^5 \to \mathbb{R}P^5$ be the universal cover. Then $h = g \circ p\colon S^5 \to \mathbb{R}P^8$ is an immersion. Let $K_i = h^{-1}(\{v_i = 0\}) \cong S^4$. Choose Morse functions $\phi_i\colon S^5 \to \mathbb{R}$ such that $\{\phi_i = 0\} = K_i$. Let $\epsilon > 0$ be small. Then $f\colon S^5 \to \mathbb{R}P^8$, given by

$$f(x) = h(x) + \epsilon \sum_{i=1}^{3} \phi_i(x) v_i(h(x)) \quad \text{for } x \in S^5,$$

is a generic immersion with $\Gamma_2(f) \cong \mathbb{R}P^2$. Hence, $j_2(f) = 1$.

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