FINITE SATISFIABILITY AND $\aleph_0$-CATEGORICAL STRUCTURES WITH TRIVIAL DEPENDENCE

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Introduction

The main subject of the article is the finite submodel property for $\aleph_0$-categorical structures, in particular under the additional assumptions that the structure is simple, 1-based and has trivial dependence. Here, a structure has the finite submodel property if every sentence which is true in the structure is true in a finite substructure of it. It will be useful to consider a couple of other finiteness properties, related to the finite submodel property, which are variants of the usual concept of saturation.

For the rest of the introduction we will assume that M is an $\aleph_0$-categorical (infinite) structure with a countable language. We also assume that there is an upper bound to the arity of the function symbols in M:s language and that, for every $0 < n < \aleph_0$ and $R \subseteq M^n$ which is definable in M without parameters, there exists a relation symbol, in the language of M, which is interpreted as R; these assumptions are not necessary for most results to be presented, but it simplifies the statement of a result which I mention in this introduction.

First we will consider 'canonically embedded' substructures of M^{eq}. Here, a structure N is canonically embedded in M^{eq} if N's universe is a subset of M^{eq} which is definable without parameters and, for every $0 < n < \aleph_0$ and $R \subseteq N^n$ which is $\emptyset$-definable in M^{eq} there is a relation symbol in the language of N which is interpreted as R; we also assume that the language of N has no other relation (or function or constant) symbols. We prove that if $N \subseteq M^{\mathrm{eq}}$ is a structure which is canonically embedded in M^{eq}, only finitely many sorts are represented in N and M is included in the algebraic closure of N (where algebraic closure is taken in M^{eq}), then M has the finite submodel property if and only if N has it; except for the assumptions on the language of M we only assumed that M is $\aleph_0$-categorical.

Then, in Section 3, we show that, under the additional assumptions that M is simple, 1-based and has trivial dependence (which implies that M has finite SU-rank), there exists a structure $N \subseteq M^{\mathrm{eq}}$ such that N is canonically embedded in M^{eq}, only finitely many sorts are represented in N, M is included in the algebraic closure of N and the algebraic closure restricted to N forms a trivial (or degenerate) pregeometry.

Let N be as above and let acl_N denote the algebraic closure in N (which is the same as the algebraic closure in M^{eq} restricted to N), so (N, acl_N) is a pregeometry. Then, to N we may apply results from [4] where $\aleph_0$-categorical structures M' such that the algebraic closure on M' forms a pregeometry are studied. We do this in Section 4 where we draw a conclusion about what happens if N does not have the finite submodel property, in terms of the the main notions studied in [4], namely 'polynomial k-saturation' and 'the k-independence hypothesis' (where $0 < k < \aleph_0$).

As Section 3 introduces the notion of 'height', we prove in Section 5 that the SU-rank of M is at least as great as the height of M. Finally, Section 6 gives a couple of examples which illustrate the notions and constructions from Section 3.

For more results on the finite submodel property in the ω-categorical setting, see [1] and [2] which treat smoothly approximable (or equivalently, Lie coordinatizable) structures. The random graph and random structure have the finite submodel property, which is proved by a probability theoretic argument (see [6], for example). A more general approach which uses a probabilistic argument to show that a structure, satisfying certain conditions, has the finite submodel property is given in [4].

1. Preliminaries

Notation and terminology. We will use more or less standard notation. By $\bar{a}, \bar{b}$ etc. we denote sequences of elements (from some structure, usually); these will always be finite. Occasionally we may consider a sequence $\bar{a}$ as a set (by disregarding the order of the elements in the sequence). With the notation $\bar{a} \in A$ we mean that each element in the sequence $\bar{a}$ belongs to A. For a sequence $\bar{a}$, $|\bar{a}|$ denotes its length; for a set A, $|A|$ denotes its cardinality. Occasionally we use the notation $\mathrm{rng}(\bar{a})$ to denote the set of all elements that occur in $\bar{a}$. Given sets A and B we sometimes write AB instead of $A \cup B$.

We will always assume that the language of any structure that we talk about is countable. For a structure M, the complete theory of M is denoted by $Th(M)$. We write $\mathrm{dcl}_M(A)$, $\mathrm{acl}_M(A)$ and $tp_M(\bar{a}/A)$ for the definable closure of A in M, the algebraic closure of A in M and the type of $\bar{a}$ over A in M; if the subscript 'M' is clear from the context we may drop it. For a complete theory T, let $S_n(T)$ be the set of complete n-types of T. For a subset $A \subseteq M$, let $S_n^M(A)$ denote the set of n-types over A (which are realized in some elementary extension of M).

We say that M is $\aleph_0$-categorical/simple/supersimple if $Th(M)$ is it. We will frequently use the well-known characterization of $\aleph_0$-categorical theories (see [6] for example). An important consequence of this characterization is that if M is $\aleph_0$-categorical and $A \subseteq M$ is finite then $\mathrm{acl}_M(A)$ is finite.

We assume familiarity with M^{eq} but since the distinction between different sorts of elements of M^{eq} will be important here, we clarify now what we mean by a 'sort' in M^{eq}. For every $0 < n < \aleph_0$ and every equivalence relation E on M^n which is $\emptyset$-definable (i.e. definable without parameters) L^{eq} contains a unary relation symbol P_E (which is not in L). By a *sort* (in M^{eq}) we mean a set of the form $S_E = \{a \in M^{\mathrm{eq}} : M^{\mathrm{eq}} \models P_E(a)\}$ for some E as above. If $A \subseteq M^{\mathrm{eq}}$ and there are only finitely many E such that $A \cap S_E \neq \emptyset$ then we say that *only finitely many sorts are represented in A.*

In the next fact (which is Lemma 6.4 of chapter III in [7]), F_E denotes the relation symbol in L^{eq} which in M^{eq} is interpreted as the graph of the function which (assuming that E is a $\emptyset$-definable equivalence relation on M^n) sends $\bar{a} \in M^n$ to the E-equivalence class that $\bar{a}$ belongs to.

Fact 1.1. [7] *For every $\varphi(\bar{x}) \in L^{\mathrm{eq}}$ there is a formula $\theta(\bar{x}) \in L^{\mathrm{eq}}$ such that $\varphi(\bar{x})$ and $\theta(\bar{x})$ are equivalent in M^{eq} and $\theta(\bar{x})$ is a boolean combination of formulas of the following forms:*
(i) $\forall x(x = x)$ *or* $\neg\forall x(x = x)$,
(ii) $x = y$,
(iii) $P_E(x)$,
(iv) $\bigwedge_{i=1}^n P_{E_i}(x_i) \to \forall \bar{y}_1, \ldots, \bar{y}_n \big(\bigwedge_{i=1}^n F_{E_i}(\bar{y}_i, x_i) \to \psi(\bar{y}_1, \ldots, \bar{y}_n)\big)$, *where* $\psi(\bar{y}_1, \ldots, \bar{y}_n)$ *is an L-formula.*

M^{eq} is not $\aleph_0$-categorical (by the well-known characterization [6] of $\aleph_0$-categorical theories), even if M is. We get the following from from Fact 1.1 and the fact that an $\aleph_0$-categorical structure has only finitely many formulas in k free variables (up to equivalence in M) for every $k < \aleph_0$:

Fact 1.2. *If M is $\aleph_0$-categorial and A is a subset of M^{eq} in which only finitely many sorts are represented, then, for any $0 < n < \aleph_0$, only finitely many types from $S_n(Th(M^{\mathrm{eq}}))$ are realized by n-tuples in A^n.*

A consequence of Fact 1.2 is the following:

Fact 1.3. *Suppose that M is $\aleph_0$-categorical and that A is a subset of M^{eq} such that only finitely many sorts are represented in A. If $B \subseteq M^{\mathrm{eq}}$ is finite then $\mathrm{acl}_{M^{\mathrm{eq}}}(B) \cap A$ is finite.*

Any definable set $N \subseteq M^{\mathrm{eq}}$ may be considered as a structure in a language which, for every $0 < n < \aleph_0$ and every relation $R \subseteq N^n$ which is $\emptyset$-definable in M^{eq}, contains a relation symbol which is interpreted as R; and we assume that the language of N has no other relation (or function or constant) symbols.

Definition 1.4. If a $\emptyset$-definable set $N \subseteq M^{\mathrm{eq}}$ is considered as a structure in the way just described above then we say that N is *canonically embedded* in M^{eq}; this definition is stronger than the one given in [1] since we require that N is definable (in M^{eq}) without parameters.

Note that if N is canonically embedded in M^{eq} then N has elimination of quantifiers. Also observe that if N is canonically embedded in M^{eq} and $\bar{a}, \bar{b} \in N$ then $tp_N(\bar{a}) = tp_N(\bar{b})$ if and only if $tp_{M^{\mathrm{eq}}}(\bar{a}) = tp_{M^{\mathrm{eq}}}(\bar{b})$. By Fact 1.1 we get:

Fact 1.5. *Suppose that M is $\aleph_0$-categorical. If $N \subseteq M^{\mathrm{eq}}$ is a canonically embedded structure in which only finitely many sorts are represented then N is $\aleph_0$-categorical.*

From Fact 1.5 we easily derive the following:

Fact 1.6. *Suppose that M is $\aleph_0$-categorical.*
(i) For any $\bar{a} \in M^{\mathrm{eq}}$, $tp_{M^{\mathrm{eq}}}(\bar{a})$ is isolated
(ii) M^{eq} is $\aleph_0$-homogeneous.

Proof. (i) Let $\bar{a} \in M^{\mathrm{eq}}$ and let

$$N = \{b \in M^{\mathrm{eq}} : \ b \text{ belongs to the same sort as some element in } \bar{a}\}.$$

Then N is $\emptyset$-definable, so we may regard N as a canonically embedded structure. By Fact 1.5, N is $\aleph_0$-categorical, so $tp_N(\bar{a})$ is isolated. As noted above, if $\bar{b}, \bar{c} \in N$ then $tp_N(\bar{b}) = tp_N(\bar{c})$ if and only if $tp_{M^{\mathrm{eq}}}(\bar{b}) = tp_{M^{\mathrm{eq}}}(\bar{c})$, so $tp_{M^{\mathrm{eq}}}(\bar{a})$ must be isolated.
(ii) is an immediate consequence of (i). □

If $\bar{a} \in M^{\mathrm{eq}}$ and $A \subseteq M^{\mathrm{eq}}$ then $\mathrm{SU}_M(\bar{a}/A)$ denotes the SU-rank of the type $tp_{M^{\mathrm{eq}}}(\bar{a}/A)$; and $\mathrm{SU}_M(\bar{a})$ means $\mathrm{SU}_M(\bar{a}/\emptyset)$; as usual we may sometimes drop the subscript 'M'.

We define the SU-*rank* of a simple structure M to be $\sup\{\mathrm{SU}(a) : a \in M\}$, if the supremum exists (which it will in the context where it will be used). We say that M has *finite* SU-*rank* if this supremum is finite. Observe that if M has finite SU-rank and is $\aleph_0$-categorical then every N such that $N \equiv M$ is $\aleph_0$-saturated and hence every such N has finite SU-rank.

An $\aleph_0$-categorical and simple (complete) theory has elimination of hyperimaginaries (by [8], Theorem 6.1.9, for instance) and therefore the bounded closure and the algebraic closure are the same thing; this is the reason why Fact 1.8 holds, although the definition of 1-basedness, given below, only speaks about algebraic closure.

Definition 1.7. Let T be an $\aleph_0$-categorical and simple (complete) theory.
(i) T is *1-based* if, whenever $M \models T$ and $A, B \subseteq M^{\mathrm{eq}}$, then A is independent from B over $\mathrm{acl}_{M^{\mathrm{eq}}}(A) \cap \mathrm{acl}_{M^{\mathrm{eq}}}(B)$.
(ii) T has *trivial dependence* if, whenever $M \models T$, $A, B, C_1, C_2 \subseteq M^{\mathrm{eq}}$ and $A \not\downarrow_B C_1 C_2$, then $A \not\downarrow_B C_i$ for $i = 1$ or for $i = 2$.
(iii) We say that M is *1-based* (or has *trivial dependence*) if $Th(M)$ is 1-based (or has trivial dependence).

By the finite character of forking, if, in (ii) of Definition 1.7, we would only ask the given condition to be fulfilled when C_1 and C_2 are finite, then the resulting definition of triviality would be equivalent with the one given. The following fact is part of Corollary 4.7 in [5], where the terminology is different from here (the term 'modular theory' is used instead of '1-based theory'):

Fact 1.8. *If M is an $\aleph_0$-categorical, simple and 1-based structure then M is supersimple and has finite SU-rank.*

For a more detailed understanding (which will not be needed here) of the relationships between 1-basedness, trivial dependence and types of SU-rank 1, the reader is refered to Corollary 4.7 in [5] and Lemma 3.22 in [3].

The Lascar inequalities ([8], Theorem 5.1.6, for instance) imply the following fact which we call the "Lascar equation":

Fact 1.9. (Lascar equation) *If M is supersimple, $\bar{a}, \bar{b} \in M^{\mathrm{eq}}$, $A \subseteq M^{\mathrm{eq}}$ and $\mathrm{SU}_M(\bar{a}, \bar{b}/A) < \aleph_0$, then*

$$\mathrm{SU}_M(\bar{a}, \bar{b}/A) = \mathrm{SU}_M(\bar{a}/A\bar{b}) + \mathrm{SU}_M(\bar{b}/A).$$

A basic fact about SU-rank is that if $\bar{a}, \bar{b} \in M^{\mathrm{eq}}$, $A \subseteq M^{\mathrm{eq}}$ and $\bar{b} \in \mathrm{acl}_{M^{\mathrm{eq}}}(\bar{a}A)$ then $\mathrm{SU}_M(\bar{a}, \bar{b}/A) = \mathrm{SU}_M(\bar{a}/A)$. Suppose that M has finite SU-rank. By the Lascar equation it follows that $\mathrm{SU}_M(\bar{a})$ is finite for every $\bar{a} \in M$. Since every $\bar{b} \in M^{\mathrm{eq}}$ is included in $\mathrm{dcl}_{M^{\mathrm{eq}}}(\bar{a})$ for some $\bar{a} \in M$, it follows from the Lascar equation that $\mathrm{SU}_M(\bar{b})$ is finite for every $\bar{b} \in M^{\mathrm{eq}}$.

Definition 1.10. An L-theory T has the *finite submodel property* if the following holds for any $M \models T$ and sentence $\varphi \in L$: If $M \models \varphi$ then there is a finite substructure $N \subseteq M$ such that $N \models \varphi$. A structure M has the *finite submodel property* if whenever φ is a sentence such that $M \models \varphi$, then there is a finite substructure $N \subseteq M$ such that $N \models \varphi$.

If $Th(M)$ has the finite submodel property then clearly M has it. The opposite direction holds if the language contains only finitely many relation, function and constant symbols; because if φ is true in a finite substructure $A = \{a_1, \ldots, a_n\} \subseteq M$ then, by the assumption on the language, there is a quantifier free formula $\psi_A(x_1, \ldots, x_n)$ which describes the isomorphism type of A, so $\exists x_1, \ldots, x_n \psi_A$ belongs to $Th(M)$ and hence A is embeddable in every model of $Th(M)$.

In Sections 3 and 4 we will speak about pregeometries; for a definition, the reader is refered to [6], for instance.

2. The finite submodel property and saturation

In this section the main result is that if there is an upper bound to the arity of all function symbols in the language of M and M is $\aleph_0$-categorical and there exists a canonically embedded structure $N \subseteq M^{\mathrm{eq}}$ such that $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$ and N has the finite submodel property, then M has the finite submodel property. If the language of M satisfies an additional condition then the reverse holds (i.e. if M has the finite submodel property then so does N). We will use a lemma which relates the finite submodel property with saturation, as we define it below.

Definition 2.1. (i) Let κ be a cardinal. A subset $B \subseteq M$ is *κ-saturated with respect to M* if, whenever $A \subseteq B$, $|A| < \kappa$ and $p(x) \in S_1^M(A)$, then there is $a \in B$ such that $M \models p(a)$. A substructure of M is κ-saturated (with respect to M) if its universe is. As indicated, the definitions depends on the model M in which we evaluate truth of formulas.
(ii) If L is a first order language and $k < \aleph_0$ then let L^k denote the set of formulas from L in which at most k variables occur (whether free or bound).

The next lemma shows how saturation is related to the finite submodel property.

Lemma 2.2. *The following are equivalent:*
(i) *For every $k < \aleph_0$, M has a finite substructure which is k-saturated with respect to M.*
(ii) *M is $\aleph_0$-categorical and if $\widehat{M}$ is the expansion of M obtained by, for every $k < \aleph_0$ and every type $p(\bar{x}) \in S_k(Th(M))$, adding a relation symbol R_p such that, for every $\bar{a} \in M^k$, $\widehat{M} \models R_p(\bar{a}) \Leftrightarrow M \models p(\bar{a})$, then $\widehat{M}$ has the finite submodel property.*

Proof. Suppose that (i) holds. If M is not $\aleph_0$-categorical then for some $n < \aleph_0$ there are infinitely many n-types (over $\emptyset$) so M has no finite n-saturated substructure, contradicting our assumption.

Now we show that $\widehat{M}$ has the finite submodel property. Let $n < \aleph_0$ be arbitrary. Let L be the language of M and let $\widetilde{M}$ be the expansion of M obtained by, for every $n' \leq n$ and every $p(\bar{x}) \in S_{n'}(Th(M))$, adding a relation symbol R_p such that, for every $\bar{a} \in M^{n'}$, $\widetilde{M} \models R_p(\bar{a}) \Leftrightarrow M \models p(\bar{a})$; so $\widetilde{M}$ is a reduct of $\widehat{M}$. Let $\widetilde{L}$ be the language of $\widetilde{M}$. Since $Th(M)$ is $\aleph_0$-categorical, every type of $S_{n'}(Th(M))$ (for $n' \leq n$) is isolated and therefore, for every $\varphi(\bar{x}) \in \widetilde{L}$ (where $\bar{x}$ may have arbitrary length) there is $\psi(\bar{x}) \in L$ such that $\widetilde{M} \models \forall \bar{x}\big(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})\big)$. So for every $k < \aleph_0$, there are up to equivalence in $Th(\widetilde{M})$ only finitely many formulas with k free variables, so $\widetilde{M}$ is $\aleph_0$-categorial.

Hence, there exists $m < \aleph_0$ such that $m \geq n$ and, for every $n' \leq n$ and every $p(\bar{x}) \in S_{n'}(Th(\widetilde{M}))$, $p(\bar{x})$ is isolated by a formula $\varphi_p(\bar{x}) \in \widetilde{L}^m$. By (i) there is a finite substructure $A \subseteq M$ which is m-saturated with respect to M. Let $\widetilde{A}$ be the substructure of $\widetilde{M}$ with the same universe as A

Claim: For every $\bar{a} \in A$ and $\psi(\bar{x}) \in \widetilde{L}^m$, $\widetilde{M} \models \psi(\bar{a})$ if and only if $\widetilde{A} \models \psi(\bar{a})$.

We prove the claim by induction on the complexity of formulas. For atomic formulas the claim is trivial. The induction step for $\wedge$ and $\neg$ is straightforward so we only do the induction step for $\exists$. Suppose that $\exists y \psi(\bar{x}, y) \in \widetilde{L}^m$, where we may assume that no variable occurs twice in $\bar{x}$ and that y is different from all variables in $\bar{x}$, so $|\bar{x}| < m$. Suppose that $\bar{a} \in A$ and $\widetilde{M} \models \exists y \psi(\bar{a}, y)$. Let $\psi^*(\bar{x}, y)$ be the formula which is obtained by, for every $n' \leq n$ and every n'-type p, replacing every occurrence of $R_p(t_1, \ldots, t_{n'})$ in $\psi(\bar{x}, y)$ by $\varphi_p(t_1, \ldots, t_{n'})$, where the t_i are terms. Since, for any $n' \leq n$ and n'-type $p \in S_{n'}(Th(M))$

$$\widetilde{M} \models \forall x_1, \ldots, x_{n'}\big(R_p(x_1, \ldots, x_{n'}) \leftrightarrow \varphi_p(x_1, \ldots, x_{n'})\big),$$

and $\widetilde{M} \models \exists y \psi(\bar{a}, y)$, it follows (see [6], Theorem 2.6.4, for instance) that $\widetilde{M} \models \exists y \psi^*(\bar{a}, y)$. As $\psi^*(\bar{x}, y) \in L$ we get $M \models \exists y \psi^*(\bar{a}, y)$. By the m-saturation of A there is $b \in A$ such that $M \models \psi^*(\bar{a}, b)$ which, by a similar reasoning as above, implies $\widetilde{M} \models \psi(\bar{a}, b)$. By the induction hypothesis we get $\widetilde{A} \models \psi(\bar{a}, b)$ and we are done. The reverse direction is similar but simpler, so we leave it to the reader.

By the claim, $\widetilde{M}$ and $\widetilde{A}$ satisfy the same $\widetilde{L}^m$-sentences, so in particular the same $\widetilde{L}^n$-sentences. Since $n < \aleph_0$ is arbitrary and $\widetilde{M}$ and $\widehat{M}$ satisfy the same $\widetilde{L}$-sentences it follows that $\widehat{M}$ has the finite submodel property.

Now assume that (ii) holds. Let $k < \aleph_0$ be arbitrary. For any $p(x_1, \ldots, x_k) \in S_k(Th(M))$ let $p_{\restriction k-1}$ be the restriction of p to the variables $x_1, \ldots, x_{k-1}$. By the $\aleph_0$-categorictity of M, $S_k(Th(M))$ is finite so

$$\phi = \bigwedge_{p(x_1,\ldots,x_k) \in S_k(Th(M))} \forall x_1, \ldots, x_{k-1}\Big(R_{p_{\restriction k-1}}(x_1, \ldots, x_{k-1}) \rightarrow \exists x_k(R_p(x_1, \ldots, x_k)\Big)$$

is a first order sentence which, by the finite submodel property of $\widehat{M}$, is true in a finite submodel $\widehat{A} \subseteq \widehat{M}$. Let $\bar{a} \in A^{k-1}$ and $b \in M$ and suppose that $M \models p(\bar{a}, b)$, where $p \in S_k(Th(M))$. Then $M \models p_{\restriction k-1}(\bar{a})$ so $\widehat{M} \models R_{p_{\restriction k-1}}(\bar{a})$ and therefore $\widehat{A} \models R_{p_{\restriction k-1}}(\bar{a})$. Since $\widehat{A} \models \phi$ there is $c \in A$ such that $\widehat{A} \models R_p(\bar{a}, c)$ and hence $\widehat{M} \models R_p(\bar{a}, c)$ and therefore $M \models p(\bar{a}, c)$. Since $k < \aleph_0$ was arbitrary (i) follows. □

Remark 2.3. I have not been able to prove that (i) in Lemma 2.2 follows only from the assumption that M has the finite submodel property, nor have I been able to show the necessity of assuming that $\widehat{M}$, and not just M, has the finite submodel property. Therefore, I don't know whether the assumptions on the language in (ii) of Corollary 2.5 are necessary.

Theorem 2.4. *Suppose that M is $\aleph_0$-categorical and that $N \subseteq M^{\mathrm{eq}}$ is a canonically embedded structure such that only finitely many sorts are represented in N and $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$. The following are equivalent:*
(i) For every $k < \aleph_0$, M has a finite subset which is k-saturated (with respect to M).
(ii) For every $k < \aleph_0$, N has a finite subset which is k-saturated (with respect to N).

Proof. Suppose that M is $\aleph_0$-categorical and that $N \subseteq M^{\mathrm{eq}}$ is a canonically embedded structure such that only finitely many sorts are represented in N and $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$. We will show that if, for every $k < \aleph_0$, N has a finite k-saturated subset, then the same is true about M. Since only finitely many sorts (only one) are represented in M and $N \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(M)$ the reverse direction can be proved by switching places of M and N in the proof that follows. In the proof we use, without mentioning it, that, since N is canonically embedded in M^{eq}, if $\bar{a}, \bar{b} \in N$ then $tp_{M^{\mathrm{eq}}}(\bar{a}) = tp_{M^{\mathrm{eq}}}(\bar{b})$ if and only if $tp_N(\bar{a}) = tp_N(\bar{b})$.

Suppose that N has a finite subset which is k-saturated with respect to N, for every $k < \aleph_0$. Let $k < \aleph_0$. We will show that M has a finite subset which is k-saturated (in M). Choose $k_1, k_2, k_3 < \aleph_0$ in such a way that:

- Any element from M is in the algebraic closure of a tuple of elements from N which has length at most k_1.
- If $\bar{a} \in N$ and $|\bar{a}| \leq kk_1$ then $\left|\mathrm{acl}_{M^{\mathrm{eq}}}(\bar{a}) \cap M\right| \leq k_2$.
- For any $A \subseteq M$ with $|A| \leq k_2$, the number of $2kk_1$-types over A (i.e. types in $S^{M^{\mathrm{eq}}}_{2kk_1}(A)$) which are realized in N is at most k_3 (recall Fact 1.2).

By assumption there exists a finite substructure $N' \subseteq N$ such that N' is $(k_3+2)kk_1$-saturated. Define

$$M' = \left\{a \in M : \text{ there is } \bar{b} \in N', |\bar{b}| \leq k_1 \text{ and } a \in \mathrm{acl}_{M^{\mathrm{eq}}}(\bar{b})\right\}.$$

Then M' is a finite subset of M. It remains to show that M' is k-saturated.

Suppose that $\bar{a} \in M'$, $|\bar{a}| < k$ and $b \in M$. We will find $c \in M'$ such that $tp_M(\bar{a}, c) = tp_M(\bar{a}, b)$. By the choice of k_1, the definition of M' and the assumptions on $\bar{a}$, there is $\bar{a}^* \in N'$ such that $\bar{a} \in \mathrm{acl}_{M^{\mathrm{eq}}}(\bar{a}^*)$ and $|\bar{a}^*| \leq kk_1$. By the choice of k_1, there is $\bar{b}^* \in N$ such that $b \in \mathrm{acl}_{M^{\mathrm{eq}}}(\bar{b}^*)$ and $|\bar{b}^*| \leq k_1$. By the choice of k_2 we have $\left|\mathrm{acl}_{M^{\mathrm{eq}}}(\bar{a}^*) \cap M\right| \leq k_2$. Now we carry on a construction which will terminate after a finite number of steps and give us the $c \in M'$ that we are looking for.

Step 1: Since N' is $(k_3+2)kk_1$-saturated there exists $\bar{c}^*_1 \in N'$ such that $tp_{M^{\mathrm{eq}}}(\bar{a}^*, \bar{c}^*_1) = tp_{M^{\mathrm{eq}}}(\bar{a}^*, \bar{b}^*)$. We get two cases:

Case (a)$_1$: Suppose that $tp_{M^{\mathrm{eq}}}(\bar{a}^*, \bar{a}, \bar{c}^*_1) = tp_{M^{\mathrm{eq}}}(\bar{a}^*, \bar{a}, \bar{b}^*)$.
By Fact 1.6, M^{eq} is $\aleph_0$-homogeneous so there exists c such that $tp_{M^{\mathrm{eq}}}(\bar{a}^*, \bar{a}, \bar{c}^*_1, c) =$

$tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{b}^*, b)$, so in particular $tp_M(\bar{a}, c) = tp_M(\bar{a}, b)$ and $c \in \mathrm{acl}_{M^{\rm eq}}(\bar{c}_1^*) \cap M$ (because $b \in \mathrm{acl}_{M^{\rm eq}}(\bar{b}^*) \cap M$). Since $\bar{c}_1^* \in N'$ and $|\bar{c}_1^*| = |\bar{b}^*| \leq k_1$ we have $c \in M'$. So we are done and the construction terminates.

Case $(\mathrm{b})_1$: Suppose that $tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{c}_1^*) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{b}^*)$.
Recall that $tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_1^*) = tp_{M^{\rm eq}}(\bar{a}^*, \bar{b}^*)$. Since $\bar{a} \in \mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M$ the assumption gives

$$tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_1^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{b}^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M).$$

Now we proceed to step 2.

Step $n+1$ *(where* $n \geq 1$*)*: Suppose that $\bar{c}_1^*, \ldots, \bar{c}_n^* \in N'$ are defined and suppose that

(1) $tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_i^*) = tp_{M^{\rm eq}}(\bar{a}^*, \bar{b}^*)$ whenever $1 \leq i \leq n$,
(2) $tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_i^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{b}^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M)$ whenever $1 \leq i \leq n$, and
(3) $tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_i^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_j^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M)$ whenever $1 \leq i < j \leq n$; if $n = 1$ then this last condition is omitted.

If $n > 1$ then from (3) and the choice of k_3 it follows that $n \leq k_3$; if $n = 1$ then, as $k_3 \geq 1$, we automatically have $n \leq k_3$. By (1), we have $|\bar{c}_i^*| = |\bar{b}^*| \leq k_1$ for $1 \leq i \leq n$. Since N' is $(k_3+2)kk_1$-saturated there exists $\bar{c}_{n+1}^* \in N'$ which realizes

$$p(\bar{x}) = tp_{M^{\rm eq}}(\bar{b}^*/\bar{a}^*, \bar{c}_1^*, \ldots, \bar{c}_n^*).$$

Since $\bar{c}_{n+1}^*$ realizes $p(\bar{x})$ it follows from (2) that

$$tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_{n+1}^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_i^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M) \text{ whenever } 1 \leq i \leq n.$$

Now we have to cases:

Case $(\mathrm{a})_{n+1}$: Suppose that $tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{c}_{n+1}^*) = tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{b}^*)$.
As is case $(\mathrm{a})_1$ we find $c \in M'$ such that $tp_M(\bar{a}, c) = tp_M(\bar{a}, b)$ and the construction terminates.

Case $(\mathrm{b})_{n+1}$: Suppose that $tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{c}_{n+1}^*) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{a}, \bar{b}^*)$.
As $\bar{a} \in \mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M$ we get

$$tp_{M^{\rm eq}}(\bar{a}^*, \bar{c}_{n+1}^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M) \neq tp_{M^{\rm eq}}(\bar{a}^*, \bar{b}^*/\mathrm{acl}_{M^{\rm eq}}(\bar{a}^*) \cap M).$$

Now (1), (2), (3) are satisfied with $n+1$ in place of n, so we may proceed with step $n+2$.

Observe that, for any $n \geq 1$, we will proceed with step $n+1$ only if (1)-(3) and cases $(\mathrm{b})_1, \ldots, (\mathrm{b})_n$ hold and in this situation we must have $n \leq k_3$, as explained above. Therefore, for some $n \leq k_3 + 1$ case $(\mathrm{a})_n$ will hold and we find $c \in M'$ such that $tp_M(\bar{a}, c) = tp_M(\bar{a}, b)$. □

Corollary 2.5. *Suppose that M is $\aleph_0$-categorical and that $N \subseteq M^{\rm eq}$ is a canonically embedded structure such that only finitely many sorts are represented in N and $M \subseteq \mathrm{acl}_{M^{\rm eq}}(N)$. Also assume that for some $r < \aleph_0$, every function symbol in the language of M has arity at most r.*
(i) If N has the finite submodel property then so does M.
(ii) Suppose that for every formula $\varphi(\bar{x})$ (without parameters) in the language of M, there is a relation symbol R in the language of M such that $R^M = \{\bar{a} : M \models \varphi(\bar{a})\}$. Then M has the finite submodel property if and only if N has the finite submodel property.

Proof. (i) Suppose that N the finite submodel property. By Lemma 2.2 (and the assumption that N is canonically embedded in $M^{\rm eq}$) it follows that, for every $k < \aleph_0$, N has a finite subset which is k-saturated with respect to N. By Theorem 2.4, for every

$k < \aleph_0$, M has a finite subset which is k-saturated with respect to M. Since there is, by assumption, an upper bound to the arity of any function symbol in the language of M, it follows that for all k large enough any k-saturated subset of M will be a substructure of M. Thus, for every $k < \aleph_0$, M has a k-saturated substructure. By Lemma 2.2, M has the finite submodel property.

(ii) Suppose that the language of M satisfies the additional condition mentioned in (ii), and that M has the finite submodel property. By Lemma 2.2 (and the assumption about M:s language), for every $k < \aleph_0$, M has a finite substructure which is k-saturated. By Theorem 2.4, for every $k < \aleph_0$, N has a finite subset which is k-saturated. Since the language of N (being canonically embedded in M^{eq}) has no constant or function symbols, any subset is also a substructure. So by Lemma 2.2 N has the finite submodel property. □

Remark 2.6. The condition on the language of M in part (ii) of Corollary 2.5 automatically holds for N in the same theorem, because N is assumed to be canonically embedded in M^{eq}; see Definition 1.4. As mentioned in Remark 2.3, I don't know whether this condition on M is necessary for the conclusions of (ii).

3. Trivial dependence and canonically embedded pregeometries

We will show that if M is $\aleph_0$-categorical, simple, 1-based with trivial dependence, then there is a canonically embedded structure $N \subseteq M^{\mathrm{eq}}$ such that $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$, only finitely many sorts are represented in N and (N, acl_N) forms a trivial pregeometry. (The terminology 'degenerate' or 'disintegrated' pregeometry is also used.) Hence, by Corollary 2.5, if M does not have the finite submodel property then neither does N.

This result will be proved via a sequence of constructions and lemmas. Throughout this section we assume that M is $\aleph_0$-categorical, simple, 1-based with trivial dependence. From Fact 1.8 it follows that M is supersimple with finite SU-rank. We assume that M is elementarily embedded in a "monster model" $\mathfrak{M}$ which is at least $|M|^+$-saturated. We may naturally identify M^{eq} with an elementary substructure of $\mathfrak{M}^{\mathrm{eq}}$ (see [6], Theorem 4.3.3).

Since we will only consider types over subsets of M^{eq} we will never have a reason to go outside of $\mathfrak{M}^{\mathrm{eq}}$ when looking for realizations of such types. Recall that, by Fact 1.6, M^{eq} and $\mathfrak{M}^{\mathrm{eq}}$ are $\aleph_0$-homogeneous.

Notation for this section. If $\bar{a} \in M^{\mathrm{eq}}$ and $A \subseteq M^{\mathrm{eq}}$ then $tp(\bar{a}/A)$, $\mathrm{acl}(A)$ and $\mathrm{SU}(\bar{a}/A)$ mean $tp_{M^{\mathrm{eq}}}(\bar{a}/A)$, $\mathrm{acl}_{M^{\mathrm{eq}}}(A)$ and $\mathrm{SU}_M(\bar{a}/A)$, respectively. If $a, b \in M^{\mathrm{eq}}$ then $a < b$ is an abbreviation for '$a \in \mathrm{acl}(b)$ and $b \notin \mathrm{acl}(a)$'.

The following special case of a well-known result (see [8] for example), which we prove for completeness, will be used:

Lemma 3.1. *If $a \in M^{\mathrm{eq}}$, $A \subseteq M^{\mathrm{eq}}$ and $\mathrm{SU}(a/A) > 1$ then there is $b \in \mathrm{acl}(a)$ such that $\mathrm{SU}(a/Ab) = 1$, and consequently $\mathrm{SU}(b/A) = \mathrm{SU}(a) - 1$.*

Proof. Suppose that $\mathrm{SU}(a/A) = r > 1$ where $a \in M^{\mathrm{eq}}$ and $A \subset M^{\mathrm{eq}}$. By the finite character of forking there is a finite set $B \subset \mathfrak{M}^{\mathrm{eq}}$ such that $\mathrm{SU}(a/AB) = 1$. By 1-basedness, a is independent from AB over $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) \cap \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(AB)$, so

$$\mathrm{SU}\big(a/\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) \cap \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(AB)\big) = \mathrm{SU}(a/AB) = 1.$$

As M is supersimple there is finite $B' \subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) \cap \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(AB)$ such that $\mathrm{SU}(a/B') = 1$. Then $1 = \mathrm{SU}(a/AB) \leq \mathrm{SU}(a/AB') \leq \mathrm{SU}(B') = 1$ so $\mathrm{SU}(a/AB') = 1$. Since $a \in M^{\mathrm{eq}}$ we have $B' \subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{acl}_{M^{\mathrm{eq}}}(a) \subseteq M^{\mathrm{eq}}$ and since B' is finite there is $b \in M^{\mathrm{eq}}$ such that $\mathrm{acl}(b) = \mathrm{acl}(B')$. Then $\mathrm{SU}(a/Ab) = 1$ and by the Lascar equation we must have $\mathrm{SU}(b/A) = r - 1$. □

Remark 3.2. Suppose that $a, b \in M^{\mathrm{eq}}$, $b \in \mathrm{acl}(a)$ and $\mathrm{SU}(a/b) = 1$. If $c \in M^{\mathrm{eq}}$ is such that $c < a$ and $c \notin \mathrm{acl}(b)$ then $a \in \mathrm{acl}(b, c)$, since otherwise we would have

$$1 = \mathrm{SU}(a/b) = \mathrm{SU}(ac/b) = \mathrm{SU}(a/cb) + \mathrm{SU}(c/b) \geq 1 + 1.$$

Definition 3.3. We will say that a set $A \subseteq M^{\mathrm{eq}}$ is *self-coordinatized* if the following holds:

(1) If $a \in A$ and $\mathrm{SU}(a) > 1$ then there is $b \in A \cap \mathrm{acl}(a)$ such that $\mathrm{SU}(a/b) = 1$ (and hence $\mathrm{SU}(b) = \mathrm{SU}(a) - 1$).

(2) If $a, b \in A$, $\mathrm{SU}(a) > 1$, $b \in A \cap \mathrm{acl}(a)$, $\mathrm{SU}(a/b) = 1$ and there exists $c \in M^{\mathrm{eq}}$ such that $c < a$ and $c \notin \mathrm{acl}(b)$ then such c exists in A.

We say that an element $a \in A$ is *coordinatized in* A if (1) and (2) hold for a; so if $\mathrm{SU}(a) = 1$ then a is coordinatized in A. If A is *not* self-coordinatized then let

$$\delta(A) = \sup \big\{ \mathrm{SU}(a) : a \in A \text{ and } a \text{ is } \textit{not} \text{ coordinatized in } A \big\},$$

and note that $\delta(A) \geq 2$ in this case. If A is self-coordinatized let $\delta(A) = 1$.

Lemma 3.4. *Suppose that $A \subseteq M^{\mathrm{eq}}$ is not self-coordinatized and that only finitely many sorts are represented in A.*
Then there is $B \subseteq M^{\mathrm{eq}}$ such that
(i) $A \subseteq B$,
(ii) $\delta(B) < \delta(A)$,
(iii) only finitely many sorts are represented in B, and
(iv) if $a \in B$, $b \in M^{\mathrm{eq}}$ and $tp(a) = tp(b)$ then $b \in B$.
In particular it follows that B is $\emptyset$-definable (in M^{eq}).

Proof. Suppose that $A \subseteq M^{\mathrm{eq}}$ is not self-coordinatized and that only finitely many sorts are represented in A. We may assume that (iv) is satisfied for A, since otherwise we can add the missing elements. Let B' include A, and in addition, elements obtained in the following way: For every $a \in A$ which does not satisfy (1) in Definition 3.3, add to B' (by Lemma 3.1) one element b so that (1) is satisfied for a, and note that $\mathrm{SU}(b) < \mathrm{SU}(a)$. Then $\delta(B') \leq \delta(A)$ and, by the $\aleph_0$-categorcity of M, we may assume that (iii) is satisfied for B' and by adding missing elements we may also assume that (iv) holds. Now let B include B' and in addition contain elements obtained in the following way: For every pair $a \in A, b \in B'$ which satisfy (1) (in Definition 3.3) and for which there is $c \in M^{\mathrm{eq}}$ such that $c < a$ and $c \notin \mathrm{acl}(b)$, but no such c exists in B', add one such c to B; from the Lascar equation it follows that $\mathrm{SU}(c) < \mathrm{SU}(a)$. Then $\delta(B) < \delta(A)$ because if $a \in A$ is not coordinatized in A then a is coordinatized in B. In the same way as for B' we may assume that (iii) and (iv) hold for B; this can only add elements with rank less than $\delta(A)$. By Fact 1.6, for any $a \in B$, $tp(a)$ is isolated. Since we made sure that (iii) and (iv) hold for B it follows from Fact 1.2 that B is $\emptyset$-definable. □

Construction 3.5. (i) By Lemma 3.4 and induction we find a self-coordinatized $\emptyset$-definable set $C \subseteq M^{\mathrm{eq}}$ such that $M \subseteq C$, only finitely many sorts are represented in C and if $c \in C$, $c' \in M^{\mathrm{eq}}$ and $tp_M(c) = tp_M(c')$ then $c' \in C$. As a consequence, only finitely many 1-types (i.e. types in $S_1(Th(M^{\mathrm{eq}}))$) are realized in C. By Fact 1.9 and the discussion following it we have $\mathrm{SU}(a) < \aleph_0$ for every $a \in M^{\mathrm{eq}}$. Therefore there is $m < \aleph_0$ such that if $c_0, \ldots, c_n \in C$ and $c_0 < \ldots < c_n$ then $n \leq m$.
(ii) We define subsets $C_n \subseteq C$ inductively by: $C_0 = \emptyset$ and if C_n is defined and $C \not\subseteq \mathrm{acl}(C_n)$ then

$$C_{n+1} = C_n \cup \big\{ c \in C - \mathrm{acl}(C_n) : \text{ there exists no } c' \in C - \mathrm{acl}(C_n) \text{ such that } c' < c \big\}.$$

If $C \subseteq \operatorname{acl}(C_n)$ then C_{n+1} is not defined. Since $C_0 = \emptyset$ (by definition) and M is infinite and $\aleph_0$-categorical it follows that C_1 is defined. As noted in part (i), there is a natural number which bounds the length of any 'descending chain' (with respect to $<$) in C, so it follows that if C_{n+1} is defined then $C_n \subset C_{n+1}$, where the inclusion is proper. From Lemma 3.7 below, it follows that there exists $0 < r < \aleph_0$ such that C_{r+1} is undefined.

Lemma 3.6. *Suppose that C_n is defined. If $c \in C_n$, $c' \in M^{\mathrm{eq}}$ and $tp(c') = tp(c)$ then $c' \in C_n$.*

Proof. Immediate since C_n is $\emptyset$-definable, which is explained in Remark 3.9 below. □

Lemma 3.7. *There exists $0 < r < \aleph_0$ such that C_r is defined but C_{r+1} is not defined, and hence $C \subseteq \operatorname{acl}(C_r)$.*

Proof. We observed in Construction 3.5 (ii) that if C_{n+1} is defined then $C_n \subset C_{n+1}$ (where the inclusion is proper). So from Lemma 3.6 it follows that at least one more 1-type is realized in C_{n+1} than in C_n. But, as noted in Construction 3.5 (i), only finitely many 1-types are realized in C and since $C_{n+1} \subseteq C$ the same holds for C_{n+1}. It follows that there exists $0 < r < \aleph_0$ such that C_r is defined but C_{r+1} is not defined. □

Construction 3.8. Let $0 < r < \aleph_0$ be such as in Lemma 3.7. In other words, C_r is defined but C_{r+1} is not defined which implies that $C \subseteq \operatorname{acl}(C_r)$. As $M \subseteq C$ we in particular have $M \subseteq \operatorname{acl}(C_r)$. We call r the *height* of C.

Remark 3.9. If $N \equiv M$ then N^{eq} contains a self-coordinatized set C' which contains N and has height r. The reason is the following. From Fact 1.6 (i), which says that $tp(\bar{a})$ is isolated for any $\bar{a} \in M^{\mathrm{eq}}$, it follows that if $A \subseteq M^{\mathrm{eq}}$ is $\emptyset$-definable then the following relations are $\emptyset$-definable:

$$\begin{aligned} P(x,y) &\iff x, y \in A \text{ and } \mathrm{SU}(x/y) = 1, \\ Q(x) &\iff x \in A \text{ and } \mathrm{SU}(x) > 1, \\ R_n(x, y_1, \ldots, y_n) &\iff x, y_1, \ldots, y_n \in A \text{ and } x \in \operatorname{acl}(y_1, \ldots, y_n), \text{ for } n < \aleph_0. \end{aligned}$$

Since only finitely many 1-types are realized in C there is $k < \aleph_0$ such that, for any $n \leq r$ and $c \in C$, if $c \in \operatorname{acl}(C_n)$ then there are $a_1, \ldots, a_k \in C_n$ such that $c \in \operatorname{acl}(a_1, \ldots, a_k)$. From the $\emptyset$-definability of P, Q, R_n, $n < \aleph_0$, in the case $A = C$, it follows that for each $n \leq r$, C_n is $\emptyset$-definable. Therefore there is a first-order sentence (in the language of M^{eq}) which belongs to $Th(M^{\mathrm{eq}})$ and which expresses that M^{eq} (or any other model of $Th(M^{\mathrm{eq}})$) includes a self-coordinatized set which satisfies the conditions of Construction 3.5 (i) and has height r. Hence, for any model N of $Th(M)$, N^{eq} includes a self-coordinatized set which satisfies the conditions of Construction 3.5 (i) and has height r.

Definition 3.10. If C is chosen so that no other self-coordinatized set, which satisfies the conditions of Construction 3.5 (i), has smaller height than C, then the height of C is called the *height of M*. By Remark 3.9, the height of M is an invariant of $Th(M)$, so we can talk about the height of $Th(M)$.

The next lemma relates height with SU-rank. Since it will not be used in proving other results, we postpone its proof until Section 5.

Lemma 3.11. *The SU-rank of M is at least as great as the height of M.*

Lemma 3.12. *If $n < r$ and $c \in C_{n+1} - C_n$ then* $\mathrm{SU}(c/C_n) = 1$.

Proof. Let $c \in C_{n+1} - C_n$. Then $c \notin \operatorname{acl}(C_n)$ so $\mathrm{SU}(c/C_n) \geq 1$. If $\mathrm{SU}(c) = 1$ then we must have $\mathrm{SU}(c/C_n) = 1$ and we are done. Now suppose that $\mathrm{SU}(c) > 1$. Since C is self-coordinatized there is $c' \in C \cap \operatorname{acl}(c)$ such that $\mathrm{SU}(c/c') = 1$ and as a consequence

$c \notin \operatorname{acl}(c')$, so $c' < c$. If $c' \notin \operatorname{acl}(C_n)$ then, since $c' < c$, we have a contradiction to the assumption that $c \in C_{n+1} - C_n$. Hence $c' \in \operatorname{acl}(C_n)$ so $\mathrm{SU}(c/C_n) = 1$. □

Construction 3.13. For $s = 1, \dots, r$ let

$$N_s = \{c \in C_s : \text{ there exists no } c' \in C_s \text{ such that } c < c' \}.$$

We will only use N_r to prove the main result (Theorem 3.19) of this section, but as the few technical lemmas which will follow can be proved for N_s for arbitrary $s \in \{1, \dots, r\}$, we will do this.

Lemma 3.14. *(i) Let $1 \leq s \leq r$. If $c, c' \in N_s$ and $c \in \operatorname{acl}(c')$ then $c' \in \operatorname{acl}(c)$.*
(ii) $M \subseteq C \subseteq \operatorname{acl}(N_r)$.

Proof. (i) Suppose that $c, c' \in N_s$ and $c \in \operatorname{acl}(c')$. If $c' \notin \operatorname{acl}(c)$ then $c < c'$. But by the definition of N_s, $c, c' \in C_s$ and since $c \in N_s$ there cannot exist any $d \in C_s$ such that $c < d$, so we have a contradiction.

(ii) From Construction 3.5 (i) we have $M \subseteq C$. By Construction 3.8 we have $C \subseteq \operatorname{acl}(C_r)$. Suppose that $a \in C$. We show that $a \in \operatorname{acl}(N_r)$. By Construction 3.8 there are $c_1, \dots, c_n \in C_r$ such that $a \in \operatorname{acl}(c_1, \dots, c_n)$. Consider c_i for each $1 \leq i \leq n$. If $c_i \notin N_r$ then $c_i < c_i'$ for some $c_i' \in C_r$. We may assume that no $c_i'' \in C_r$ exists such that $c_i' < c_i''$, for otherwise we can replace c_i' with c_i''. But then $c_i' \in N_r$. So if $c_i \notin N_r$ then let $c_i' \in N_r$ be such that $c_i < c_i'$ and otherwise let $c_i' = c_i$. Then $c_1', \dots, c_n' \in N_r$ and $a \in \operatorname{acl}(c_1, \dots, c_n) \subseteq \operatorname{acl}(c_1', \dots, c_n')$. □

Lemma 3.15. *If $a \in C_r$, $b \in C$, $A \subseteq M^{\mathrm{eq}}$, $b < a$, $\mathrm{SU}(a/b) = 1$ and $a \not\downarrow_b A$ then $a \in \operatorname{acl}(A)$.*

Proof. Let a, b, A satisfy the assumptions of the lemma. By Construction 3.5, $C_0 = \emptyset$ and $C_i \subset C_{i+1}$ whenever $i < r$, so there is $n < r$ such that $a \in C_{n+1} - C_n$. We must have $b \in \operatorname{acl}(C_n)$ because otherwise, as $b < a$, we would have a contradiction to the assumption that $a \in C_{n+1} - C_n$ (see Construction 3.5). The assumptions that $\mathrm{SU}(a/b) = 1$ and $a \not\downarrow_b A$ imply that $a \in \operatorname{acl}(bA)$. If $\mathrm{SU}(b) = 0$ then $a \in \operatorname{acl}(A)$ and we are done. So suppose that $\mathrm{SU}(b) > 0$. Since $b < a$ it follows (from the Lascar equation) that $\mathrm{SU}(a) > 1$.

Since M is 1-based, a is independent from $\operatorname{acl}(A)$ over $\operatorname{acl}(a) \cap \operatorname{acl}(A)$ and since M is supersimple there is a finite $C \subseteq \operatorname{acl}(a) \cap \operatorname{acl}(A)$ such that a is independent from $\operatorname{acl}(a) \cap \operatorname{acl}(A)$ over C. Transitivity implies that $a \downarrow_C A$. As C is finite we may replace it by an element $c \in M^{\mathrm{eq}}$. Thus we find $c \in \operatorname{acl}(a) \cap \operatorname{acl}(A)$ such that $a \downarrow_c A$. If $a \in \operatorname{acl}(c)$ then $a \in \operatorname{acl}(A)$ and we are done. Therefore it is sufficient to show that $a \in \operatorname{acl}(c)$.

So suppose, for a contradiction, that $a \notin \operatorname{acl}(c)$. First we show that $c \notin \operatorname{acl}(b)$. If $c \in \operatorname{acl}(b)$ then we have $\operatorname{acl}(c) \subseteq \operatorname{acl}(b) \subseteq \operatorname{acl}(a)$ and, since $A \downarrow_c a$ (by the choice of c) we get $A \downarrow_{\operatorname{acl}(c)} \operatorname{acl}(a)$, so transitivity of independence gives $A \downarrow_{\operatorname{acl}(b)} \operatorname{acl}(a)$ and hence $A \downarrow_b a$ which contradicts the assumption that $a \not\downarrow_b A$.

Hence $c \notin \operatorname{acl}(b)$. To sum up, we now have $\mathrm{SU}(a/b) = 1$, $b < a$, $c < a$ (by the assumption that $a \notin \operatorname{acl}(c)$) and $c \notin \operatorname{acl}(b)$. By assumption $\mathrm{SU}(b) > 0$. Since $b < a$ and $\mathrm{SU}(a/b) = 1$ we get $\mathrm{SU}(a) > 1$ (by the Lascar equation). Also $b \in \operatorname{acl}(a) \cap C$ and C is self-coordinatized, so part (2) of Definition 3.3 gives us $c' \in C$ such that $c' < a$ and $c' \notin \operatorname{acl}(b)$. As shown in Remark 3.2 we must have $a \in \operatorname{acl}(b, c')$. Recall that n was chosen so that $a \in C_{n+1} - C_n$. If $c' \notin \operatorname{acl}(C_n)$ then, since $c' < a$, this contradicts that $a \in C_{n+1} - C_n$ (see Construction 3.5). If $c' \in \operatorname{acl}(C_n)$ then, since $b \in \operatorname{acl}(C_n)$ (as we concluded earlier in the proof), we have $a \in \operatorname{acl}(C_n)$ which also contradicts that $a \in C_{n+1} - C_n$. Now we have shown that $a \in \operatorname{acl}(c)$ so we are finished. □

Lemma 3.16. *If $a \in C_r$, $d_1, \ldots, d_k \in M^{\mathrm{eq}}$ and $a \in \mathrm{acl}(d_1, \ldots, d_k)$ then $a \in \mathrm{acl}(d_i)$ for some $1 \leq i \leq k$.*

Proof. Suppose that $a \in C_r, d_1, \ldots, d_k \in M^{\mathrm{eq}}$ and $a \in \mathrm{acl}(d_1, \ldots, d_k)$. First suppose that $\mathrm{SU}(a) = 1$. Then $tp(a/d_1, \ldots, d_k)$ forks over $\emptyset$, so by the triviality of dependence there is $1 \leq i \leq k$ such that $tp(a/d_i)$ forks over $\emptyset$, which means that $a \in \mathrm{acl}(d_i)$.

Now suppose that $\mathrm{SU}(a) > 1$. By Construction 3.5, $C_0 = \emptyset$ and $C_i \subset C_{i+1}$ whenever $i < r$, so there is $n < r$ such that $a \in C_{n+1} - C_n$. Since $a \in C$ (because $C_r \subseteq C$) and C is self-coordinatized it follows that there is $b \in C$ such that $b \in \mathrm{acl}(a)$ and $\mathrm{SU}(a/b) = 1$; we must have $b \in \mathrm{acl}(C_n)$ because otherwise we would have a contradiction to the assumption that $a \in C_{n+1} - C_n$ (see Construction 3.5). By assumption, $tp(a/b, d_1, \ldots, d_k)$ forks over b and since dependence is trivial there exists $1 \leq i \leq k$ such that $tp(a/b, d_i)$ forks over b, that is, $a \not\mathrel{\smile}_b d_i$. Since we also have $a \in C_r$, $b \in C$, $d_i \in M^{\mathrm{eq}}$, $b < a$ and $\mathrm{SU}(a/b) = 1$ we get $a \in \mathrm{acl}(d_i)$ from Lemma 3.15. $\square$

Construction 3.17. Fix any $s \in \{1, \ldots, r\}$. It follows from the argument in Remark 3.9 that N_s is a $\emptyset$-definable subset of M^{eq}. From now on we will consider N_s as a structure which is canonically embedded in M^{eq}. In other words, for every $0 < n < \omega$ and every relation $R \subseteq N^n$ which is $\emptyset$-definable in M^{eq}, the language of N contains a relation symbol which is interpreted as R; and we assume that the language of N has no other relation (or function or constant) symbols.

Lemma 3.18. *Let $1 \leq s \leq r$.*
(i) N_s is ω-categorical.
(ii) If $A \subseteq N_s$ then $\mathrm{acl}_{N_s}(A) = \mathrm{acl}_{M^{\mathrm{eq}}}(A) \cap N_s$.
(iii) $(N_s, \mathrm{acl}_{N_s})$ is a trivial pregeometry.

Proof. (i) By its definition as a structure, N_s is canonically embedded in M^{eq}. Only finitely many sorts are represented in C (by its construction) so the same holds for N_s, since $N_s \subseteq C$. So by Fact 1.5, N_s is ω-categorical.
(ii) This is a straightforward consequence of the fact that N_s is canonically embedded in M^{eq}.
(iii) This is a consequence of lemmas 3.14 and 3.16 and part (ii). $\square$

Theorem 3.19. *Suppose that M is an ω-categorical, simple, 1-based structure with trivial dependence. Then there is an ω-categorical structure N which is canonically embedded in M^{eq} and (N, acl_N) forms a trivial pregeometry and $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$; moreover, only finitely many sorts (from M^{eq}) are represented in N.*

Proof. The lemmas in this section have been obtained under the assumption that M is ω-categorical, simple, 1-based structure with trivial dependence. If N_r is the canonically embedded (in M^{eq}) structure from Constructions 3.13 and 3.17 then, by lemmas 3.14 and 3.18, only finitely many sorts are represented in N_r, N_r is ω-categorical, $(N_r, \mathrm{acl}_{N_r})$ forms a trivial pregeometry and $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N_r)$. $\square$

Remark 3.20. By Corollary 2.5, if N in Theorem 3.19 has the finite submodel property then so does M. We don't claim that it is, in practise, easier to determine if N (as in Theorem 3.19) has the finite submodel property than if M has it. The point is rather that if M does not have the finite submodel property then M^{eq} canonically embeds a structure N as in Theorem 3.19, which in some sense is less complicated than M (for example (N, acl_N) is a trivial pregeometry while (M, acl_M) need not necessarily be a pregeometry, as the 'exchange property' may fail), but still N does not have the finite submodel property.

Remark 3.21. (i) The proof of Lemma 3.16 is the only place where we use the assumption that dependence is trivial. Lemmas 3.14 and 3.16 together imply that $\mathrm{acl}_{M^{\mathrm{eq}}}$ forms

a (trivial) pregeometry on N_s, $1 \le s \le r$. Without the assumption that dependence is trivial the given arguments even fail to show that $\mathrm{acl}_{M^{\mathrm{eq}}}$ forms a pregeometry on N_s (whether trivial or not).
(ii) One may consider the following question: Suppose that M satisfies all conditions of Theorem 3.19 except that we don't insist that dependence is trivial. Is it the case that M^{eq} canonically embedds a (not necessarily trivial) pregeometry? The author guesses that the answer is 'no', but has not been able to prove it.

4. Polynomial k-saturation

In Theorem 3.19 we found a canonically embedded $N \subseteq M^{\mathrm{eq}}$ such that (N, acl_N) is a pregeometry and if N has the finite submodel property then so does M. As N is $\aleph_0$-categorical and (N, acl_N) is a pregeometry we can apply results from [4] to N and get some information about what happens, in terms of two notions studied in [4] (defined below), if N does not have the finite submodel property.

In this section N will be an $\aleph_0$-categorical structure such that (N, acl_N) forms a trivial pregeometry; recall that this is the case for the strucure N from Theorem 3.19.

Assumptions on the language of N. Throughout this section L will be the language of N, although all the definitions, and a couple of the results, which will follow could be given for any language. Since N is $\aleph_0$-categorical there exists $m < \aleph_0$ such that $\big|\mathrm{acl}_N(a)\big| \le m$ for every $a \in N$. We will suppose that L has symbols P, Q and $R_1, \ldots, R_m$ which are interpreted in the following way:

$$\begin{aligned} P^N &= \big\{a \in N : a \in \mathrm{acl}_N(\emptyset)\big\}, \\ Q^N &= \big\{(a,b) \in N^2 : a \in \mathrm{acl}_N(b)\big\}, \\ R_i^N &= \big\{a \in N - \mathrm{acl}_N(\emptyset) : \big|\mathrm{acl}_N(a)\big| = i\big\} \quad \text{for } i = 1, \ldots, m. \end{aligned}$$

If, originally, the language of N does not have such symbols (with interpretations as above) then we can expand N to N' so that the above holds for N' and N' is $\aleph_0$-categorical and $(N', \mathrm{acl}_{N'})$ forms a pregeometry.

Note that the above assumptions on the language of N hold by definition if N is canonically embedded in M^{eq} (for some $\aleph_0$-categorical M).

Definition 4.1. L_{acl} denotes the sublanguage of L which has as its relation symbols exactly the symbols P, Q, $R_1, \ldots, R_m$ (and $=$) and which has no function or constant symbols.

If $\mathcal{L}$ and $\mathcal{L}'$ are first-order languages, $\mathcal{L}'$ is a sublanguage of $\mathcal{L}$ and M is an $\mathcal{L}$-structure then $M{\restriction}\mathcal{L}'$ is the reduct of M to $\mathcal{L}'$. The following fact appears as Claim 1.13 in [4]:

Fact 4.2. *(i) $N{\restriction}L_{\mathrm{acl}}$ has elimination of quantifiers.*
(ii) For any subset $A \subseteq N$, $\mathrm{acl}_{N\restriction L_{\mathrm{acl}}}(A) = \mathrm{acl}_N(A)$; in other words, $\mathrm{acl}_{N\restriction L_{\mathrm{acl}}}$ an acl_N coincide.

Definition 4.3. (i) If M is a structure such that (M, acl_M) forms a pregeometry and $A \subseteq M$ then we define the *dimension of* A to be

$$\dim_M(A) = \inf\big\{|B| : B \subseteq A \text{ and } A \subseteq \mathrm{acl}_M(B)\big\}.$$

(ii) For a structure M and a type $p(\bar{x})$ over $A \subseteq M$, we say that $p(\bar{x})$ is *algebraic* if it has only finitely many realizations; otherwise we say that $p(\bar{x})$ is *non-algebraic.*

Definition 4.4. Let $0 < k < \aleph_0$ and suppose that M is a structure (in any language) such that (M, acl_M) forms a pregeometry. We say that M is *polynomially k-saturated* if there is a polynomial $F(x)$ such that for every $n_0 < \aleph_0$ there is a natural number $n \geq n_0$ and a finite substructure $A \subseteq M$ such that:

(1) $n \leq |A| \leq F(n)$.
(2) A is algebraically closed (in M).
(3) Whenever $\bar{a} \in A$, $\dim_M(\bar{a}) < k$ and $q(x) \in S_1^M(\bar{a})$ is non-algebraic there are distinct $b_1, \ldots, b_n \in A$ such that $M \models q(b_i)$ for each $1 \leq i \leq n$.

Examples of structures which are polynomially k-saturated for every $0 < k < \aleph_0$ include infinite vector spaces over a finite field and the random graph; this is shown in [4]. From [4] we also have the following:

Fact 4.5. *If M is polynomially k-saturated for every $0 < k < \aleph_0$, then M has the finite submodel property.*

The following fact appears as Claim 1.14 in [4]:

Fact 4.6. *$N{\restriction}L_{\mathrm{acl}}$ is polynomially k-saturated for every $0 < k < \aleph_0$.*

Notation. (i) If $\bar{s} = (s_1, \ldots, s_n)$ is a sequence of objects and $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, where we assume $i_1 < \ldots < i_m$, then $\bar{s}_I$ denotes the sequence $(s_{i_1}, \ldots, s_{i_m})$. (ii) If $p(\bar{x})$ is a type and $\bar{x}'$ is a subsequence of $\bar{x}$, then $p{\restriction}\{\bar{x}'\}$ is the set of all formulas $\varphi(\bar{x}')$ such that $\varphi(\bar{x}') \in p(\bar{x})$; so in particular, $p{\restriction}\{\bar{x}'\}$ is a type.

In the following definition and in Fact 4.8 it is not essential that L is the language of N (which is a standing assumption of this section).

Definition 4.7. Suppose that M is an $\aleph_0$-categorical L-structure such that (M, acl_M) is a pregeometry. Let $\mathcal{L}$ be a sublanguage of L. We say that M *satisfies the k-independence hypothesis over* $\mathcal{L}$ if the following holds for any $\bar{a} = (a_1, \ldots, a_n) \in M^n$ such that $\dim_M(\bar{a}) \leq k$:
If $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ and $p(\bar{x}_I) \in S_m(Th(M))$ (where $\bar{x}_I = (x_{i_1}, \ldots x_{i_m})$) are such that

(a) $\mathrm{acl}_M(\bar{a}_I) = \mathrm{rng}(\bar{a}_I)$, $\dim_M(\bar{a}_I) < k$, $p(\bar{x}_I) \cap \mathcal{L} = tp_{M{\restriction}\mathcal{L}}(\bar{a}_I)$ and for every $J \subset I$ with $\dim_M(\bar{a}_J) < \dim_M(\bar{a}_I)$, $p{\restriction}\{\bar{x}_J\} = tp_M(\bar{a}_J)$,

then there is $\bar{b} = (b_1, \ldots, b_n) \in M^n$ such that

(b) $tp_{M{\restriction}\mathcal{L}}(\bar{b}) = tp_{M{\restriction}\mathcal{L}}(\bar{a})$, $tp_M(\bar{b}_I) = p(\bar{x}_I)$ and, for every $J \subset \{1, \ldots, n\}$ such that $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$, $tp_M(\bar{a}_J) = tp_M(\bar{b}_J)$.

In [4] examples are given of structures which either satisfy or fail to satisfy the k-independence hypothesis for various k. Also, it is shown that if M is an $\aleph_0$-categorical L-structure which is simple with SU-rank 1 and (M, acl_M) forms a pregeometry, then M satisfies the 3-independence hypothesis over L_{acl}; as a consequence of Theorem 2.1 in [4], any sentence in which at most 3 distinct variables occur and which is true in M has a finite model. From [4] (Theorem 2.2) we have the following:

Fact 4.8. *Let M be an $\aleph_0$-categorical L-structure such that (M, acl_M) forms a pregeometry. Suppose that there is a sublanguage $\mathcal{L} \subseteq L$ such that $\mathrm{acl}_{M{\restriction}\mathcal{L}}$ coincides with acl_M and, for every $0 < k < \aleph_0$, $M{\restriction}\mathcal{L}$ is polynomially k-saturated and M satisfies the k-independence hypothesis over $\mathcal{L}$. Then M is polynomially k-saturated, for every $0 < k < \aleph_0$, and M has the finite submodel property.*

From now on, suppose that N does not have the finite submodel property. Then, by Fact 4.5, N is not polynomially k-saturated for some $0 < k < \aleph_0$. Recall that L is the language of N. Sublanguages of L can be partially ordered by inclusion. By Fact 4.6, $N{\restriction}L_{\mathrm{acl}}$ is polynomially k-saturated for every $0 < k < \aleph_0$. Suppose that $\mathcal{L}$ is a maximal sublanguage of L, with respect to $\subseteq$, such that $L_{\text{acl}} \subseteq \mathcal{L}$ and $N{\restriction}\mathcal{L}$ is polynomially k-saturated for every $0 < k < \aleph_0$. By assumption $\mathcal{L}$ is a proper sublanguage of L. Now let S be any symbol which occurs in the vocabulary (i.e. signature) of L but not in the vocabulary of $\mathcal{L}$. Let $\mathcal{L}'$ be the language obtained by adding S to (the vocabulary of) $\mathcal{L}$, so we have $L_{\text{acl}} \subseteq \mathcal{L} \subset \mathcal{L}' \subseteq L$. By the maximality of $\mathcal{L}$, for some $0 < k_1 < \aleph_0$, $N{\restriction}\mathcal{L}'$ is not polynomially k_1-saturated. From Fact 4.2 it follows that $\text{acl}_{N\restriction\mathcal{L}}$ and $\text{acl}_{N\restriction\mathcal{L}'}$ coincide. Therefore, by Fact 4.8, there exists $0 < k_2 < \aleph_0$ such that $N{\restriction}\mathcal{L}'$ does not satisfy the k_2-independence hypothesis over $\mathcal{L}$. The argument just given proves the following:

Theorem 4.9. *Let N be an $\aleph_0$-categorical L-structure such that (N, acl_N) forms a trivial pregeometry. Suppose that the language L is subject to the assumptions made in the beginning of this section and let L_{acl} be defined as in Definition 4.1. If N does not have the finite submodel property, then there are sublanguages $\mathcal{L}$ and $\mathcal{L}'$ of L such that*

(1) *$L_{\text{acl}} \subseteq \mathcal{L} \subset \mathcal{L}' \subseteq L$ and there is exactly one symbol which occurs in $\mathcal{L}'$ but not in $\mathcal{L}$,*
(2) *$N{\restriction}\mathcal{L}$ is polynomially k-saturated for every $0 < k < \aleph_0$, and*
(3) *$N{\restriction}\mathcal{L}'$ does not satisfy the k'-independence hypothesis over $\mathcal{L}$ for some $0 < k' < \aleph_0$.*

5. Height and rank

In this section we adopt all assumptions from Section 3, so M is $\aleph_0$-categorical, simple and 1-based with trivial dependence; we also use the same notation as in that section. By Lemma 3.4 and Construction 3.5 it follows that there is a self-coordinatized set $C \subseteq M^{\text{eq}}$ which satisfies the conditions of Construction 3.5 (i). The number r and the sets $C_0, \ldots, C_r$ are as given by Construction 3.5 (ii). We may assume that r is the height of M (see Construction 3.5 (ii) and Definition 3.10). We will show that the SU-rank of M is always at least as big as the height of M. For this we need two lemmas.

Lemma 5.1. *If $1 \leq s \leq r$ and $a \in \text{acl}(C_s) \cap C$ then $a \in \text{acl}\big(\text{acl}(a) \cap C_s\big)$.*

Proof. Recall that $C_i \subset C_j$ if $i < j$ and that $C_i \subseteq C$ for every i. Let $a \in \text{acl}(C_s) \cap C$. Suppose for a contradiction that $a \notin \text{acl}\big(\text{acl}(a) \cap C_s\big)$. Let $\bar{b} = \text{acl}(a) \cap C_s$. Then $a \underset{\bar{b}}{\not\downarrow} C_s$ by the assumption that $a \in \text{acl}(C_s)$. Let $t \leq s$ be minimal such that $a \underset{\bar{b}}{\not\downarrow} C_t$, so $C_t \neq \emptyset$ and $t > 0$. By the minimality of t, $a \underset{\bar{b}C_{t-1}}{\not\downarrow} C_t$ so by the triviality of dependence there is $c \in C_t$ such that

(1) $a \underset{\bar{b}C_{t-1}}{\not\downarrow} c$, and hence
(2) $a\bar{b} \underset{C_{t-1}}{\not\downarrow} c$.

Thus $c \notin C_{t-1}$ so, by Lemma 3.12, $\text{SU}(c/C_{t-1}) = 1$; this together with (1) implies that $\bar{b} \underset{C_{t-1}}{\downarrow} c$. So by (2) and the triviality of dependence we must have $a \underset{C_{t-1}}{\not\downarrow} c$, so

(3) $c \in \text{acl}(aC_{t-1})$.

We have $\text{SU}(c) \geq 1$ (by (1) for example). First suppose that $\text{SU}(c) = 1$. Then $c \notin \text{acl}(\emptyset) = \text{acl}(C_0)$, and there is no $c' \in C - \text{acl}(C_0)$ such that $c' < c$, because this would (by the Lascar equation) imply that $\text{SU}(c) > 1$. Therefore $c \in C_1$. This together with (1) implies that $t - 1 = 0$, so (3) implies that $c \in \text{acl}(a)$. But $c \in C_t \subseteq C_s$ (because $t \leq s$) so we get $c \in \text{acl}(a) \cap C_s = \bar{b}$, which contradicts (1).

Now suppose that $\text{SU}(c) > 1$. From (1) it follows that $c \notin C_{t-1}$, so $c \in C_t - C_{t-1}$. Since C is self-coordinatized there is $c' \in C$ such that $c' < c$ and $\text{SU}(c/c') = 1$, and because $c \in C_t - C_{t-1}$ we get $c' \in \text{acl}(C_{t-1})$ (see Construction 3.5). From $\text{SU}(c/C_{t-1}) = 1$ (which we concluded above), $\text{SU}(c/c') = 1$ and $c' \in \text{acl}(C_{t-1})$ it follows that

(4) $c \mathop{\smile\!\!\!\!\!\!\mid}\limits_{c'} C_{t-1}$.

Together with (3) this means that $c \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{c'} aC_{t-1}$; this together with (4) and the triviality of dependence implies that $c \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{c'} a$. By the choices of the elements we have $c \in C_t$, $a, c' \in C$, $c' < c$ and $\mathrm{SU}(c/c') = 1$, so we can apply Lemma 3.15 to get $c \in \mathrm{acl}(a)$. Since $c \in C_t \subseteq C_s$ we have $c \in \mathrm{acl}(a) \cap C_s = \bar{b}$, but this contradicts (1). □

Lemma 5.2. *If $n < r$ and $a \in C - \mathrm{acl}(C_n)$ then $a \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{C_n} C_{n+1}$.*

Proof. By induction on $\mathrm{SU}(a)$. If $a \in C$ and $\mathrm{SU}(a) = 0$ then $a \in \mathrm{acl}(C_n)$, for every $n < r$, so the assertion of the lemma is vacuously satisfied. Now suppose that $a \in C$ and that the assertion of the lemma holds for every $a' \in C$ such that $\mathrm{SU}(a') < \mathrm{SU}(a)$. Let $n < r$ and suppose that $a \notin \mathrm{acl}(C_n)$. If $a \in \mathrm{acl}(C_{n+1})$ then clearly $a \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{C_n} C_{n+1}$. So suppose that $a \notin \mathrm{acl}(C_{n+1})$. Recall that by Lemma 3.7 and Construction 3.8, $C \subseteq \mathrm{acl}(C_r)$ so $a \in \mathrm{acl}(C_r)$. Let $s \leq r$ be minimal such that $a \in \mathrm{acl}(C_s)$ and let $\bar{b} = \mathrm{acl}(a) \cap C_s$. Since we assume that $a \notin \mathrm{acl}(C_n)$ we have $s \geq 1$. By Lemma 5.1, we have $a \in \mathrm{acl}(\bar{b})$. Let $\bar{b}'$ be a subsequence of $\bar{b}$ of minimal length such that $a \in \mathrm{acl}(\bar{b}')$.

First suppose that $|\bar{b}'| \geq 2$. For any b_i in the sequence $\bar{b}'$, if $\bar{b}''$ is the subsequence of $\bar{b}'$ which contains all elements of $\bar{b}'$ except b_i, then we have

$$\mathrm{SU}(a) = \mathrm{SU}(a, \bar{b}'', b_i) = \mathrm{SU}(a/\bar{b}''b_i) + \mathrm{SU}(\bar{b}'', b_i) = 0 + \mathrm{SU}(\bar{b}''/b_i) + \mathrm{SU}(b_i).$$

By the minimality of $|\bar{b}'|$ we have $\mathrm{SU}(\bar{b}''/b_i) \geq 1$, which together with the above equation implies that $\mathrm{SU}(b_i) < \mathrm{SU}(a)$. There must exist b_i in $\bar{b}'$ such that $b_i \notin \mathrm{acl}(C_n)$ because otherwise we would have $a \in \mathrm{acl}(\bar{b}') \subseteq \mathrm{acl}(C_n)$ which would contradict one of the assumptions on a. So let b_i in $\bar{b}'$ be such that $b_i \notin \mathrm{acl}(C_n)$. As noted above, we have $\mathrm{SU}(b_i) < \mathrm{SU}(a)$. By applying the induction hypothesis to b_i we get $b_i \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{C_n} C_{n+1}$. It follows that $\bar{b}' \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{C_n} C_{n+1}$. Since $\mathrm{acl}(a) = \mathrm{acl}(\bar{b}')$ we get $a \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{C_n} C_{n+1}$, which is what we wanted to prove.

Now suppose that $|\bar{b}'| = 1$, so $\bar{b}'$ consists of a single element from C_s which we call b'. Then $\mathrm{acl}(a) = \mathrm{acl}(b')$ and therefore $a \in C_s$ (see Construction 3.5). The assumption that $a \notin \mathrm{acl}(C_{n+1})$ implies that $\mathrm{SU}(a) \geq 1$. By assumption, $a \in C$ so if $\mathrm{SU}(a) = 1$ then, by Construction 3.5 (ii), $a \in C_1 \subseteq C_{n+1}$ which contradicts that $a \notin \mathrm{acl}(C_{n+1})$. Hence $\mathrm{SU}(a) > 1$. Since C is self-coordinatized there is $b \in C$ such that $b < a$ and $\mathrm{SU}(a/b) = 1$. As s was chosen to be minimal such that $a \in \mathrm{acl}(C_s)$, it follows that $a \in C_s - C_{s-1}$; therefore it must be the case that $b \in \mathrm{acl}(C_{s-1})$. If it would be the case that $\mathrm{SU}(a/bC_{n+1}) = 0$ then $a \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{b} C_{n+1}$, so Lemma 3.15 would imply that $a \in \mathrm{acl}(C_{n+1})$, contradicting one of the assumptions on a. Hence, $\mathrm{SU}(a/bC_{n+1}) = 1$ and therefore $\mathrm{SU}(a/bC_n) = 1$ (since $\mathrm{SU}(a/b) = 1$).

Claim. $b \notin \mathrm{acl}(C_n)$.

Suppose for a contradiction that $b \in \mathrm{acl}(C_n)$. We will show that $a \in C_{n+1}$ which contradicts the assumption that $a \notin C_{n+1}$. Since, by assumption, $a \notin \mathrm{acl}(C_n)$ we need to show that there is no $c \in C$ such that $c < a$ and $c \notin \mathrm{acl}(C_n)$. Suppose that such c exists. By assumption, $b \in \mathrm{acl}(C_n)$, so $c \notin \mathrm{acl}(b)$ and hence $\mathrm{SU}(c/b) \geq 1$. We also have

$$1 = \mathrm{SU}(a/b) = \mathrm{SU}(a, c/b) = \mathrm{SU}(a/cb) + \mathrm{SU}(c/b),$$

so $\mathrm{SU}(a/cb) = 0$, which gives $a \mathop{\not\!\smile\!\!\!\!\!\!\mid}\limits_{b} c$. We concluded above that $\mathrm{SU}(a) > 1$ and $a \in C_s \subseteq C_r$. By the choice of $b \in C$ we also have $b < a$ and $\mathrm{SU}(a/b) = 1$. Hence, Lemma 3.15 implies that $a \in \mathrm{acl}(c)$ which contradicts the assumption that $c < a$. Hence no $c \in C$ exists such that $c < a$ and $c \notin \mathrm{acl}(C_n)$, and therefore $a \in C_{n+1}$. But this contradicts the assumption that $a \notin \mathrm{acl}(C_{n+1})$, so we have proved the claim.

Recall that we have proved that $\mathrm{SU}(a/bC_{n+1}) = \mathrm{SU}(a/bC_n) = \mathrm{SU}(a/b) = 1$. We now get

$$\begin{aligned}
\mathrm{SU}(a/C_{n+1}) &= \mathrm{SU}(a, b/C_{n+1}) \\
&= \mathrm{SU}(a/bC_{n+1}) + \mathrm{SU}(b/C_{n+1}) = 1 + \mathrm{SU}(b/C_{n+1}) \\
&< 1 + \mathrm{SU}(b/C_n) \quad \text{by the induction hypothesis, since } \mathrm{SU}(b) < \mathrm{SU}(a) \\
&\qquad\qquad\qquad\quad \text{and, by the claim, } b \notin \mathrm{acl}(C_n) \\
&= \mathrm{SU}(a/bC_n) + \mathrm{SU}(b/C_n) = \mathrm{SU}(a, b/C_n) = \mathrm{SU}(a/C_n),
\end{aligned}$$

so $a \not\perp_{C_n} C_{n+1}$, which is what we wanted to prove. □

Now we can prove:

Lemma 3.11 The SU-rank of M is at least as great as the height of M.

Proof. We are assuming that $C \supseteq M$ is chosen so that the height of C, which is r, equals the height of M. Hence, it is sufficient to show that there exists $a \in M$ such that $\mathrm{SU}(a) \geq r$. By the construction of C_n, $n = 1, \ldots, r$, there is $a \in C$ such that $a \notin \mathrm{acl}(C_n)$ whenever $n < r$. Since $M \subseteq C$ and $C \subseteq M^{\mathrm{eq}} \subseteq \mathrm{acl}(M)$ there is, in fact, $a \in M$ such that $a \notin \mathrm{acl}(C_n)$ whenever $n < r$. By Lemma 5.2, $\mathrm{SU}(a/C_n) < \mathrm{SU}(a/C_{n-1})$ for $n = 1, \ldots, r$, so $\mathrm{SU}(a) \geq r$. □

6. Examples

We illustrate the constructions made in Section 3 (Constructions 3.5 and 3.13) with a couple of examples. These examples will be $\aleph_0$-categorical, simple, 1-based with trivial dependence, but not stable. The notation $C_0, \ldots, C_r$ and $N_1, \ldots, N_r$ is like in Section 3 for each M and C considered below. All examples that follow have the finite submodel property, which is left for the reader to verify, but it essentially follows from the fact that the random graph has it. By 'acl' we mean '$\mathrm{acl}_{M^{\mathrm{eq}}}$' for the structure M under consideration.

Example 6.1. If M is the (infinite countable) random graph (see [6]) then it is easy to see that if $C = M$ then C is self-coordinatized, $C_0 = \emptyset$, $C_1 = C = M$ and, of course, $C \subseteq \mathrm{acl}(C_1)$, so the height of M is 1 (since it cannot be less than 1, for we always have $C_0 = \emptyset$ by definition). It is well-known that the SU-rank of M is 1 so SU-rank and height coincide for M. We also have $N_1 = C_1 = M$.

Example 6.2. We construct M with SU-rank 3 and height 2. Let the language L have as its relation symbols P, Q, R and E, where the two first are unary and the other two are binary, and assume that L has no function or constant symbols. Let M be an L-structure satisfying:

(1) M is the disjoint union of P^M and Q^M where $|P^M| = \aleph_0$.
(2) $(a, b) \in R^M \implies a, b \in P^M$.
(3) $(a, b) \in E^M \implies a \in P^M,\ b \in Q^M$.
(4) If A is the substructure of M with universe P^M then the reduct of A to the language with symbols R and $=$ is the random graph. (This point serves only to make M unstable.)
(5) For every $b \in Q^M$, $\big|\{a \in P^M : (a, b) \in E^M\}\big| = 2$.
(6) For any two distinct $a_1, a_2 \in P^M$,

$$\big|\{b \in Q^M : (a_1, b) \in E^M \text{ and } (a_2, b) \in E^M\}\big| = \aleph_0.$$

For any $A \subseteq M$, define

$$\mathrm{cl}(A) = A \cup \big\{a \in P^M : \exists b \in A \cap Q^M,\ (a,b) \in E^M\big\}.$$

Observe that $|\mathrm{cl}(A)| \leq 2|A|$ for every $A \subseteq M$. We say that A is *closed* if $\mathrm{cl}(A) = A$. Any subset A of M will also be considered as an L-structure, namely the substructure of M which has A as its universe.
The following is easy to prove, by a back and forth argument (as M is countable):

Claim. If $A, B \subseteq M$ are finite and closed and $\sigma : A \to B$ is an isomorphism, then there is an automorphism $\tau : M \to M$ which extends σ.

From the claim and the already mentioned fact that $|\mathrm{cl}(A)| \leq 2|A|$ it follows that for any $0 < n < \aleph_0$, up to equivalence in M, there are only finitely many formulas in the free variables $x_1, \dots, x_n$. Hence M is $\aleph_0$-categorical. It also follows that if $\bar{a} \in M$ is closed then $tp(\bar{a})$ is isolated by a quantifier free formula.

Now suppose that $A \subseteq M$ is finite and $a \in M - \mathrm{cl}(A)$. It is not hard to see that there are distinct b_i, $i < \aleph_0$, and isomorphisms $\tau_i : \mathrm{cl}(A \cup \{a\}) \to \mathrm{cl}(A \cup \{b_i\})$ such that τ_i extends the identity map on $\mathrm{cl}(A)$ and sends a to b_i. By the claim, there are automorphisms $\tau_i' : M \to M$, $i < \aleph_0$, such that τ_i' extends τ_i. It follows that $a \notin \mathrm{acl}_M(A)$. If on the other hand $a \in \mathrm{cl}(A)$ then, by the definition of cl and (5), we have $a \in \mathrm{acl}_M(A)$. So we have proved that $\mathrm{cl}(A) = \mathrm{acl}_M(A)$ for any finite $A \subseteq M$. But since for any $A \subseteq M$, $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(B) : B \subseteq A,\ B \text{ is finite}\}$, we have $\mathrm{cl}(A) = \mathrm{acl}_M(A)$ for any $A \subseteq M$.

We outline a proof that M is supersimple with SU-rank 3. From here on we may replace the original M with any structure which is elementarily equivalent to it. The main step is to prove that if $A \subseteq B \subset M$ are finite and closed then

(∗) $tp(\bar{a}/B)$ divides over A if and only if $\mathrm{cl}(\bar{a}) \cap B \not\subseteq A$.

From (∗) it follows that there are *no* $\bar{a} \in M$ and $\bar{b}_i \in M$, $i < \aleph_0$, such that $tp(\bar{a}/\bar{b}_0 \dots \bar{b}_{i+1})$ divides over $\bar{b}_0 \dots \bar{b}_i$ for every $i < \aleph_0$. By Proposition 2.8.13 in [8], M is supersimple. From (∗) one also deduces, in a similar way, that $\mathrm{SU}(a) \leq 3$ for all $a \in M$ and $\mathrm{SU}(a) = 3$ if $a \in Q^M$. We leave it to the reader to verify that M has trivial dependence; from this it follows that all types with SU-rank 1 are trivial and hence 1-based; by Proposition 4.6 in [5] it follows that M is 1-based.

Now we find a self-coordinatized set C. For any $a, b \in M$ define

$$a \sim b \iff a, b \in P^M \text{ or } a, b \in Q^M \text{ and } \mathrm{acl}_M(a) \cap P^M = \mathrm{acl}_M(b) \cap P^M.$$

Then $\sim$ is a $\emptyset$-definable equivalence relation on M, so the equivalence classes of $\sim$ are elements of M^{eq}. Let $C = P^M \ \cup\ Q^M/\!\sim\ \cup Q^M$ where $Q^M/\!\sim$ is the set of $\sim$-classes which contain (only) elements from Q^M. It is straightforward to verify that C is self-coordinatized and $M = P^M \cup Q^M \subseteq C$. By Construction 3.5 we have $C_0 = \emptyset$, $C_1 = P^M$, $C_2 = P^M \cup Q^M$ and $C \subseteq \mathrm{acl}(C_2)$ so C_3 is not defined and the height of C is 2; hence the height of M is at most 2. We leave it to the reader to show that the height of M is exactly 2. By Construction 3.13, we have $N_1 = P^M$ and $N_2 = Q^M$.

Example 6.3. Suppose that we construct M in the same way as in Example 6.2 except that we replace (5) and (6) by

(5)' For every $b \in Q^M$, $\big|\{a \in P^M : (a,b) \in E^M\}\big| = 1$.
(6)' For every $a \in P^M$, $\big|\{b \in Q^M : (a,b) \in E^M\}\big| = \aleph_0$.

Then one can show, in a similar way as in Example 6.2, that M has height 2 and SU-rank 2. In this case the set $C = P^M \cup Q^M$ is self-coordinatized and we get $C_1 = P^M$, $C_2 = P^M \cup Q^M$, $N_1 = P^M$ and $N_2 = Q^M$.

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