Primitive recursive and recursive functions

Initial functions:

(I) The zero function $z(x) = 0$ for all x.

(II) The successor function $s(x) = x+1$ for all x.

(III) The projection functions:

for all positive integers n and i

P_i^n is the function

$$P_i^n(x_1, \ldots, x_n) = x_i \quad .$$

Ways of defining new functions from given functions.

(IV) Substitution:

If $g(y_1, \ldots, y_m)$ and $h_1(x_1, \ldots x_n), \ldots, h_m(x_1, \ldots x_n)$ are given functions then we say that the function f defined by

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots x_n), \ldots, h_m(x_1, \ldots, x_n))$$

is defined from $g, h_1, \ldots, h_m$ by substitution.

(V) Recursion:

If $g(x_1, \dots, x_n)$ and $h(x_1 \dots x_n\ x_{n+1}\ x_{n+2})$ are given functions and f is defined by

$$f(x_1, \dots, x_n, 0) = g(x_1, \dots, x_n)$$

$$f(x_1, \dots, x_n, y+1) = h(x_1, \dots, x_n, y, f(x_1, \dots, x_n, y))$$

then we say that f is defined from g and h by recursion.

If $k \in \mathbb{N}$ and $h(x_1, x_2)$ are given then we say that f defined by

$f(0) = k$

$f(y+1) = h(y, f(y))$

is defined from k and h by recursion.

(VI) μ-operator:

Let $g(x_1, \dots, x_n, y)$ be a function such that for every $x_1, \dots, x_n$ there is y such that $g(x_1, \dots, x_n, y) = 0$. Then we say that f defined by

$f(x_1, \dots, x_n) =$ least y such that $g(x_1, \dots, x_n, y) = 0$

is defined from g by the μ-operator.

If $f(x_1 \ldots x_n)$ is obtained from $g(x_1 \ldots x_n, y)$ by the μ-operator then we write

$$f(x_1 \ldots x_n) = \mu y(g(x_1, \ldots, x_n, y) = 0).$$

We say that a function is <u>primitive recursive</u> if it can be defined from the initial functions by a finite number of applications of substitution and recursion.

We say that a function is <u>recursive</u> if it can be defined from the initial functions by a finite number of applications of substitution, recursion and the μ-operator.

Proposition 1.1 Let $g(y_1, \dots, y_k)$ be primitive recursive (or recursive) and let $x_1, \dots, x_n$ be distinct variables such that for every $1 \le i \le k$, z_i is one of $x_1, \dots, x_n$. Then the function

$$f(x_1, \dots, x_n) = g(z_1, \dots, z_k)$$

is primitive recursive (or recursive).

Proof. Let $z_i = x_{j_i}$ where $(1 \le j_i \le n)$.

Then $P^n_{j_i}(x_1, \dots, x_n) = z_i$ for every $1 \le i \le k$,

so $f(x_1, \dots, x_n) = g(z_1, \dots, z_k) =$

$$= g(P^n_{j_1}(x_1, \dots, x_n), \dots, P^n_{j_k}(x_1, \dots, x_n))$$

and hence f is primitive recursive (or recursive) $\square$

We give some examples to illustrade how Proposition 1.1 can be used.

Examples:

1. If $g(x, y)$ is primitive recursive and we define $f(x, y, z) = g(x, y)$ then f is primitive recursive.

2. If $g(x, y, z)$ is primitive recursive and we define $f(x, y, z) = g(y, z, x)$ then f is primitive recursive.

3. If $g(x,y,z)$ is primitive recursive and $f(x,y) = g(x,y,y)$ then f is primitive recursive.

Corollary 1.2

(a) The function $Z_n(x_1, \dots, x_n) = 0$ is primitive recursive, for every $n \geq 1$.

(b) For every $k \in \mathbb{N}$ and $n \geq 1$ the constant function $C_k^n(x_1, \dots, x_n) = k$ is primitive recursive.

Proof. (a) We have $Z_n(x_1, \dots, x_n) = Z(x_1)$ so letting $z_1 = x_1$ we get $Z_n(x_1, \dots, x_k) = Z(z_1)$ so Z_n is primitive recursive by Proposition 1.1.

(b) By induction on k.

$C_0^n(x_1, \dots, x_n) = Z_n(x_1, \dots, x_n)$ so C_0^n is primitive recursive by (a).

Suppose that $C_k^n(x_1, \dots, x_n)$ is primitive recursive. Since $C_{k+1}^n(x_1, \dots, x_n) = S(C_k^n(x_1, \dots, x_n))$ it follows by substitution that C_{k+1}^n is primitive recursive.

Proposition 1.3 The following functions are primitive recursive.

(a) $x+y$ (b) $x \cdot y$ (c) x^y

(d) $\delta = \begin{cases} x-1 & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$

(e) $x \dot{-} y = \begin{cases} x-y & \text{if } x \geq y \\ 0 & \text{if } x<y \end{cases}$

(f) $|x-y| = \begin{cases} x-y & \text{if } x \geq y \\ y-x & \text{if } x<y \end{cases}$

(g) $sg(x) = \begin{cases} 0 & \text{if } x=0 \\ 1 & \text{if } x \neq 0 \end{cases}$

(h) $\overline{sg}(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x \neq 0 \end{cases}$

(i) $x!$

(j) $\min(x,y)$ = minimum of x and y

(k) $\min(x_1,\dots,x_n)$ = minimum of $x_1,\dots,x_n$

(l) $\max(x,y)$ = maximum of x and y

(m) $\max(x_1,\dots,x_n)$ = maximum of $x_1,\dots,x_n$

(n) $rm(x,y)$ = remainder when y is divided by x. (There are unique q, r such that $y = q \cdot x + r$ and $r < x$; $rm(x,y) = r$.)

(o) $qt(x,y)$ = quotient when y is divided by x. ($qt(x,y) = q$, where q, r are as above)

Proof. (a) Observe that $x + 0 = x$, $x + (y+1) = (x+y)+1$.
Hence we define $f(x,0) = P_1^1(x)$
and $f(x, y+1) = S(P_3^3(x, y, f(x,y)))$,
and it follows by substitution (IV) and recursion (V) that f is primitive recursive, and we also have $f(n,m) = n+m$ for all $n, m \in \mathbb{N}$.

(b) Observe that $x \cdot 0 = 0$ and $x \cdot (y+1) = x \cdot y + x$.
If we define $g(x,0) = Z(x)$ and
$g(x, y+1) = f(g(x,y), x) = f(P_3^3(x, y, g(x,y)), x)$

(where f is addition)
then g is primitive recursive and $g(n,m) = n \cdot m$ for all $n, m \in \mathbb{N}$.

(c) $x^0 = 1$, $x^{(y+1)} = x^y \cdot x$, so define

$h(x,0) = C_1^1(x)$, $h(x,y+1) = g(h(x,y),x)$
where g is multiplication.

(d) $\delta(0) = 0$ and $\delta(y+1) = y$ so
$\delta(y+1) = P_1^2(y, \delta(y))$. By (V) δ is primitive recursive.

For (e) – (o) we give indications.

(e) $x \dot{-} 0 = x$, $x \dot{-} (y+1) = \delta(x \dot{-} y)$

(f) $|x-y| = (x \dot{-} y) + (y \dot{-} x)$

(g) $sg(x) = x \dot{-} \delta(x)$

(h) $\overline{sg}(x) = 1 \dot{-} sg(x)$

(i) $0! = 1$, $(y+1)! = (y!) \cdot (y+1)$

(j) $min(x,y) = x \dot{-} (x \dot{-} y)$

(k) By induction on n : $min(x_1, \ldots, x_n, x_{n+1}) = min(min(x_1, \ldots, x_n), x_{n+1})$

(l) $max(x,y) = y + (x \dot{-} y)$

(m) like (k)

(n) $rm(x,0) = 0$
$rm(x,y+1) = s(rm(x,y)) \cdot sg(|x - s(rm(x,y))|)$

(o) $qt(x,y) = 0$
$qt(x,y+1) = qt(x,y) + \overline{sg}(|x - s(rm(x,y))|)$.

$\square$

Bounded sums and products

We define:

$$\sum_{y<z} f(x_1,\ldots,x_n,y) = \begin{cases} 0 & \text{if } z=0 \\ f(x_1,\ldots,x_n,0)+\ldots+f(x_1,\ldots x_n,z-1) & \text{if } z>0 \end{cases}$$

$$\sum_{y\le z} f(x_1,\ldots,x_n,y) = \sum_{y<z+1} f(x_1,\ldots,x_n,y)$$

$$\prod_{y<z} f(x_1,\ldots,x_n,y) = \begin{cases} 1 & \text{if } z=0 \\ f(x_1\ldots x_n,0)\cdot\ldots\cdot f(x_1\ldots x_n,z-1) & \text{if } z>0 \end{cases}$$

$$\prod_{y\le z} f(x_1,\ldots,x_n,y) = \prod_{y<z+1} f(x_1,\ldots,x_n,y)$$

Doubly bounded sums and products

We define:

$$\sum_{u<y<v} f(x_1,\ldots,x_n,y) = \sum_{y<(v\dot{-}u)\dot{-}1} f(x_1\ldots x_n,y+u+1)$$

$$\left(= f(x_1\ldots x_n,u+1)+\ldots+f(x_1\ldots x_n,v-1)\right)$$

$$\prod_{u<y<v} f(x_1\ldots x_n,y) = \prod_{y<(v\dot{-}u)\dot{-}1} f(x_1\ldots x_n,y+u+1)$$

Proposition 1.4 If $f(x_1, \ldots, x_n, y)$ is primitive recursive (or recursive) then all bounded and doubly bounded sums and products are primitive recursive (or recursive).

Proof. Let $g(x_1, \ldots, x_n, 0) = 0$, and

$$g(x_1, \ldots, x_n, z+1) = g(x_1, \ldots, x_n, z) + f(x_1, \ldots, x_n, z).$$

Then g is primitive recursive (or recursive) by (V) and $g(x_1, \ldots, x_n, z) = \sum_{y<z} f(x_1, \ldots x_n, y)$.

Let $h(x_1, \ldots, x_n, z) = g(x_1, \ldots, x_n, z+1)$.
Then h is primitive recursive (or recursive) by (IV) and $h(x_1, \ldots, x_n, z) = \sum_{y \leq z} f(x_1, \ldots x_n, y)$.
The rest is left as an exercise. $\square$

By an n-ary relation we mean a subset of $\mathbb{N}^n$. If R is an n-ary relation then we often write $R(x_1, \ldots, x_n)$, $R(y_1, \ldots, y_n)$ etc. to show this and if $(k_1, \ldots, k_n) \in \mathbb{N}^n$ then $R(k_1, \ldots, k_n)$ means $(k_1, \ldots, k_n) \in R$.

The characteristic function of R is the function $C_R : \mathbb{N}^n \to \mathbb{N}$ such that

$$C_R(k_1, \ldots, k_n) = \begin{cases} 1 & \text{if } R(k_1, \ldots, k_n) \\ 0 & \text{if not } R(k_1, \ldots, k_n) \end{cases}.$$

We say that a relation is <u>primitive recursive</u> (or <u>recursive</u>) if its characteristic function is primitive recursive (or recursive).

<u>Examples</u>.

1. $x = y$ (or $\{(n, m) \in \mathbb{N}^2 : n = m\}$) is a primitive recursive relation because its characteristic function is $\overline{sg}(|x - y|)$.

2. $x < y$ is primitive recursive because its characteristic function is $\overline{sg}(y \dot{-} x)$

3. $x | y$ (x divides y) has characteristic function $\overline{sg}(rm(x, y))$ so is primitive recursive.

4. Let

$$Pr(x) \Leftrightarrow x \text{ is a prime}.$$

We show that Pr is primitive recursive.
First, let

$$\tau(x) = \begin{cases} 1 & \text{if } x = 0 \\ \text{the number of divisors of } x, & \text{if } x > 0. \end{cases}$$

Then $\tau(x) = \sum_{y \leq x} \overline{sg}(rm(y, x))$, so by .

Proposition 1.4 $\tau(x)$ is primitive recursive.

Since $C_{Pr}(x) =$

$$\overline{sg}\big((\tau(x) \dot{-} 2) + \overline{sg}(|x-1|) + \overline{sg}(|x-0|)\big)$$

it follows that Pr is primitive recursive.

If $R(x_1, \dots, x_n, y)$ is a relation then we defina a function

$$\mu y < z\, R(x_1, \dots, x_n, y) = \begin{cases} \text{the least } y < z \text{ such that } R(x_1, \dots, x_n, y) \text{ is true if such } y < z \text{ exists} \\ z \text{ otherwise} \end{cases}$$

We define $\mu y \leq z\, R(x_1, \dots, x_n, y)$ in the same way, by replacing $<$ by $\leq$.

<u>Proposition 1.5</u> Let $P(x_1, \dots, x_n)$, $Q(x_1, \dots, x_n)$ and $R(x_1, \dots, x_n, y)$ be primitive recursive (or recursive) relations. Then the following relations and functions are primitive recursive (or recursive):

$\neg P(x_1, \dots, x_n)$

$P(x_1, \dots, x_n) \wedge Q(x_1, \dots, x_n)$

$P(x_1, \ldots, x_n) \vee Q(x_1, \ldots, x_n)$

$P(x_1, \ldots, x_n) \rightarrow Q(x_1, \ldots, x_n)$

$P(x_1, \ldots, x_n) \longleftrightarrow Q(x_1, \ldots, x_n)$

$\forall y < z \; R(x_1, \ldots, x_n, y)$

$\forall y \leq z \; R(x_1, \ldots, x_n, y)$

$\exists y < z \; R(x_1, \ldots, x_n, y)$

$\exists y \leq z \; R(x_1, \ldots, x_n, y)$

$\mu y < z \; R(x_1, \ldots, x_n, y)$

$\mu y \leq z \; R(x_1, \ldots, x_n, y)$

Proof. Let C_P, C_Q and C_R be the characteristic functions of P, Q and R, respectively. Then

$$C_{\neg P}(x_1, \ldots, x_n) = 1 \dot{-} C_P(x_1, \ldots, x_n)$$

$$C_{P \wedge Q}(x_1, \ldots, x_n) = C_P(x_1, \ldots, x_n) \cdot C_Q(x_1, \ldots, x_n)$$

$$C_{P \vee Q} = C_{\neg(\neg P \wedge \neg Q)}, \quad C_{P \rightarrow Q} = C_{\neg P \vee Q}$$

$$C_{P \leftrightarrow Q} = C_{(P \rightarrow Q) \wedge (Q \rightarrow P)}$$

The characteristic function of $\forall y < z\, R(x_1, \ldots, x_n, y)$ is $\prod_{y<z} C_R(x_1, \ldots, x_n, y)$.

The characteristic function of $\exists y < z\, R(x_1, \ldots, x_n, y)$ is $\overline{sg}\left(\prod_{y<z} C_{\neg R}(x_1, \ldots, x_n, y)\right)$.

Similarly for $\exists y \leq z$ and $\forall y \leq z$.

We have

$$\mu y < z\, R(x_1, \ldots, x_n, y) = \sum_{y<z} \overline{sg}(C_R(x_1, \ldots, x_n, y))$$

and similarly for $\mu y \leq z$.

$\square$

We will now define some functions which will be usefull later when we want to code expressions (such as formulas etc.)

Let $p(x)$ = xth prime number in ascending order, so $p(0) = 2$, $p(1) = 3$, $p(2) = 5, \ldots$.

We will show that p is primitive recursive.

First observe that $p(x+1) \leq p(x)! + 1$.

Proof of this: Let q be a prime in the prime factorization of $p(x)! + 1$. If q divides

$p(x)!$ then q divides $(p(x)!+1) - p(x)! = 1$, so $q = 1$, contradicting the assumption that q is prime. Hence $q \nmid p(x)!$ so $q \neq p(y)$ for all $y \leq x$. Then $p(x+1) \leq q \leq p(x)!+1$.

Since $f(u,v) = \mu y \leq u\,(v < y \wedge Pr(y))$ is primitive recursive,

$g(v) = \mu y \leq (v!+1)\,(v < y \wedge Pr(y))$

is also primitive recursive (by substitution).

Since $p(0) = 2$ and

$p(x+1) = \mu y \leq (p(x)!+1)\,\big(p(x) < y \wedge Pr(y)\big)$

it follows that p is primitive recursive.

We will often denote $p(x)$ by p_x.

Now we define functions $f_i(x)$, for $i \in \mathbb{N}$, by $f_i(0) = 0$ and for $x > 0$ $f_i(x) = a_i$ where $x = p_0^{a_0} \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$ is the unique prime factorization of x into prime powers.

$f_i(x)$ will be denoted by $(x)_i$.

$(x)_i$ is primitive recursive because

$(x)_i = \mu y < x\,\Big(\,(p_i)^y \mid x \;\wedge\; \neg\big(\,(p_i)^{y+1} \mid x\,\big)\Big)$.

Define a function $lh(x)$ by $lh(0) = 0$, $lh(1) = 1$ and for $x > 1$ $lh(x) =$ the number y such that $p(y)$ is the largest prime that divides x.

Then $lh(x) = \mu y \leq x \left[p(y) \mid x \wedge \forall z \leq x \left(y < z \to \neg(p(z) \mid x) \right) \right]$

so lh is primitive recursive.

Let's define $x * y = x \cdot \prod_{i < lh(y)} \left(p(lh(x) + i) \right)^{(y)_i}$.

Then $*$ is primitive recursive and

$0 * y = 0$, $x * 0 = x * 1 = x$ and

$x * (y * z) = (x * y) * z$, if $y \neq 0$.

If $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m \in \mathbb{N}$, $a_n \neq 0$,

$x = p_0^{a_0} \cdot p_1^{a_1} \cdot \ldots \cdot p_n^{a_n}$ and $y = p_0^{b_0} \cdot p_1^{b_1} \cdot \ldots \cdot p_m^{b_m}$

then

$$x * y = p_0^{a_0} \cdot p_1^{a_1} \cdot \ldots \cdot p_n^{a_n} \cdot p_{n+1}^{b_0} \cdot p_{n+2}^{b_1} \cdot \ldots \cdot p_{n+m+1}^{b_m}$$

We call $*$ the <u>juxtaposition</u> function.

Proposition 1.6 Let $R_i(x_1, \ldots, x_n)$, for $1 \le i \le k$, be primitive recursive (or recursive) relations and let $g_i(x_1, \ldots, x_n)$, for $1 \le i \le k$, be primitive recursive (or recursive) functions. Suppose that for any $x_1, \ldots, x_n$ exactly one of $R_i(x_1, \ldots, x_n)$, $1 \le i \le k$, is true.

Then the following function is primitive recursive (or recursive).

$$f(x_1, \ldots, x_n) = \begin{cases} g_1(x_1, \ldots, x_n) & \text{if } R_1(x_1, \ldots, x_n) \text{ is true} \\ \vdots & \\ g_k(x_1, \ldots, x_n) & \text{if } R_k(x_1, \ldots, x_n) \text{ is true} \end{cases}$$

Proof. $f(x_1, \ldots, x_n) = g_1(x_1, \ldots, x_n) \cdot C_{R_1}(x_1, \ldots, x_n) +$

$$\vdots$$

$$+ g_k(x_1, \ldots, x_n) \cdot C_{R_k}(x_1, \ldots, x_n).$$

$\square$

Remark. All the recursive functions that we have seen are in fact primitive recursive. We may ask if all recursive functions are primitive recursive. The answer is no.

The Ackermann function defined by

$A(0, x) = 2^x$, for every $x \in \mathbb{N}$.

$A(y, 0) = 1$, for every $y \in \mathbb{N}$.

$A(y+1, x+1) = A(y, A(y+1, x))$ for all $x, y \in \mathbb{N}$.

is recursive but not primitive recursive.

We do not prove this here, since the proof is a bit involved.

A bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$

$$\begin{array}{lllllll}
(0,0) & \rightarrow & (0,1) & & (0,2) & & (0,3) \\
& \swarrow & & \swarrow & & \swarrow & \\
(1,0) & & (1,1) & & (1,2) & & \\
& \swarrow & & \swarrow & & & \\
(2,0) & & (2,1) & & & & \\
& \swarrow & & & & & \\
(3,0) & & & & & &
\end{array}
\qquad
\begin{array}{l}
(0,0) \mapsto 0 \\
(0,1) \mapsto 1 \\
(1,0) \mapsto 2 \\
(0,2) \mapsto 3 \\
(1,1) \mapsto 4 \\
\vdots
\end{array}$$

Call this bijection σ^2.

There are $z+1$ pairs (x,y) such that $x+y=z$.

Hence $\sigma^2(x,y) = \left(\sum_{k=0}^{x+y} k \right) + x$ $\qquad \boxed{\sum_{k=0}^{n} k = \frac{n^2+n}{2}}$

$$= \frac{(x+y)^2 + (x+y)}{2} + x$$

$$= \frac{(x+y)^2 + 3x + y}{2}$$

We see that σ^2 is primitive recursive.

Let $\sigma_1^2(z)$ = the first coordinate of $(\sigma^2)^{-1}(z)$

(that is, if $\sigma^2(x,y) = z$ then $\sigma_1^2(z) = x$)

and $\sigma_2^2(z)$ = the second coordinate of $(\sigma^2)^{-1}(z)$

(that is, if $\sigma^2(x,y) = z$ then $\sigma_2^2(z) = y$)

σ_1^2 and σ_2^2 are primitive recursive because

$$\sigma_1^2(z) = \mu x \leq z \left(\exists y \leq z \left(z = \sigma^2(x,y) \right) \right)$$

$$\sigma_2^2(z) = \mu y \leq z \left(\exists x \leq z \left(z = \sigma^2(x,y) \right) \right)$$

Observe that $\sigma_1^2(\sigma^2(x,y)) = x$

$$\sigma_2^2(\sigma^2(x,y)) = y$$

$$\sigma^2(\sigma_1^2(z), \sigma_2^2(z)) = z \, .$$

(1) Suppose that $n \geq 2$ and that there are a prim. rec. bijection $\sigma^n : \mathbb{N}^n \to \mathbb{N}$ and for every $1 \leq k \leq n$ prim. rec. injections $\sigma_k^n : \mathbb{N} \to \mathbb{N}$ such that

$$\sigma_k^n(\sigma^n(x_1 \ldots x_n)) = x_k$$

and $\sigma^n(\sigma_1^n(x), \ldots, \sigma_n^n(x)) = x$

Then we can define σ^{n+1} and σ_k^{n+1} for every $1 \leq k \leq n+1$ as follows

$$\sigma^{n+1}(x_1 \ldots x_{n+1}) = \sigma^2(\sigma^n(x_1 \ldots x_n), x_{n+1})$$

$$\sigma_{n+1}^{n+1}(x) = \sigma_2^2(x)$$

for $1 \leq k \leq n$

$$\sigma_k^{n+1}(x) = \sigma_k^n(\sigma_1^2(x))$$

Then it is easy to see that for every $1 \leq k \leq n+1$

$$\sigma_k^{n+1}(\sigma^{n+1}(x_1 \ldots x_{n+1})) = x_k$$

$$\sigma^{n+1}(\sigma_1^{n+1}(x), \ldots, \sigma_{n+1}^{n+1}(x)) = x$$

and that σ^{n+1} and σ_k^{n+1} are prim. rec.

Since (1) is true for $n=2$, we can, by induction, conclude that (1) holds for every n.

If we define a function $f(x_1 \ldots x_n, y)$ in a way such that

$f(x_1 \ldots x_n, y+1)$ depends not only on $f(x_1 \ldots x_n, y)$ but on several of the values $f(x_1 \ldots x_n, 0), \ldots, f(x_1 \ldots x_n, y)$

then we say that f is defined by <u>course of values recursion</u>.

For any function $f(x_1 \ldots x_n, y)$ let

$$f\#(x_1 \ldots x_n, y) = \prod_{u<y} p(u)^{f(x_1 \ldots x_n, u)}$$

Observe that $f(x_1 \ldots x_n, y) = (f\#(x_1 \ldots x_n, y+1))_y$

<u>Proposition 1.7</u> If $h(x_1 \ldots x_n, y, z)$ is primitive recursive (or recursive) and

$$f(x_1 \ldots x_n, y) = h(x_1 \ldots x_n, y, f\#(x_1 \ldots x_n, y))$$

then f is primitive recursive (or recursive)

Proof. We have

$$f\#(x_1 \ldots x_n, 0) = 1$$

$$f\#(x_1 \ldots x_n, y+1) = f\#(x_1 \ldots x_n, y) \cdot p(y)^{f(x_1 \ldots x_n, y)}$$

$$= f\#(x_1 \ldots x_n, y) \cdot p(y)^{h(x_1 \ldots x_n, y, f\#(x_1 \ldots x_n, y))}$$

so $f\#$ is prim. rec. (or rec.) and since

$$f(x_1 \ldots x_n, y) = (f\#(x_1 \ldots x_n, y+1))_y$$

it follows (by substitution (IV)) that f is prim. rec. (or rec.). $\square$

Corollary 1.8 If $P(x_1 \ldots x_n, y, z)$ is a primitive recursive (or recursive) relation and the relation $R(x_1 \ldots x_n, y)$ holds if and only if $P(x_1 \ldots x_n, y, C_R\#(x_1 \ldots x_n, y))$ holds (where C_R is the characteristic function of R) then R is primitive recursive (or recursive).

Proof. $C_R(x_1 \ldots x_n, y) = C_P(x_1 \ldots x_n, y, C_R\#(x_1 \ldots x_n, y))$

so by Prop. 1.7 C_R is prim. rec. (or rec.) $\square$

Example

Define $f(0) = f(1) = 1$, $f(x+2) = f(x) + f(x+1)$.
We show that f is primitive recursive.

Define $g(x) = \sigma^2(f(x), f(x+1))$.

Then $g(0) = \sigma^2(1,1)$

$$g(x+1) = \sigma^2(\sigma_2^2(g(x)), \sigma_1^2(g(x)) + \sigma_2^2(g(x)))$$

so g is primitive recursive.

Since $f(x) = \sigma_1^2(g(x))$ it follows that f is prim. rec.

Alternative solution:

Define $h(y,z) = \begin{cases} 1 & \text{if } y=0 \vee y=1 \\ (z)_{y-2} + (z)_{y-1} & \text{if } y \geq 2 \end{cases}$

Then h is prim. rec. by Proposition 1.6,
and $f(x) = h(x, f\#(x))$ so by Proposition 1.7
f is prim. rec.