Arithmetization of the syntax

To each one of the symbols

$\neg \quad \wedge \quad \rightarrow \quad \forall \quad (\quad) \quad 0 \quad s \quad + \quad \cdot \quad = \quad , \quad x_i$.

(where $i = 1, 2, 3, \ldots$) we associate a number $g(\ )$ (called Gödel number) as follows:

$g(\neg) = 1$, $g(\wedge) = 2$, $g(\rightarrow) = 3$, $g(\forall) = 4$,

$g(() = 5$, $g()) = 6$, $g(0) = 7$, $g(s) = 8$, $g(+) = 9$,

$g(\cdot) = 10$, $g(=) = 11$, $g(,) = 12$, $g(x_i) = 12 + i$.

A finite sequence of symbols (as given above) will be called an expression.

To every expression $u_0, u_1, \ldots, u_k$ we associate a number (called Gödel number) by

$$g(u_0, u_1, \ldots, u_k) = p_0^{g(u_0)} \cdot p_1^{g(u_1)} \cdot \ldots \cdot p_k^{g(u_k)}.$$

We will now define several functions and relations which are primitive recursive.

$Gd(x) \Leftrightarrow x$ is the Gödel number of an expression.

Gd is primitive recursive because

$Gd(x) \Leftrightarrow x > 1 \wedge \forall y < x \left[\forall z < y \left(p(y) | x \rightarrow p(z) | x \right) \right]$.

For any symbol u, let <u>$\#u$</u> denote the Gödel number of u (so for example, $\#\neg = 1$, $\#\wedge = 2$).

$\mathrm{Trm}(x) \iff x$ is the Gödel number of a term

$$\iff \exists y<x\,[y>12 \wedge x=2^y]$$

$$\vee\ x=2^{\#0} \vee \exists y<x\left[x=2^{\#S} * 2^{\#(} * y * 2^{\#)} \wedge \mathrm{Trm}(y)\right]$$

$$\vee \exists y<x\,\exists z<x\left[x=2^{\#+} * 2^{\#(} * y * 2^{\#,} * z * 2^{\#)} \wedge \mathrm{Trm}(y) \wedge \mathrm{Trm}(z)\right]$$

$$\vee \exists y<x\,\exists z<x\left[x=2^{\#\cdot} * 2^{\#(} * y * 2^{\#,} * z * 2^{\#)} \wedge \mathrm{Trm}(y) \wedge \mathrm{Trm}(z)\right]$$

Here we used course of values recursion but by Proposition 1.7 (or Corollary 1.8) Trm is primitive recursive.

$\mathrm{Atom}(x) \iff x$ is the Gödel number of an atomic formula.

$$\iff \exists y<x\,\exists z<x\left[x=2^{\#=} * 2^{\#(} * y * 2^{\#,} * z * 2^{\#)} \wedge \mathrm{Trm}(y) \wedge \mathrm{Trm}(z)\right].$$

$\mathrm{Form}(x) \iff x$ is the Gödel number of a formula $\iff$

$$\mathrm{Atom}(x) \vee \exists y<x\,\exists z<x\Big[\left(x=2^{\#\neg} * y \wedge \mathrm{Form}(y)\right)$$

$$\vee \left(x=2^{\#(} * y * 2^{\#\wedge} * z * 2^{\#)} \wedge \mathrm{Form}(y) \wedge \mathrm{Form}(z)\right)$$

$$\vee \left(x=2^{\#(} * y * 2^{\#\to} * z * 2^{\#)} \wedge \mathrm{Form}(y) \wedge \mathrm{Form}(z)\right)$$

$$\vee \left(x = 2^{\#(} * 2^{\#\forall} * 2^{y} * z * 2^{\#)} \wedge Form(z) \wedge y > 12 \right) \Big]$$

$Subst(x, y, u, v) \Longleftrightarrow$ x is the Gödel number of the result of substituting in the expression with Gödel number y the term with Gödel number u for all free occurences of the variable with Gödel number v.

$$\Longleftrightarrow \; Gd(y) \wedge Trm(u) \wedge v > 12$$

$$\wedge \Big[(y = v \wedge x = u) \vee$$

$$\vee \exists w < y \left(y = 2^{w} \wedge y \neq 2^{v} \wedge x = y \right)$$

$$\vee \exists z < y \, \exists w < y \Big(Form(w) \wedge y = 2^{\#(} * 2^{\#\forall} * 2^{v} * w * 2^{\#)} * z$$

$$\wedge \exists \alpha < x \left(x = 2^{\#(} * 2^{\#\forall} * 2^{v} * w * 2^{\#)} * \alpha \wedge Subst(\alpha, z, u, v) \right) \Big)$$

$$\vee \Big(\neg \exists z < y \, \exists w < y \left(Form(w) \wedge y = 2^{\#(} * 2^{\#\forall} * 2^{v} * w * 2^{\#)} * z \right)$$

$$\wedge \exists \alpha < x \, \exists \beta < x \, \exists z < y \Big(z > 1 \wedge y = 2^{(y)_0} * z \wedge x = \alpha * \beta$$

$$\wedge Subst(\alpha, 2^{(y)_0}, u, v)$$

$$\wedge Subst(\beta, z, u, v) \Big) \Big) \Big]$$

$Sub(y, u, v) =$ the Gödel number of the result of substituting in the expression with Gödel number y the term with Gödel number u for all free occurences of the variable with Gödel number v.

$$Sub(y, u, v) = \mu x < (p(u \cdot y)!)^{u \cdot v} \; Subst(x, y, u, v).$$

$Fr(y,v) \iff y$ is the Gödel number of a formula or a term that contains free occurences of the variable with Gödel number v.

$$\iff (Form(y) \vee Trm(y)) \wedge v > 12 \wedge \neg Subst(y, y, 2^{v+1}, v)$$

$Ff(u,v,w) \iff u$ is the Gödel number of a term that is free for the variable with Gödel number v in the formula with Gödel number w.

$$\iff Trm(u) \wedge v > 12 \wedge Form(w) \wedge$$

$$\Big[\, Atom(w)$$

$$\vee\ \exists y < w\,(w = 2^{\#\neg} * y \wedge Ff(u,v,y))$$

$$\vee\ \exists y < w\, \exists z < w\,(w = 2^{\#(} * y * 2^{\#\wedge} * z * 2^{\#)} \wedge Ff(u,v,y) \wedge Ff(u,v,z))$$

$$\vee\ \exists y < w\, \exists z < w\,(w = 2^{\#(} * y * 2^{\#\to} * z * 2^{\#)} \wedge Ff(u,v,y) \wedge Ff(u,v,z))$$

$$\vee\ \exists y < w\, \exists z < w\,(w = 2^{\#(} * 2^{\#\forall} * 2^{z} * y * 2^{\#)} \wedge z > 12$$

$$\wedge\,(z = v \vee \neg Fr(y,v) \vee (Ff(u,v,y) \wedge \neg Fr(u,z))))\Big]$$

We will now code derivations in natural deduction. First we give the derivation rules (for a language with the logical symbols $\neg\ \wedge\ \to\ \forall$):

$$\wedge I \quad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad\qquad \wedge E \quad \frac{\varphi \wedge \psi}{\varphi} \text{ and } \frac{\varphi \wedge \psi}{\psi}$$

$$\to I \quad \frac{\begin{matrix}\cancel{\varphi}\\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \qquad\qquad \to E \quad \frac{\varphi \to \psi \quad \varphi}{\psi}$$

$$\neg I \quad \frac{\begin{matrix}\cancel{\varphi}\\ \vdots \\ \theta \wedge \neg\theta\end{matrix}}{\neg \varphi} \qquad\qquad \neg E \quad \frac{\neg\neg\varphi}{\varphi}$$

$\forall I \quad \dfrac{\varphi}{\forall x \varphi}$ where x may not occur free in any hypothesis in the derivation of φ.

$\forall E \quad \dfrac{\forall x \varphi}{\varphi[t/x]}$ where the term t is free for x in φ.

(See for example 'Logic and structure' by Van Dalen for more about natural deduction, but note that instead of $\neg I$ and $\neg E$, above, he has two other rules which give an equivalent system.)

Example of derivation :

$$\textcircled{2}\;\dfrac{\textcircled{1}\;\dfrac{\dfrac{\textcircled{1}\;\dfrac{\cancel{\varphi\wedge\psi}}{\psi}\wedge E \qquad \dfrac{\textcircled{1}\;\dfrac{\cancel{\varphi\wedge\psi}}{\varphi}\wedge E \qquad \cancel{\varphi\to(\psi\to\sigma)}\;\textcircled{2}}{\psi\to\sigma}\to E}{\sigma}}{\varphi\wedge\psi\to\sigma}\to I}{(\varphi\to(\psi\to\sigma))\to(\varphi\wedge\psi\to\sigma)}\to I$$

We call the formulas at "the top" <u>hypotheses</u>
The numbers indicate how a hypothesis was <u>cancelled</u>.

Note that a derivation has the form ψ (just a hypothesis)

or $\dfrac{\Pi_1'}{\psi}$ or $\dfrac{\Pi_1' \quad \Pi_2'}{\psi}$ where Π_1 and Π_2 are derivations

and Π_i' (for $i=1,2$) is obtained by eventually cancelling some hypotheses in Π_i (if the rule used to get ψ is $\to I$ or $\neg I$) and ψ is obtained by applying some derivation rule to the last formula in Π_1 (or the last formulas in Π_1 and Π_2).
We say that Π_1' is a <u>subderivation</u> of

$\dfrac{\Pi_1'}{\psi}$ and that Π_1' and Π_2' are <u>subderivations</u> of

$\dfrac{\Pi_1' \quad \Pi_2'}{\psi}$.

To every uncancelled hypothesis (in a derivation) we associate the number 0 and to every cancelled hypothesis we associate a number greater than 0 (as in the example) which corresponds to the particular application of a rule which allow us to cancel it.

Inductively, we now associate a Gödel number, $\#\Pi$, for every subderivation Π (of some derivation).

- If Π is an uncancelled hypothesis φ, then
$$\#\Pi = \sigma^4(0,0,0,\#\varphi)$$
where $\#\varphi$ is the Gödel number of φ.

- If Π is a cancelled hypothesis with which is associated the number $k > 0$, then
$$\#\Pi = \sigma^4(0,0,k,\#\varphi)$$

- If Π is $\dfrac{\Pi_1 \quad \Pi_2}{\varphi \wedge \psi}\wedge I$ then
$$\#\Pi = \sigma^4(1, \#\Pi_1, \#\Pi_2, \#(\varphi\wedge\psi))$$

- If Π is $\dfrac{\Pi_1}{\varphi}\wedge E$ then
$$\#\Pi = \sigma^4(2, \#\Pi_1, 0, \#\varphi)$$

- If Π is $k\ \dfrac{\Pi_1}{\varphi\to\psi}\to I$ then
$$\#\Pi = \sigma^4(3, \#\Pi_1, k, \#(\varphi\to\psi))$$

- If Π is $\dfrac{\Pi_1 \quad \Pi_2}{\psi}\to E$ then
$$\#\Pi = \sigma^4(4, \#\Pi_1, \#\Pi_2, \#\psi)$$

- If Π is $\textcircled{k}\dfrac{\Pi_1}{\neg\varphi}\neg I$ then

$$\#\Pi = \sigma^4(5, \#\Pi_1, k, \#\neg\varphi)$$

- If Π is $\dfrac{\Pi_1}{\varphi}\neg E$ then

$$\#\Pi = \sigma^4(6, \#\Pi_1, 0, \#\varphi)$$

- If Π is $\dfrac{\Pi_1}{\forall x\varphi}\forall I$ then

$$\#\Pi = \sigma^4(7, \#\Pi_1, 0, \#(\forall x\varphi))$$

- If Π is $\dfrac{\Pi_1}{\varphi[t/x]}\forall E$ then

$$\#\Pi = \sigma^4(8, \#\Pi_1, 0, \#\varphi[t/x])$$

Define a primitive recursive function by

$$Unc(x,y) = \begin{cases} x & \text{if } \sigma_1^4(x)=0 \text{ and } \sigma_3^4(x)\neq y \\ \sigma^4(0,0,0,\sigma_4^4(x)) & \text{if } \sigma_1^4(x)=0 \text{ and } \sigma_3^4(x)=y \\ \sigma^4(1, Unc(\sigma_2^4(x),y), Unc(\sigma_3^4(x),y), \sigma_4^4(x)) & \\ \quad \text{if } \sigma_1^4(x)=1 & \\ \sigma^4(2, Unc(\sigma_2^4(x),y), 0, \sigma_4^4(x)) & \text{if } \sigma_1^4(x)=2 \\ \sigma^4(3, Unc(\sigma_2^4(x),y), \sigma_3^4(x), \sigma_4^4(x)) & \text{if } \sigma_1^4(x)=3 \\ \sigma^4(4, Unc(\sigma_2^4(x),y), Unc(\sigma_3^4(x),y), \sigma_4^4(x)) & \text{if } \sigma_1^4(x)=4 \end{cases}$$

$$\begin{cases} \sigma^4(5, Unc(\sigma^4_2(x), y), \sigma^4_3(x), \sigma^4_4(x)) & \text{if } \sigma^4_1(x) = 5 \\ \sigma^4(6, Unc(\sigma^4_2(x), y), 0, \sigma^4_4(x)) & \text{if } \sigma^4_1(x) = 6 \\ \sigma^4(7, Unc(\sigma^4_2(x), y), 0, \sigma^4_4(x)) & \text{if } \sigma^4_1(x) = 7 \\ \sigma^4(8, Unc(\sigma^4_2(x), y), 0, \sigma^4_4(x)) & \text{if } \sigma^4_1(x) = 8 \\ x & \text{if } \sigma^4_1(x) > 8 \end{cases}$$

Unc is primitive recursive because $\sigma^4_i(x) < x$ for all $1 \le i \le 4$ and $x > 0$.

For any subderivation Π, if Π is obtained from Π' by associating to some uncalled hypotheses in Π the number y (hence, making them cancelled), then $Unc(\#\Pi, y) = \#\Pi'$.
Also note that $Unc(x, y) \le x$ for all x, y.

Define a primitive recursive relation by

$$\begin{aligned} Free(x, v) \iff & (\sigma^4_1(x) = 0 \wedge \sigma^4_3(x) > 0 \wedge Fr(\sigma^4_4(x), v)) \\ & \vee (\sigma^4_1(x) = 1 \wedge (Free(\sigma^4_2(x), v) \vee Free(\sigma^4_3(x), v))) \\ & \vee (\sigma^4_1(x) = 2 \wedge Free(\sigma^4_2(x), v)) \\ & \vdots \quad \text{similarly for every derivation rule} \\ & \vee (\sigma^4_1(x) = 8 \wedge Free(\sigma^4_2(x), v)). \end{aligned}$$

Then for every subderivation Π, $Free(\#\Pi, v)$ if and only if v is a Gödel number of a variable that occurs free in some uncancelled hypothesis of Π.

It will be convenient to use the following primitive recursive functions:

$$Neg(x) = 2^{\#\neg} * x$$

$$Imp(x, y) = 2^{\#(} * x * 2^{\#\rightarrow} * y * 2^{\#)}$$

$$Conj(x, y) = 2^{\#(} * x * 2^{\#\wedge} * y * 2^{\#)}$$

$$Gen(x, v) = 2^{\#(} * 2^{\#\forall} * 2^{v} * x * 2^{\#)}$$

Let $Der(x) \Leftrightarrow x$ is the Gödel number of a derivation (in natural deduction).

Then Der is primitive recursive because:

$Der(x) \Leftrightarrow$

$$\left[x = \sigma^4(0, 0, 0, \sigma^4_4(x)) \wedge Form(\sigma^4_4(x))\right]$$

$$\vee \left[\sigma^4_1(x) = 1 \wedge Der(\sigma^4_2(x)) \wedge Der(\sigma^4_3(x)) \wedge Form(\sigma^4_4(x)) \wedge \sigma^4_4(x) = Conj(\sigma^4_4(\sigma^4_2(x)), \sigma^4_4(\sigma^4_3(x)))\right]$$

$$\vee \left[\sigma^4_1(x) = 2 \wedge Der(\sigma^4_2(x)) \wedge \exists y < x (Form(y) \wedge (\sigma^4_4(\sigma^4_2(x)) = Conj(y, \sigma^4_4(x)) \vee \sigma^4_4(\sigma^4_2(x)) = Conj(\sigma^4_4(x), y))\right]$$

$$\vee \left[\sigma^4_1(x) = 3 \wedge Der(Unc(\sigma^4_2(x), \sigma^4_3(x))) \wedge \exists y < x (Form(y) \wedge \sigma^4_4(x) = Imp(y, \sigma^4_2(\sigma^4_4(x)))\right].$$

$$\vee \Big[\sigma_1^4(x) = 4 \wedge Der(\sigma_2^4(x)) \wedge Der(\sigma_3^4(x)) \wedge Imp(\sigma_4^4(\sigma_3^4(x)), \sigma_4^4(x)) = \sigma_4^4(\sigma_2^4(x)) \Big]$$

$$\vee \Big[\sigma_1^4(x) = 5 \wedge Der(Unc(\sigma_2^4(x), \sigma_3^4(x)) \wedge \exists y < x \, \exists z < x (Form(y) \wedge \sigma_4^4(x) = Neg(y) \wedge Hyp(Unc(\sigma_2^4(x), \sigma_3^4(x)), y) \wedge Form(z) \wedge \sigma_4^4(\sigma_2^4(x)) = Conj(z, Neg(z)) \Big]$$

* see below

$$\vee \Big[\sigma_1^4(x) = 6 \wedge Der(\sigma_2^4(x)) \wedge \sigma_3^4(x) = 0 \wedge \sigma_4^4(\sigma_2^4(x)) = Neg(Neg(\sigma_4^4(x))) \Big]$$

$$\vee \Big[\sigma_1^4(x) = 7 \wedge Der(\sigma_2^4(x)) \wedge \sigma_3^4(x) = 0 \wedge \exists v < x (v > 12 \wedge \sigma_4^4(x) = Gen(\sigma_4^4(\sigma_2^4(x)), v) \wedge \neg Free(\sigma_2^4(x), v)) \Big]$$

$$\vee \Big[\sigma_1^4(x) = 8 \wedge Der(\sigma_2^4(x)) \wedge \sigma_3^4(x) = 0 \wedge \exists y < x \, \exists v < x \, \exists z < x (\sigma_4^4(\sigma_2^4(x)) = Gen(y, v) \wedge Form(y) \wedge Subst(\sigma_4^4(x), y, z, v) \wedge Ff(z, v, y)) \Big]$$

* Hyp is defined on the next page and has the property that if Π is a subderivation and φ is a formula, then $Hyp(\#\Pi, \#\varphi)$ if and only

if φ is an uncancelled hypothesis in Π.
Now we define

$$Hyp(x,y) \iff [\sigma_1^4(x)=0 \wedge \sigma_3^4(x)=0 \wedge \sigma_4^4(x)=y]$$

$$\vee [\sigma_1^4(x)=1 \wedge (Hyp(\sigma_2^4(x),y) \vee Hyp(\sigma_3^4(x),y))]$$

$$\vee [\sigma_1^4(x)=2 \wedge Hyp(\sigma_2^4(x),y)]$$

$\vdots$ and similarly for every derivation rule

$$\vee [\sigma_1^4(x)=8 \wedge Hyp(\sigma_2^4(x),y)].$$

<u>Remark 3.1</u> Note that the symbols $\vee$ $\leftrightarrow$ and $\exists$ have not been given Gödel numbers, but, as logical symbols, these can be expressed in terms of $\neg$ $\wedge$ $\rightarrow$ and $\forall$. If we assume that

- $\varphi \vee \psi$ abbreviates $\neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- $\exists x \varphi$ abbreviates $\neg \forall x \neg \varphi$

then we can still formaly derive (within natural deduction, for example) (1)–(32) and we can still prove the

Representation Theorem (Theorem 2.7).

If Π is a derivation and the last formula in Π is φ and all uncancelled hypotheses in Π belong to T, then we say that Π is <u>a proof of φ from T</u> (or that Π is a <u>derivation of φ from T</u>).

We say that a theory $T \subseteq L_A$ is <u>recursive</u> if the relation

$T(x) \Leftrightarrow x$ is the Gödel number of a sentence in T

is recursive.

<u>Proposition 3.2</u> If $T \subseteq L_A$ is a recursive theory then the relation

$Pf_T(x, y) \Leftrightarrow x$ is the Gödel number of a proof of the formula with Gödel number y from T

is recursive.

Proof. $Pf_T(x,y) \Leftrightarrow Der(x) \wedge \sigma_4^4(x) = y$

$$\wedge \forall z < x \left[Form(z) \wedge Hyp(x,z) \rightarrow T(z) \right]$$

where $T(z) \Leftrightarrow z$ is the Gödel number of a sentence in T. $\square$

Remark 3.3 For any countable vocabulary (i.e. any countable set of constant symbols, function symbols and relation symbols), we can give a Gödel number to every symbol (but the Gödel numbering may become more complicated than the one given) such that all functions and relations defined in this section are still primitive recursive. From this it follows that if $T \subseteq L$, where L is a countable first order languge then Proposition 3.2 still holds if we replace L_A by L.

Proposition 3.4 $\mathcal{P}$ is a recursive theory.

Proof. Tedious, but in principle not difficult. The axioms of equality for L_A and axioms $A_1 - A_7$ are straightforward.* Taking care of the scheme of induction requires a little bit more thought.
(* assume that $x = x_1$, $y = x_2$, $z = x_3$, $y_1 = x_4$, $y_2 = x_5$.) $\square$

The following function and relation are primitive recursive.

Num(x) = the Gödel number of the numeral $\underline{x}$.

$$Num(0) = 2^{\#0}$$

$$Num(x+1) = 2^{\#S} * 2^{\#(} * Num(x) * 2^{\#)}$$

$Nu(x) \iff x$ is the Gödel number of a numeral

$\iff \exists y < x\, (x = Num(y))$.

<u>Theorem 3.5</u> If $\mathcal{P}$ is consistent then a function is recursive if and only if it is representable.

Proof. By Theorem 2.7 we need only prove that every function that is representable is recursive. Let $f(x_1, \ldots, x_k)$ be represented by $F(x_1, \ldots, x_k, x_{k+1})$. Define a relation R by

$R(n_1, \ldots, n_k, n, m) \iff m$ is the Gödel number of a proof of $F(\underline{n}_1, \ldots, \underline{n}_k, \underline{n})$ from $\mathcal{P}$.

First we show that, for all $n_1, \ldots, n_k, n, m \in \mathbb{N}$,

(*) if $R(n_1, \ldots, n_k, n, m)$ is true then $f(n_1, \ldots, n_k) = n$.

Suppose that $R(n_1, \ldots, n_k, n, m)$ is true.
Then $\mathcal{P} \vdash F(\underline{n}_1, \ldots, \underline{n}_k, \underline{n})$ and since f is represented by F we have

$$\mathcal{P} \vdash \forall x_{k+1}(F(\underline{n}_1, \ldots, \underline{n}_k, x_{k+1}) \leftrightarrow x_{k+1} = \underline{f(n_1, \ldots, n_k)})$$

Therefore $\mathcal{P} \vdash \underline{n} = \underline{f(n_1, \ldots, n_k)}$. If $n \neq f(n_1, \ldots, n_k)$ then (by (29)) $\mathcal{P} \vdash \underline{n} \neq \underline{f(n_1, \ldots, n_k)}$ which contradicts the assumption that $\mathcal{P}$ is consistent. Hence $n = f(n_1, \ldots, n_k)$ and the proof of (*) is complete.

Let l be the Gödel number of $F(x_1, \ldots, x_k, x_{k+1})$.

Then $R(n_1, \ldots, n_k, n, m) \Longleftrightarrow$

$$\exists z_1 < m, \ldots, \exists z_{k+1} < m \Big[$$

$$z_1 = Sub(l, Num(n_1), 13)$$

$$\wedge\ z_2 = Sub(z_1, Num(n_2), 14)$$

$$\vdots$$

$$\wedge\ z_k = Sub(z_{k-1}, Num(n_k), 13+k-1)$$

$$\wedge\ z_{k+1} = Sub(z_k, Num(n), 13+k)$$

$$\wedge\ Pf_{\mathcal{P}}(m, z_{k+1}) \Big]$$

Since $Pf_{\mathcal{P}}$ is recursive, by Proposition 3.2 and Proposition 3.4, it follows that R is recursive.

For arbitrary $n_1, \ldots, n_k \in \mathbb{N}$ we have $\mathcal{P} \vdash F(\underline{n}_1, \ldots, \underline{n}_k, \underline{f(n_1, \ldots, n_k)})$, since f is represented by F, and hence there is some $m \in \mathbb{N}$ such that $R(n_1, \ldots, n_k, f(n_1, \ldots, n_k), m)$ holds, which implies that there is some $r \in \mathbb{N}$ such that $R(n_1, \ldots, n_k, (r)_0, (r)_1)$ holds. Hence if g is the characteristic function of the relation $P(x_1, \ldots, x_k, y) \Leftrightarrow R(x_1, \ldots, x_k, (y)_0, (y)_1)$ then $\mu y(\overline{sg}(g(x_1, \ldots, x_k, y)) = 0)$ is a recursive function, and from $(*)$ it follows that $f(x_1, \ldots, x_k) = (\mu y(\overline{sg}(g(x_1, \ldots, x_k, y)) = 0))_0$ $\square$

Undecidability and Incompleteness

We say that a theory $T \subseteq L$, where L is any countable first order language, is <u>decidable</u> if the set (or equivalently, unary relation)

$$\{\#\varphi : \varphi \in L \text{ is a sentence and } T \vdash \varphi\}$$

is recursive. Otherwise we say that T is <u>undecidable</u>.

Recall that we say that T is recursive if the set

$$\{\#\varphi : \varphi \in T\}$$

is recursive.

We say that T is <u>complete</u> if for every sentence $\varphi \in L$, $T \vdash \varphi$ or $T \vdash \neg\varphi$. Otherwise T is incomplete.

<u>Proposition 4.1</u> If L is a countable first order language and $T \subseteq L$ is a complete and recursive theory then T is decidable.

Proof. Suppose that T is complete, recursive <u>and consistent</u>. By Remark 3.3 the relation

$Pf_T(x,y) \iff$ x is the Gödel number of a proof of the formula with Gödel number y from T.

is recursive. Since T is complete and consistent, (for every sentence $\varphi \in L$) there exists $n \in \mathbb{N}$ such that <u>exactly one</u> of $Pf_T(n, \#\varphi)$ and $Pf_T(n, \#\neg\varphi)$ holds,

and therefore, by applying the μ-operator,

if $R(x,y) \iff \neg Form(y) \vee Pf_T(x,y) \vee Pf_T(x, Neg(y))$,
then the function $f(y) = \mu x\,(\overline{sg}\,(C_R(x,y) \dot{-} 0)$
(where C_R is the characteristic function of R)
is recursive. If, $T \vdash \varphi$ then
$Pf_T(f(\#\varphi), \#\varphi)$ holds. If $T \nvdash \varphi$ then
$T \vdash \neg\varphi$ (by completeness) so $f(\#\varphi)$ is the
Gödel number of a proof of $\neg\varphi$ from T
and therefore $Pf_T(f(\#\varphi), \#\varphi)$ is false.
Hence $T \vdash \varphi$ if and only if $Pf_T(f(\#\varphi), \#\varphi)$ holds,
and since $Form(x) \wedge Pf_T(f(x), x)$ is recursive
it follows that T is decidable.
If T is inconsistent then $T \vdash \varphi$ for all sentences
and then $\{\#\varphi : \varphi \in L \text{ is a sentence and } T \vdash \varphi\} =$
$= \{\#\varphi : \varphi \in L \text{ is a sentence}\}$ which
by earlier results (adapted to T) a recursive
set.

$\square$

In the rest of this section let $\mathcal{L}$ be a countable first order language such that $\mathcal{L}_A \subseteq \mathcal{L}$. By a theory we will mean an $\mathcal{L}$-theory (i.e. a set of sentences from $\mathcal{L}$).

Theorem 4.2 (Church's theorem)

If T is a consistent theory such that $T \vdash \mathcal{P}$ then T is undecidable.

Proof. Suppose for a contradiction, that $T \vdash \mathcal{P}$ and T is consistent and decidable. Then the relation

$$\mathrm{Prov}_T(x) \iff x \text{ is the Gödel number of a sentence } \varphi \in \mathcal{L}_A \text{ such that } T \vdash \varphi$$

is recursive. Define

$$R(x,y) \iff \mathrm{Prov}_T(\mathrm{Sub}(y, \mathrm{Num}(x), \# x_1))$$

and

$$P(x) \iff \neg R(x,x).$$

$\boxed{\text{So } R(n,m) \iff m = \#\varphi(x_1) \text{ for some } \varphi(x_1) \in \mathcal{L}_A \text{ such that } T \vdash \varphi(\underline{n}).}$

Then P is a recursive relation, so P is represented by a formula $\varphi(x_1)$. Let $q = \#\varphi(x_1)$. Suppose that $P(q)$ is true. Then, since $\varphi(x_1)$ represents P, $\mathcal{P} \vdash \varphi(\underline{q})$, so $T \vdash \varphi(\underline{q})$. But, by the definition of P, $R(q,q)$ is false, which means that $T \nvdash \varphi(\underline{q})$, a contradiction. Hence $P(q)$ must be false. But then, since $\varphi(x_1)$ represents P, $\mathcal{P} \vdash \neg\varphi(\underline{q})$, so $T \vdash \neg\varphi(\underline{q})$, and by the definition of P, $R(q,q)$ is true so $Prov_T(Sub(q, Num(q), \#x_1)$ is true which means that $T \vdash \varphi(\underline{q})$, so T is inconsistent. Hence, we conclude that if $\mathcal{P} \subseteq T$ and T is consistent then T is undecidable. $\square$

Theorem 4.3. (Gödel-Rosser incompleteness theorem) If T is a recursive and consistent theory such that $T \vdash \mathcal{P}$ then T is incomplete.

Proof. If T is recursive, consistent, complete and $T \vdash \mathcal{P}$ then by Proposition 4.1 T is decidable which contradicts Theorem 4.2. $\square$

Corollary 4.4 $\mathcal{P}$ is undecidable and incomplete.

Proof. Follows from Proposition 3.4, Theorem 4.2 and Theorem 4.3. $\square$

Remark 4.5 If we replace $\mathcal{P}$ by $\mathcal{P}_0$ in Theorem 4.2, Theorem 4.3 and Corollary 4.4 then they remain true, but the proofs become more complicated, the main obstacle being to prove an analogue of the representation theorem, i.e to prove that for every recursive function $f(x_1, \ldots, x_k)$ there is a formula $F(x_1, \ldots, x_k, y)$

such that for all $n_1, \ldots, n_k \in \mathbb{N}$.

$$(*) \qquad \mathcal{P}_0 \vdash \forall y \left(F(\underline{n}_1, \ldots, \underline{n}_k, y) \leftrightarrow y = \underline{f(n_1, \ldots, n_k)} \right).$$

However, if we let $\mathcal{P}_1$ consist of the formulas in $\mathcal{P}_0$ together with (32), the result of replacing $\mathcal{P}_0$ by $\mathcal{P}_1$ in $(*)$ can be proved in a similar fashion as the proof of Theorem 2.7, and it follows that Theorem 4.2, Theorem 4.3 and Corollary 4.4 remain true if we replace $\mathcal{P}$ by $\mathcal{P}_1$. Note that $\mathcal{P}_0$ and $\mathcal{P}_1$ are finite theories.

Let T be a recursive theory such that $T \vdash \mathcal{P}$. By Proposition 3.2 and Remark 3.3 the relation

$Pf_T(x_2, x_1) \iff x_2$ is the Gödel number of a proof of the formula with Gödel number x_1 from T.

is recursive, and hence, by Theorem 2.7, is represented by a formula $\mathcal{Pf}_T(x_2, x_1)$. Let $\mathcal{P}rov_T(x_1)$ be the formula $\exists x_2\, \mathcal{Pf}_T(x_2, x_1)$, and let $\mathcal{C}on_T$ be the sentence $\neg \mathcal{P}rov_T(\ulcorner \underline{0} = \underline{1} \urcorner)$ where, for any expression E, $\ulcorner E \urcorner$ is the numeral $\underline{n}$ where $n = \#E$ ($\#E$ is the Gödel number of E).

Intuitively $\mathcal{C}on_T$ expresses that $\underline{0} = \underline{1}$ is not provable from T.

Let $\mathcal{N}$ be the L_A-structure which universe is $\mathbb{N}$ (the set of natural numbers) and in which

- the constant symbol 0 is interpreted as the natural number $0 \in \mathbb{N}$.
- the symbol s is interpreted as the successor function on $\mathbb{N}$.
- the symbols $+$ and $\cdot$ are interpreted as addition and multiplication, respectively on $\mathbb{N}$.
- the symbol $=$ is interpreted as the identity relation on $\mathbb{N}^2$.

Then we see that $\mathcal{N}$ is a model of $\mathcal{P}$ (if we assume that the successor function, addition and multiplication satisfies the properties formalized by $A_1 - A_7$ and if we assume that the principle of induction holds for the natural numbers), so in particular $\mathcal{P}$ is consistent. This can, however, not be proved with methods which can be formalized in L_A, as we will see.

$\mathcal{N}$ is called the <u>standard model</u> of $\mathcal{P}$ (or the standard model of arithmetic). All other models of $\mathcal{P}$ are called <u>nonstandard</u> models of $\mathcal{P}$.

Observe the following, where we assume $\mathcal{N} \models \mathcal{P}$:
If $T \nvdash \underline{0}=\underline{1}$ then for all $n \in \mathbb{N}$, $Pf_T(n, \#\underline{0}=\underline{1})$ is false so $\mathcal{P} \vdash \neg Pf_T(\underline{n}, \ulcorner \underline{0}=\underline{1} \urcorner)$ and $\mathcal{N} \models \neg Pf_T(\underline{n}, \ulcorner \underline{0}=\underline{1} \urcorner)$

for all $n \in \mathbb{N}$, and hence $\mathcal{N} \models \neg \exists x_2 \mathcal{P}\!f_T(x_2, \ulcorner \underline{0} = \underline{1} \urcorner)$ so $\mathcal{N} \models Con_T$.

If $T \vdash \underline{0} = \underline{1}$ then $Pf_T(n, \# \underline{0} = \underline{1})$ is true for some $n \in \mathbb{N}$, so $\mathcal{P} \vdash \mathcal{P}\!f_T(\underline{n}, \ulcorner \underline{0} = \underline{1} \urcorner)$ and $\mathcal{N} \models \mathcal{P}\!f_T(\underline{n}, \ulcorner \underline{0} = \underline{1} \urcorner)$ which gives $\mathcal{N} \models \neg Con_T$.

Hence, Con_T is interpreted as being true in $\mathcal{N}$ if and only if T is consistent.

The following, so called Hilbert-Bernays derivability conditions, hold for any $\mathcal{L}_A$-formulas φ and ψ.

(HB1) If $T \vdash \varphi$ then $\mathcal{P} \vdash \mathcal{P}rov_T(\ulcorner \varphi \urcorner)$

(HB2) $\mathcal{P} \vdash \mathcal{P}rov_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathcal{P}rov_T(\ulcorner \varphi \urcorner) \to \mathcal{P}rov_T(\ulcorner \psi \urcorner))$

(HB3) $\mathcal{P} \vdash \mathcal{P}rov_T(\ulcorner \varphi \urcorner) \to \mathcal{P}rov_T(\ulcorner \mathcal{P}rov_T(\ulcorner \varphi \urcorner) \urcorner)$

(HB1) and (HB2) are not too hard to verify by using the properties of Pf_T and $\mathcal{P}\!f_T$. (HB3) is more difficult to prove and requires a careful analysis of the informal arguments used in showing (1) – (32) and the Representation Theorem.

Lemma 4.6 For every $\varphi(x_1) \in L_A$ (in which x_1 is the only free variable) there is a sentence $\psi \in L_A$ such that $\mathcal{P} \vdash \psi \leftrightarrow \varphi(\ulcorner\psi\urcorner)$.

Proof. Define $D(x) = Sub(x, Num(x), \# x_1)$.
Then D is primitive recursive and for every $\varphi(x_1) \in L_A$, $D(\#\varphi(x_1)) = \#\varphi(\ulcorner\varphi(x_1)\urcorner)$.
Hence D is represented by a formula $\mathcal{D}(x_1, y) \in L_A$, where we may assume that y does not occur in $\varphi(x_1)$. Let $\chi(x_1)$ be $\forall y(\mathcal{D}(x_1, y) \to \varphi(y))$.
Let m be the Gödel number of $\chi(x_1)$ and let ψ be the sentence $\chi(\underline{m})$.
Then $D(m) = D(\#\chi(x_1)) = \#\chi(\underline{m}) = \#\psi$

$$(*) \quad \mathcal{P} \vdash \forall y(\mathcal{D}(\underline{m}, y) \leftrightarrow y = \ulcorner\psi\urcorner)$$

and using the definition of ψ and $(*)$ we get

$$\mathcal{P} \vdash \psi \to \varphi(\ulcorner\psi\urcorner).$$

From $\mathcal{P}$, $\varphi(\ulcorner\psi\urcorner)$ and $\mathcal{D}(\underline{m}, y)$ we can, using $(*)$, formally derive $y = \ulcorner\psi\urcorner$ which, by using the assumption $\varphi(\ulcorner\psi\urcorner)$ gives $\varphi(y)$. Hence, since y does not occur free in $\mathcal{P}$ or $\varphi(\ulcorner\psi\urcorner)$, we get

$$\mathcal{P}, \varphi(\ulcorner\psi\urcorner) \vdash \forall y(\mathcal{D}(\underline{m}, y) \to \varphi(y)),$$

and from this we have $\mathcal{P} \vdash \varphi(\ulcorner\psi\urcorner) \to \ulcorner\psi\urcorner$ $\square$

__Theorem 4.7__ (Gödel's second incompleteness theorem) If $T \subseteq L$ is a recursive and consistent theory such that $T \vdash \mathcal{P}$ then $T \nvdash Con_T$.

Proof. By Lemma 4.6 there exists $\psi_T \in L_A$ such that

(a) $\mathcal{P} \vdash \psi_T \leftrightarrow \neg Prov_T(\ulcorner \psi_T \urcorner)$.

We first show that $T \nvdash \psi_T$.

If there would be a formal proof of ψ_T from T then let k be the Gödel number of such a proof. Then $Pf_T(k, \# \psi_T)$ (is true) so $\mathcal{P} \vdash \mathcal{P}f_T(\underline{k}, \ulcorner \psi_T \urcorner)$ and hence

(b) $T \vdash \mathcal{P}f_T(\underline{k}, \ulcorner \psi_T \urcorner)$.

But from the assumption $T \vdash \psi_T$ and (a) we get $T \vdash \neg \exists x_2 \, \mathcal{P}f_T(x_2, \ulcorner \psi_T \urcorner)$ which together with (b) contradicts the assumption that T is consistent. Hence we conclude that $T \nvdash \psi_T$.

Now we will prove that $T \vdash Con_T \to \psi_T$ and from this it follows that $T \nvdash Con_T$.

By (HB 3) we have

(c) $\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \to \mathrm{Prov}_T(\ulcorner \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \urcorner)$.

From (a),

$$\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \to \neg \psi_T$$

and by (HB 1),

$$\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \to \neg \psi_T \urcorner)$$

and by (HB 2),

(d) $\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \urcorner) \to \mathrm{Prov}_T(\ulcorner \neg \psi_T \urcorner)$.

Now (c) and (d) gives

(e) $\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \to \mathrm{Prov}_T(\ulcorner \neg \psi_T \urcorner)$.

Since $\mathcal{P} \vdash \psi_T \to (\neg \psi_T \to \underline{0} = \underline{1})$ we get, by (HB 1),

(f) $\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \psi_T \to (\neg \psi_T \to \underline{0} = \underline{1}) \urcorner)$.

By (e), (f) and two applications of (HB 2) we get $\mathcal{P} \vdash \mathrm{Prov}_T(\ulcorner \psi_T \urcorner) \to \mathrm{Prov}_T(\ulcorner \underline{0} = \underline{1} \urcorner)$

and since Con_T is $\neg \mathrm{Prov}_T(\ulcorner \underline{0} = \underline{1} \urcorner)$ we have

(g) $\mathcal{P} \vdash \mathrm{Con}_T \to \neg \mathrm{Prov}_T(\ulcorner \psi_T \urcorner)$.

Combining (a) and (g) gives $\mathcal{P} \vdash \mathrm{Con}_T \to \psi_T$

so $T \vdash \mathrm{Con}_T \to \psi_T$. $\square$