Assignment

In the next few pages a formal proof system, with 7 rules, is defined. In this assignment, let '$A_1, \ldots, A_n \vdash B$' mean that there is a formal proof, in the defined system, of B from the assumptions $A_1, \ldots, A_n$.

As usual '$A_1, \ldots, A_n \models B$' means that, for every structure $\mathcal{M}$,

$$\text{if } \mathcal{M} \models A_1 \wedge \ldots \wedge A_n \text{, then } \mathcal{M} \models B.$$

Prove soundness for the (below) defined proof system, that is, prove that

$$\text{if } A_1, \ldots, A_n \vdash B \text{, then } A_1, \ldots, A_n \models B$$

(where $n \in \mathbb{N}$).

<u>Hint</u>: prove it by induction on the number of applications of proof rules in a proof.

A formal proof system is defined as follows. Every part of the definition corresponds to a proof rule as well as a way of building a new, more complex, proof from simpler proofs.

Every proof (building) rule is given a name, indicated above the description of the rule.

<u>Repetition</u>

If $A_1, \dots, A_n$ are formulas and $B \in \{A_1, \dots, A_n\}$, then

$$\begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline B \end{array}$$

is a proof of B from the assumptions/premisses $A_1, \dots, A_n$.

$\wedge$ Introduction $\quad \wedge I$

$$\Pi_1 = \left|\begin{array}{l} A_1 \\ \vdots \\ \underline{A_n} \\ \vdots \\ B \end{array}\right. \quad \text{and} \quad \Pi_2 = \left|\begin{array}{l} A_1 \\ \vdots \\ \underline{A_n} \\ \vdots \\ C \end{array}\right.$$

are proofs of B, respectively C, from the assumptions $A_1, \dots, A_n$, then

$$\left|\begin{array}{l} A_1 \\ \vdots \\ \underline{A_n} \\ \left.\begin{array}{l} \vdots \\ B \end{array}\right\} \text{as in } \Pi_1 \text{ (i.e. same rules in the same order as in } \Pi_1\text{)} \\ \left.\begin{array}{l} \vdots \\ C \end{array}\right\} \text{as in } \Pi_2 \\ B \wedge C \end{array}\right.$$

is a proof of $B \wedge C$ from the assumptions $A_1, \dots, A_n$.

$\wedge$ Elimination

If

$$\Pi = \begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline \vdots \\ B \wedge C \end{array}$$

is a proof of $B \wedge C$ from the assumptions $A_1, \dots, A_n$, then

$$\begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline \left.\begin{array}{l} \vdots \\ B \wedge C \end{array}\right\} \text{as in } \Pi \\ B \end{array}$$

is a proof of B from the assumptions $A_1, \dots, A_n$. The same thing holds if B on the last (lowest) line is replaced by C.

$\neg$ Introduction

If

$$\Pi = \begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \underline{B} \\ \vdots \\ C \wedge \neg C \end{array}$$

is a proof of $C \wedge \neg C$ from the assumptions $A_1, \ldots, A_n, B$, then

$$\begin{array}{|l} A_1 \\ \vdots \\ \underline{A_n} \\ \begin{array}{|l} \underline{B} \\ \vdots \\ C \wedge \neg C \end{array} \Bigg\} \text{ as in } \Pi \\ \neg B \end{array}$$

is a proof of $\neg B$ from the assumptions $A_1, \ldots, A_n$.

$\neg$Elimination

If

$$\Pi = \begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline \vdots \\ \neg\neg B \end{array}$$

is a proof of $\neg\neg B$ from the assumptions $A_1, \ldots, A_n$, then

$$\begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline \left.\begin{array}{l} \vdots \\ \neg\neg B \end{array}\right\} \text{as in } \Pi \\ B \end{array}$$

is a proof of B from the assumptions $A_1, \ldots, A_n$.

$\forall$ Introduction $\forall I$

$$\Pi = \left|\begin{array}{l} A_1 \\ \vdots \\ A_n \\ \hline \vdots \\ B \end{array}\right.$$

is a proof of B from the assumptions $A_1, \ldots, A_n$, and x is a variable which does not occur in any of $A_1, \ldots, A_n$, then

$$\left|\begin{array}{l} A_1 \\ \vdots \\ A_n \\ \hline \left.\begin{array}{l} \vdots \\ B \end{array}\right\} \text{as in } \Pi \\ \forall x\, B \end{array}\right.$$

is a proof of $\forall x\, B$ from the assumptions $A_1, \ldots, A_n$.

$\forall$Elimination   IF

$$\Pi = \begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline \vdots \\ \forall x B \end{array}$$

is a proof of $\forall x B$ from the assumptions $A_1, \dots, A_n$ and t is a term such that none of its variables occur in B.

$$\begin{array}{|l} A_1 \\ \vdots \\ A_n \\ \hline \vdots \\ \forall x B \\ B[t/x] \end{array}$$

is a proof of $B[t/x]$ from the assumptions $A_1, \dots, A_n$, where $B[t/x]$ denotes the formula obtained by replacing every <u>free</u> occurence of x in B with t.