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PROV I MATEMATIK
Logic II

The assignments marked with * must be correctly completed in order to pass the course.

1*. In this assignment you will prove the completeness theorem for propositional logic. Let the smallest building blocks of the propositional language, called *atomic sentences*, or *propositional variables*, be the symbols P_n, $n \in \mathbb{N}$. Let F be the set of sentences that can be built from the atomic sentences and the connectives $\wedge$, $\vee$ and $\neg$. (We skip the connectives $\rightarrow$ and $\leftrightarrow$ in order to have fewer cases to deal with in some assingments below.)

We assume that we have a formal proof system and, as usual, if $T \subseteq F$ and $A \in F$, then '$T \vdash A$' means that there is a formal proof Π such that A is the conclusion of Π and Π uses only assumptions from T.

By '$T \models A$' we mean that every truth assignment $\sigma : \{P_n : n \in \mathbb{N}\} \rightarrow \{t, f\}$ to the atomic sentences which makes all sentences in T true also makes A true (in other words, A cannot be false if all sentences in T are true). We say that T is *satisfiable* if there is some truth assignment which makes all sentences in T true.

We also assume the following about the formal proof system:

Assumptions
(1) The formal proof system is sound, that is, if $T \vdash A$ then $T \models A$.
(2) $A \in T \Longrightarrow T \vdash A$.
(3) $T \vdash A$ and $T' \vdash B \Longrightarrow T \cup T' \vdash A \wedge B$.
(4) $T \vdash A \wedge B \Longrightarrow T \vdash A$ and $T \vdash B$.
(5) $T \vdash \neg\neg A \Longrightarrow T \vdash A$.
(6) $T \cup \{A\} \vdash B \wedge \neg B \Longrightarrow T \vdash \neg A$.
(7) $T \vdash A \wedge \neg A \Longrightarrow T \vdash B$ for every $B \in F$.
(8) $T \vdash A \Longrightarrow T \vdash A \vee B$ and $T \vdash B \vee A$.
(9) $T \cup \{A\} \vdash C$ and $T \cup \{B\} \vdash C \Longrightarrow T \cup \{A \vee B\} \vdash C$.

Definitions
(i) A subset $T \subseteq F$ is also called a *theory*
(ii) A theory $T \subseteq F$ is *inconsistent* if for some $A \in F$, $T \vdash A \wedge \neg A$.
(iii) A theory $T \subseteq F$ is *consistent* if it is not inconsistent.
(iv) A theory $T \subseteq F$ is *maximal consistent* if it is consistent and for every $A \in F - T$, $T \cup \{A\}$ is inconsistent.

Let $T \subseteq F$ be a theory and let $A, B \in F$.
(a) Prove that $T \cup \{A\}$ is inconsistent $\Longleftrightarrow T \vdash \neg A$.
(b) Prove that $T \cup \{\neg A\}$ is inconsistent $\Longleftrightarrow T \vdash A$.
(c) Prove that if T is *maximal* consistent, then:

(c_1) $T \vdash A \Longleftrightarrow A \in T$,

(c_2) $T \vdash \neg A \Longleftrightarrow A \notin T$,

(c_3) $T \vdash A \wedge B \Longleftrightarrow \{A, B\} \subseteq T$,

(c_4) $T \vdash A \vee B \Longleftrightarrow A \in T$ or $B \in T$.

(d) Prove that if $T \subseteq F$ is consistent, then there is $T' \subseteq F$ such that $T \subseteq T'$ and T' is maximal consistent.
(e) Prove that if $T \subseteq F$ is maximal consistent then T is satisfiable. (Use induction on the complexity of formulas and part (c).)

(f) Prove that if $T \subseteq F$, then: T is satisfiable $\Longleftrightarrow T$ is consistent.
(g) Prove the completeness theorem: $T \models A \Longrightarrow T \vdash A$.

2. In this assignment we use the same notation and terminology as in the previous assignment. Describe an algorithm which, given a finite sequence of sentences $A_1, \ldots, A_n, B \in F$, finds a formal proof with conclusion B and assumptions $A_1, \ldots, A_n$, if such proof exists, and otherwise the algorithm replies that no such proof exists.

3. In this assignment we work with first-order logic. Let V be some vocabulary, F_V the set of all first-order formulas that can be built up from V, and let $T \subseteq F_V$ be a theory (i.e. a set of sentences). Suppose that $\varphi \in F_V$ is a sentence which is true in every infinite model of T. Prove that there is $n \in \mathbb{N}$ such that whenever $\mathcal{M}$ is a finite model of T with at least n elements in its universe, then $\mathcal{M} \models \varphi$; in other words, φ is true in every sufficiently large finite model of T. (Observe that the statement is (trivially) true if T has no finite model.)
Hint: use the compactness theorem.