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PROV I MATEMATIK
**Logic II**

*The assignments marked with * must be correctly completed in order to pass the course. You may use all results from the course literature, including my notes which are available on the course home page; but everything else must be proved. Observe that sometimes some part, but not all parts, of an assignment is *-marked. When computing your grade (that is, the percentage of not *-marked assignments), different parts (for example, (i), (ii) etc.) are counted as different assignments.*

*Some of the assignments below coincide more or less with some assignments in the course book, which have (not necessarily complete) solutions at the end of the book. Before starting with assignment 9, and those that follow, it may help to do exercises 12–14 in chapter 5 of the book, so that you get a feeling for how things are done.*

**Your solutions must be well written and all details explained** *(except of course, proofs of results that you may use).*

1*. Let $\lambda_0(x), \lambda_1(x), h_0(x_1, x_2, x_3, x_4), h_1(x_1, x_2, x_3, x_4)$ be primitive recursive functions and let $f$ and $g$ be defined by

$$\begin{aligned} f(x,0) &= \lambda_0(x) \\ g(x,0) &= \lambda_1(x) \\ f(x,y+1) &= h_0(x,y,f(x,y),g(x,y)) \\ g(x,y+1) &= h_1(x,y,f(x,y+1),g(x,y)) \end{aligned}$$

Show that $f$ and $g$ are primitive recursive.
Hint: Define a function $\alpha(x,y) = 2^{f(x,y)} \cdot 3^{g(x,y)}$ and show that $\alpha$ is primitive recursive.

2*. Show that the function $f$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is even} \\ x+1 & \text{if } x \text{ is odd} \end{cases}$$

is primitive recursive.

3*. Show that the function $g$ defined by

$$g(0) = 2, g(1) = 4,$$
$$g(k+2) = 3 \cdot g(k+1) - (2 \cdot g(k) + 1)$$

is primitive recursive.

4*. Show that every finite subset of $\mathbb{N}$ is recursive.

5*. (Diagonal Lemma) Let $R(x,y)$ be a relation and for every number $y$ define a new relation $R_y$ by $R_y(x) \Leftrightarrow R(x,y)$. Prove that if $P$ is defined by $P(x) \Leftrightarrow \neg R(x,x)$ then $P$ is different from all the $R_y$.

6.* Let $L_A$ be the language of arithmetic. Remember that the relation $\mathrm{Pf}_{\mathcal{P}}(x, y)$ defined by

$$\mathrm{Pf}_{\mathcal{P}}(m, n) \iff n \text{ is the Gödel number of a a proof from } \mathcal{P} \text{ of the } L_A\text{-formula with Gödel number } m$$

is recursive and hence represented by an $L_A$-formula which we denote by $\mathcal{P}f_{\mathcal{P}}(x_1, x_2)$. Let $\mathcal{N}$ be the standard model of $\mathcal{P}$ (so we assume that $\mathcal{P}$ is consistent). Answer the following questions and prove that your answer is correct.

- Is it the case that, for every sentence $\varphi \in L_A$, we have $\mathcal{N} \models \exists x_2 \mathcal{P}f_{\mathcal{P}}(\underline{\sharp\varphi}, x_2) \to \varphi$, where $\sharp\varphi$ is the Gödel number of $\varphi$?

- Is it the case that, for every sentence $\varphi \in L_A$, we have $\mathcal{N} \models \varphi \to \exists x_2 \mathcal{P}f_{\mathcal{P}}(\underline{\sharp\varphi}, x_2)$, where $\sharp\varphi$ is the Gödel number of $\varphi$?

7. Let $L_A$ be the language of arithmetic. A theory $T \subseteq L_A$ is also called an $L_A$-*theory.* We say that a formula $\theta_T(x) \in L_A$ is a truth definition for an $L_A$-theory $T$ if for *every* sentence $\varphi \in L_A$, $T \vdash \varphi \leftrightarrow \theta_T(\underline{\sharp\varphi})$. Show that if $T$ is a consistent $L_A$-theory such that $T \vdash \mathcal{P}$, then there is *no* truth definition for $T$.
Hint: Let $T' \supseteq T$ be a *complete* and consistent $L_A$-theory. Show that if $T$ has a truth definition then the relation $\mathrm{Prov}_{T'}(x)$ is representable, and then use results from the course to arrive at a contradiction.

8. We say that a relation $R(x_1, \ldots, x_k)$ is *weakly representable* if there exists a formula $\varphi(x_1, \ldots, x_k) \in L_A$ (the language of arithmetic) such that for all $n_1, \ldots, n_k \in \mathbb{N}$

$$R(n_1, \ldots, n_k) \text{ is true } \iff \mathcal{P} \vdash \varphi(\underline{n_1}, \ldots, \underline{n_k})$$

We say that $\mathcal{P}$ is $\omega$-*consistent* if whenever $\psi(x) \in L_A$ is such that $\mathcal{P} \vdash \neg\psi(\underline{n})$ for all $n \in \mathbb{N}$, then $\mathcal{P} \nvdash \exists x \psi(x)$. Show that for every relation, it is weakly representable if and only if it is recursively enumerable.

9*. Prove that if $A \subseteq \mathbb{N}$ is a recursive nonempty set, then there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $A = \{f(0), f(1), f(2), \ldots\}$.

10*. Prove that if $f : \mathbb{N} \to \mathbb{N}$ is recursive and $\mathrm{ran}(f)$ is infinite, then there is an *injective* recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{ran}(f) = \mathrm{ran}(g)$.

11*. If $A \subseteq \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ is a function, then let

$$f(A) = \{n \in \mathbb{N} : \text{ there exists } m \in A \text{ such that } f(m) = n\} \text{ and}$$
$$f^{-1}(A) = \{n \in \mathbb{N} : \ f(n) \in A\}.$$

Prove that:
(i)* If $f$ is a recursive function and $A \subseteq \mathbb{N}$ recursive, then $f^{-1}(A)$ is recursive.
(ii)* If $f$ is a recursive function and $A \subseteq \mathbb{N}$ is recursively enumerable, then $f(A)$ and $f^{-1}(A)$ are recursively enumerable.

The notation $\{n\}^k$ is explained in my notes (on the course home page) and means the same as the notation $\phi_n^k$ in the course literature. We abbreviate $\{n\}^1$ with $\{n\}$.

12. Let $A = \{n \in \mathbb{N} : \ \{n\} \text{ is a constant function}\}$.
(i)* Prove that $A$ and $\mathbb{N} - A$ are not recursive.
Hint: Rice's theorem.
(ii) Prove that $\mathbb{N} - A$ is recursively enumerable, and conclude (explain why) that $A$ is not recursively enumerable.

13. Let $A$ and $B$ be subsets of $\mathbb{N}$. We say that $A$ is *reducible* to $B$, abbreviated $A \leq B$, if there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \in A \Leftrightarrow f(n) \in B$.
Prove that:
(i) If $A \leq B$ and $B$ is recursive then $A$ is recursive.
(ii) If $A \leq B$ and $B$ is r.e. then $A$ is r.e.
(iii) $A$ is r.e. if and only if $A \leq K$ (where $K = \{x \in \mathbb{N}\text{: } \{x\}(x) \downarrow\}$).

14. Let

$$A = \{x \in \mathbb{N} : \ \{x\}(0) \downarrow\},$$
$$B = \{x \in \mathbb{N} : \ \{x\} \text{ is a total function (i.e. defined for all natural numbers)}\}.$$

(i) Show that $A$ is r.e. but not recursive.
(ii) Show that $\mathbb{N} - A$ is not r.e.
(iii) Show that $\mathbb{N} - B$ is not r.e.
(Hint: Show that $\mathbb{N} - A \ \leq \ \mathbb{N} - B$ and use 2.(ii) and 3.(ii))

15. Let $A$ and $B$ be subsets of $\mathbb{N}$. We say that $A$ and $B$ are *recursively inseparable* if there does not exist a recursive set $C \subseteq \mathbb{N}$ such that $A \subseteq C$ and $B \cap C = \emptyset$.
(i) Show that the sets $A_0 = \{x \in \mathbb{N} \ \ \{x\}(x) \simeq 0\}$ and $A_1 = \{x \in \mathbb{N} \ \ \{x\}(x) \simeq 1\}$ are recursively inseparable. (Hint: Show that if $A_0$ and $A_1$ would not be recursively inseparable then $K$ would be recursive)
(ii) The notation $W_a$ (for $a \in \mathbb{N}$) is explained in my notes (on the course home page). Show that $A$ and $B$ are recursively inseparable if and only if for all $a, b \in \mathbb{N}$, if $A \subseteq W_a$ and $B \subseteq W_b$ and $W_a \cap W_b = \emptyset$ , then there exists $x \in \mathbb{N}$ such that $x \notin W_a \cup W_b$.

16. If $f(x, y)$ is a function then let $f_x$ be the function defined by $f_x(y) = f(x, y)$. Let $\mathcal{F}$ be the set of recursive functions. Show that there does not exist a recursive function $f(x, y)$ such that $\mathcal{F} = \{f_x \ : \ x \in \mathbb{N}\}$.

17. Show that there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$, $W_x \cup W_y = W_{f(x,y)}$.
Hint: use the normal form theorem to show that the ternary relation $R(x, y, z) \Leftrightarrow z \in W_x \cup W_y$ is recursively enumerable; then use Corollary 5.12 in my notes.