UPPSALA UNIVERSITET — PROV I MATEMATIK

Matematiska institutionen — **Logic II**

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*The assignments marked with * must be correctly completed in order to pass the course. You may use all results from the course literature, including my notes which are available on the course home page; but everything else must be proved. When computing your grade (that is, the percentage of not *-marked assignments), different parts (for example, (i), (ii) etc.) are counted as different assignments.*

Your solutions must be well written and all details explained *(except of course, proofs of results that you may use).*

The letters A, B, sometimes with subscripts, refer to sets in the following assignments.

1.* Show that if A is infinite, then $|A^n| = |A|$, for every $n < \omega$, where

$$A^n = \underbrace{A \times \ldots \times A}_{n \text{ times}}.$$

Hint: use induction on n.
(In set theory, the natural numbers $0, 1, 2, \ldots$ are identified with the finite ordinals, that is, the ordinals less than (or belonging to) ω, which is also called $\aleph_0$. Since ω is an ordinal, every nonempty subset of ω has a least element, and a consequence of this is that we may use induction on $\omega = \{0, 1, 2, \ldots\}$ in the usual way.)

2.* Show that if A is infinite, then $\left|\bigcup\{A^n : n < \omega\}\right| = |A|$. (Use 1.)

3. Show that if A is infinite and $\mathcal{P}_f(A)$ is the set of finite subsets of A then $|\mathcal{P}_f(A)| = |A|$.
Hint: first show that $|\mathcal{P}_f(A)| \leq \left|\bigcup\{A^n : n < \omega\}\right|$.

4.* Show that if B is infinite and $|B| \leq |A|$ then $|{}^A B| = |{}^A 2|$. Before starting, observe that ${}^A B \subseteq \mathcal{P}(A \times B)$, because in set theory a function $f : A \to B$ is identified with its graph, which is a subset of $A \times B$. The conclusion of this assignment is usefull when determining cardinalities in some exercises below.

5. For each of the following sets, determine its cardinality (ω, 2^ω, or something else) and prove that your answer is correct. (3)

- $A_1 = \{f \in {}^\omega\omega : \forall n, m \in \omega \, (n < m \Rightarrow f(n) < f(m))\}$
- $A_2 = \{f \in {}^\omega\omega : \exists n \in \omega \, \forall m \in \omega \, f(m) \leq n\}$
- $A_3 = \{f \in {}^\omega\omega : \exists n \in \omega \, \forall m \in \omega \, (n \leq m \Rightarrow f(n) = f(m))\}$.

6.* Show that $|{}^\omega\mathbb{R}| = |\mathbb{R}| = 2^\omega$ and that $|{}^\mathbb{R}\omega| = 2^{2^\omega}$.

7.* Let A and B be two infinite sets and let $\lambda = |A|$ and $\mu = |B|$. Suppose that $\lambda > \mu$ and that g is an injective function from B into A. For each of the following sets, determine its cardinality (λ, μ, 2^λ or 2^μ) and prove that your answer is correct. (3)

- $A_4 = \{f \in {}^A B : |f(A)| = 1\}$
- $A_5 = \{f \in {}^A B : |f(A)| = 2\}$
- $A_6 = A - g(B)$.

8. Show that every partial ordering can be extended to a linear ordering; in other words, show that if $(A, <)$ is a partial ordering, then there is a linear ordering $(A, \prec)$ such that for all $x, y \in A$, if $x < y$ then $x \prec y$.
Hint: use Zorn's lemma, or treat $(A, <)$ as a structure in first-order logic and use a compactness argument.

9. By definition, ω is the least limit ordinal.
(i) Prove that every ordinal $\alpha < \omega$ is a cardinal.
(ii) Prove that ω is a cardinal.