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PROV I MATEMATIK
Logic II

*The assignments marked with * must be correctly completed in order to pass the course. You may use all results from the course literature, including my notes which are available on the course home page; but everything else must be proved. When computing your grade (that is, the percentage of not *-marked assignments), different parts (for example, (i), (ii) etc.) are counted as different assignments.*

Your solutions must be well written and all details explained *(except of course, proofs of results that you may use).*

1.* Let T be an L-theory. Show that T is complete if and only if all models of T are elementarily equivalent (that is, T is complete if and only if $\mathfrak{M}, \mathfrak{N} \models T \implies \mathfrak{M} \equiv \mathfrak{N}$).

Let L_o be the language over the vocabulary $\{=, <\}$ (where $<$ is a binary relation symbol). In the following exercises, a symbol, like $<$, may also be used to denote its interpretation in a structure (a standard convention in model theory).

2.* Let N be the L_o-structure $(\mathbb{N}, <)$ and let M be the L_o-structure $(\mathbb{N} - \{0\}, <)$ (where $<$ is the usual inequality relation on the natural numbers $\mathbb{N}$). Show that
(i)* $M \cong N$
(ii)* $M \equiv N$
(iii)* M is a substructure of N
(iv)* M is *not* an elementary substructure of N

3. Let $\mathcal{C}$ be the class of of all L_o-structures $\mathfrak{M}$ such that $<$ is interpreted as a well ordering on M. Show that $\mathcal{C}$ is *not* axiomatizable.
Hint: Suppose for a contradiction that $\mathcal{C}$ is axiomatizable. Add new constant symbols, intended to be interpreted so that the well-ordering property fails, and use compactness.

4.* Let L_A be the language of arithmetic, as defined in the course, and let $\mathfrak{N} = (\mathbb{N}, s, +, \cdot, 0)$, where $s(n) = n + 1$ and the other symbols are interpreted as usual; so $\mathfrak{N}$ is the so-called standard model of Peano arithmetic. The relation $<$ on $\mathbb{N}$ can be expressed by an L_A-formula $\varphi_<(x, y)$ (as explained in the literature), that is, for all $m, n \in \mathbb{N}$, $n < m$ if and only if $\mathfrak{N} \models \varphi_<(n, m)$. Recall the definition of the (terms called) numerals $\underline{n}$, for $n \in \mathbb{N}$. Let $T = Th(\mathfrak{N})$ be the set of all L_A-sentences which are true in $\mathfrak{N}$.
(i)* Show that there is a *countable* $\mathfrak{M} \models T$ with an element $a \in M$ such that $\mathfrak{M} \models \varphi_<(\underline{n}, a)$ for every $n \in \mathbb{N}$.
Hint: Show, with compactness, that the theory consisting of T together with sentences expressing that c (a new constant symbol) is larger than every natural number is consistent.
(ii)* Are $\mathfrak{N}$ and $\mathfrak{M}$ isomorphic? Show that your anser is correct.
(iii)* Show that for every model $\mathfrak{M} \models T$, $\mathfrak{N}$ can be elementarily embedded into $\mathfrak{M}$.
Hint: Every element in the universe ($\mathbb{N}$) of $\mathfrak{N}$ is the interpretation of some numeral $\underline{n}$.
(iv)* By its definition, T is complete. Is T decidable? Motivate your answer.

5. Let f be a unary function symbol. Inductively, define the following terms: $\mathrm{f}^1(x)$ denotes $\mathrm{f}(x)$ and for $n > 1$, $\mathrm{f}^{n+1}(x)$ denotes $\mathrm{f}(\mathrm{f}^n(x))$. Let

$$T_1 = \{\ \forall x, y(\mathrm{f}(x) = \mathrm{f}(y) \to x = y),\ \forall x \exists y(\mathrm{f}(y) = x)\ \} \\ \cup \{\forall x(\neg \mathrm{f}^n(x) = x) :\ n = 1, 2, 3, \dots \}$$

First, try to see what the models of T_1 look like, and note that T_1 has no finite model. Then answer the following questions and *show* why your answer is correct.
(i) For which infinite cardinals κ is T_1 κ-categorical?
(ii) Is T_1 complete?
(iii) Is T_1 finitely axiomatisable?
(iv) Is T_1 decidable?.

6. Let P be a unary relation symbol and let

$$T_2 = T_1 \cup \{\ \exists x \mathrm{P}(x),\ \exists x \neg \mathrm{P}(x),\ \forall x(\ \mathrm{P}(x) \leftrightarrow \mathrm{P}(\mathrm{f}(x))\ )\ \}$$

where T_1 is the theory from assignment 5 (and f is still a unary function symbol).
Show that T_2 is complete.
Hint: Show that for every $M \models T_2$ with $|M| = \aleph_1$ there is $N \succcurlyeq M$ with $|N| = \aleph_1$ such that

$$|\{a \in N :\ N \models \mathrm{P}(a)\}| = |\{a \in N :\ N \models \neg \mathrm{P}(a)\}| = \aleph_1 \tag{$*$}$$

and show that any two models of T_2 that satisfy $(*)$ are isomorphic. Then use the Löwenheim-Skolem theorem(s) and assignment 1.

7. Let L be the language over the vocabulary $\{=, E\}$ where E is a binary relation symbol. Let $\mathfrak{M}$ be an L-structure in which E is interpreted as an equivalence relation which, for every positive $n \in \mathbb{N}$, has exactly one equivalence class with exactly n elements; and there are no other equivalence classes. Then let $\mathfrak{N}$ be an L-structure in which E is interpreted as in $\mathfrak{M}$ with the (only) exception that, in $\mathfrak{N}$, E also has (exactly) one infinite equivalence class, in addition to the finite classes.
(i) Show that, for every positive $r \in \mathbb{N}$, Duplicator has a winning strategy for $EF_r(\mathfrak{M}, \mathfrak{N})$, and from this conclude that $\mathfrak{M} \equiv \mathfrak{N}$. (If we allowed $\mathfrak{N}$ to have more than one, possibly infinitely many, infinite equivalence classes, the result would nevertheless be the same; but the proof would have to consider more cases.)
(ii) Let $T = Th(\mathfrak{M})$ be the set of all L-sentences which are true in $\mathfrak{M}$. Describe all models of T.
(iii) Is T categorical in any cardinality? Motivate your answer.
(iv) Is T finitely axiomatizable, that is, can the class of models of T be described with only finitely many L-sentences?