Recursion Theorem 5.10

For any $n > 0$ and any $(n+1)$-ary partial recursive function f, there exists $e \in \mathbb{N}$ such that for all $x_1, \dots x_n \in \mathbb{N}$

$$\{e\}^n(x_1, \dots, x_n) \simeq f(e, x_1, \dots, x_n)$$

Proof. Define $g(y, x_1, \dots x_n) \simeq f(S^1_n(y, y), x_1, \dots x_n)$.

Then g is partial recursive, and hence has an index, say d. Let $e = S^1_n(d, d)$.

Then $\{e\}^n(x_1, \dots x_n) \simeq \{S^1_n(d, d)\}^n(x_1, \dots x_n)$

$$\underset{\substack{\uparrow \\ \text{by the s-m-n theorem}}}{\simeq} \{d\}^{n+1}(d, x_1, \dots x_n) \simeq g(d, x_1, \dots x_n) \simeq f(S^1_n(d, d), x_1, \dots x_n)$$

$$\simeq f(e, x_1, \dots x_n)$$

$\square$

Lemma 5.11

If $f(y_1, \dots y_n)$ is a recursive function then for any $m > 0$ the partial function g defined by $g(y_1, \dots y_n, x_1, \dots x_m) \simeq \{f(y_1, \dots y_n)\}^m(x_1, \dots x_m)$ is partial recursive.

Proof. Since $\{f(y_1, \dots y_n)\}^m(x_1, \dots x_m) \simeq (\mu z\, T_m(f(y_1, \dots y_n), x_1, \dots x_m, z))_0$ and the right hand side is partial recursive it follows that g is partial recursive.

$\square$

Corollary to s-m-n theorem 5.12

For any $m \geq 1$ and $n \geq 1$ and any $(m+n)$-ary partial recursive function f there is a m-ary recursive function g such that for all $x_1, \dots x_m, y_1, \dots y_n \in \mathbb{N}$

$$\{g(x_1, \dots x_m)\}^n(y_1, \dots y_n) \simeq f(x_1, \dots x_m, y_1, \dots y_n).$$

Fixed Point Theorem 5.13

For any recursive function $f(x)$ and $n \geq 1$, there exists a natural number e such that

$$\{e\}^n = \{f(e)\}^n$$

Proof. Define $g(y, x_1 .. x_n) \simeq \{f(y)\}^n(x_1 .. x_n)$. Then by lemma 5.11 g is partial rec. By the recursion thm. there exists $e \in \mathbb{N}$ such that $\{e\}^n(x_1 .. x_n) \simeq g(e, x_1 .. x_n)$ for all $x \in \mathbb{N}$. But then $\{e\}^n(x_1 .. x_n) \simeq g(e, x_1 .. x_n) \simeq \{f(e)\}^n(x_1 .. x_n)$ for all $x \in \mathbb{N}$, so $\{e\}^n = \{f(e)\}^n$. □

Examples 1. There is a number e such that for all x $\{e\}(x) \simeq x^e$.

Proof. Let $g(y, x) = x^y$ for all x, y. Then g is recursive. Hence, by the recursion theorem there exists e such that $g(e, x) \simeq \{e\}(x)$ for all x. Thus $\{e\}(x) \simeq x^e$ for all x. □

2. There is a number e such that $\mathrm{dom}(\{e\}) = \{e\}$.

Proof. Let $f(x, y) \simeq \mu z(z = x \wedge z = y)$. Then f is partial recursive so by the (corollary to the) s-m-n theorem there is a recursive function g such that $f(x, y) \simeq \{g(x)\}(y)$. By the fixed point theorem there exists e such that $\{e\} = \{g(e)\}$, and $\mathrm{dom}(\{e\}) = \mathrm{dom}(\{g(e)\}) = \{e\}$ □

Proposition 5.14 For every relation R, the following are equivalent

(i) R is r.e.

(ii) R is the domain of a partial recursive function.

Proof. Suppose that $R(x_1 \ldots x_n)$ is r.e.

Then $R(x_1 \ldots x_n) \iff \exists y\, P(x_1 \ldots x_n, y)$ where P is a recursive relation.

Define $f(x_1 \ldots x_n) \simeq \mu y\, P(x_1 \ldots x_n, y)$.

Then $R = \{(x_1, \ldots, x_n) \in \mathbb{N}^n : \exists y\, P(x_1 \ldots x_n, y)\} = dom(f)$

Now suppose that $R = dom(f)$ where f is a partial recursive function. Then by the normal form theorem $f(x_1 \ldots x_n) \simeq (\mu y\, T_n(e, x_1 \ldots x_n, y))_0$ for some $e \in \mathbb{N}$. Define a relation P by

$P(x_1 \ldots x_n, y) \iff T_n(e, x_1 \ldots x_n, y)$.

Then P is recursive and

$$\begin{aligned} R(x_1, \ldots, x_n) &\iff f(x_1 \ldots x_n)\downarrow \\ &\iff (\mu y\, T_n(e, x_1 \ldots x_n, y))_0 \downarrow \\ &\iff \exists y\, T_n(e, x_1 \ldots x_n, y) \\ &\iff \exists y\, P(x_1 \ldots x_n, y) \end{aligned}$$

, so R is r.e. $\square$

Proposition 5.15 If $A \subseteq \mathbb{N}$ then the following are equivalent

(i) A is r.e.

(ii) A is the domain of a partial recursive function.

(iii) A is the range of a unary partial recursive function.

If $A \neq \emptyset$ then in (iii) we can change "partial recursive" to "recursive".

Proof. If $A = \emptyset$ then the proposition follows from the fact that the nowhere defined function is partial recursive. So assume that $A \neq \emptyset$. Then (i) $\Leftrightarrow$ (ii) follows from proposition 5.14, so it is sufficient to prove (i) $\Leftrightarrow$ (iii).

Suppose that A is r.e. Then there is a recursive relation $R(x,y)$ such that $x \in A \Leftrightarrow \exists y R(x,y)$. Let $k \in A$ and define

$$f(z) \simeq \begin{cases} (z)_0 & \text{if } R((z)_0,(z)_1) \text{ is true} \\ k & \text{if } R((z)_0,(z)_1) \text{ is false} \end{cases}$$

Then f is recursive and the range of f is A.

Now suppose that A is the range of a recursive function f. Then G_f is recursive and

$$x \in A \Leftrightarrow \exists y\, G_f(y,x) \quad \text{so } A \text{ is r.e.}$$

$\square$

Proposition 5.16 For a relation R the following are equivalent

(i) R is recursive

(ii) R and $\neg R$ are r.e.

Proof. If R is recurse then $\neg R$ is recursive and hence (by lemma 5.0) R and $\neg R$ are r.e.

Suppose that $R(x_1 \ldots x_n)$ and $\neg R(x_1 \ldots x_n)$ are r.e. Then there are recursive relations $P(x_1 \ldots x_n, x_{n+1})$ and $Q(x_1 \ldots x_n x_{n+1})$ such that

$$R(x_1 \ldots x_n) \iff \exists y\, P(x_1 \ldots x_n, y) \quad \text{and}$$

$$\neg R(x_1 \ldots x_n) \iff \exists y\, Q(x_1 \ldots x_n, y)$$

Define $f(x_1 \ldots x_n) = \mu y\,(P(x_1 \ldots x_n, y) \vee Q(x_1 \ldots x_n, y))$

Then f is a recursive function and

$$R(x_1 \ldots x_n) \iff P(x_1 \ldots x_n, f(x_1 \ldots x_n))$$

so R is recursive. $\square$

For any $e, n \in \mathbb{N}$, let $W_e^n = dom(\{e\}^n)$.
If $n=1$, we write W_e instead of W_e^1.
By proposition 5.14, for every n-ary r.e. relation R there is $e \in \mathbb{N}$ such that $R = W_e^n$, and in particular, for every r.e set A there is $e \in \mathbb{N}$ such that $A = W_e$.

We say that such a number e is an index for R (or A). By Corollary 5.8 it follows that every r.e. relation (or set) has infinitely many indices.

Rice's Theorem 5.17

(i) Let $\mathcal{C}$ be a set of partial recursive functions and let $E = \{e \in \mathbb{N} : \{e\} \in \mathcal{C}\}$.
If $E \neq \emptyset$ and $E \neq \mathbb{N}$ then E is <u>not</u> recursive

(ii) Let $\mathcal{B}$ be a set of r.e. sets, and let $F = \{e \in \mathbb{N} : W_e \in \mathcal{B}\}$.
If $F \neq \emptyset$ and $F \neq \mathbb{N}$ then F is <u>not</u> recursive.

Proof. (i) First observe that for any $e_0, e_1 \in \mathbb{N}$,

(1) if $e_0 \in E$ and $\{e_0\} = \{e_1\}$ then $e_1 \in E$.

Suppose that $E \neq \emptyset$, $E \neq \mathbb{N}$ and E is recursive.
Then choose $a \in E$ and $b \notin E$ and define a function f by
$$f(x) = \begin{cases} a & \text{if } x \notin E \\ b & \text{if } x \in E \end{cases}$$

Then for all x

$$(2) \qquad x \in E \iff f(x) \notin E$$

and f is recursive so by the fixed point theorem there exists e such that $\{e\} = \{f(e)\}$. By (1) we have $e \in E \iff f(e) \in E$, but this contradicts (2).

(ii) is proved in a similar way. $\square$

<u>Corollary 5.18</u> For any partial recursive function f, the set $\{e \in \mathbb{N} : \{e\} = f\}$ is <u>not</u> recursive.

<u>Proposition 5.19</u> Let $K = \{e \in \mathbb{N} : \{e\}(e)\downarrow\}$. Then K is r.e. but <u>not</u> recursive.

Proof. We have:

$$e \in K \iff \{e\}(e)\downarrow \iff (\mu y\, T_1(e,e,y))_0 \downarrow$$

$$\iff \exists y\, T_1(e,e,y) \quad \text{so } K \text{ is r.e.}$$

Suppose that K is recursive.

Define $$f(x) \simeq \begin{cases} \{x\}(x)+1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

Then f is a recursive function. Let e be an index for f. Then $\{e\} = f$ so $\{e\}(e) \simeq f(e)$, but we also have $f(e) \simeq \{e\}(e)+1$ which is a contradiction.
Hence K is not recursive. $\square$

<u>Proposition 5.20</u> Define a binary relation K_0 by $K_0(x,y) \Longleftrightarrow \{x\}(y)\downarrow$. Then K_0 is r.e. but not recursive

Proof. We have $x \in K \Longleftrightarrow K_0(x,x)$ so if K_0 would be recursive, so would K, which would be a contradiction $\square$

<u>Example</u> Let $K_1 = \{e : W_e \neq \emptyset\}$.
Let $\mathcal{B} = \{A \subseteq \mathbb{N} : A$ is r.e. and $A \neq \emptyset\}$
Then $K_1 \neq \emptyset$ and $K_1 \neq \mathbb{N}$ and $K_1 = \{e : W_e \in \mathcal{B}\}$, so by Rice's theorem K_1 is not recursive.
But since $e \in K_1 \Longleftrightarrow \exists x y\, T_1(e,x,y)$, K_1 is r.e.
Let $\overline{K_1} = \mathbb{N} - K_1$. If $\overline{K_1}$ would be r.e then by prop. 5.16 K_1 would be recursive, a contradiction.
Hence $\overline{K_1}$ is not r.e.