A bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[dl] & (0,2) \arrow[dl] & (0,3) \arrow[dl] \\
(1,0) & (1,1) \arrow[dl] & (1,2) \arrow[dl] \\
(2,0) & (2,1) \arrow[dl] \\
(3,0)
\end{tikzcd}$$

$$\begin{aligned}
(0,0) &\longmapsto 0 \\
(0,1) &\longmapsto 1 \\
(1,0) &\longmapsto 2 \\
(0,2) &\longmapsto 3 \\
(1,1) &\longmapsto 4 \\
&\vdots
\end{aligned}$$

Call this bijection σ^2.

There are $z+1$ pairs (x,y) such that $x+y=z$.

Hence $\sigma^2(x,y) = \left(\sum_{k=0}^{x+y} k \right) + x$ $\quad \boxed{\sum_{k=0}^{n} k = \frac{n^2+n}{2}}$

$$= \frac{(x+y)^2 + (x+y)}{2} + x$$

$$= \frac{(x+y)^2 + 3x + y}{2}$$

We see that σ^2 is primitive recursive.

Let $\sigma_1^2(z) =$ the first coordinate of $(\sigma^2)^{-1}(z)$

(that is, if $\sigma^2(x,y) = z$ then $\sigma_1^2(z) = x$)

and $\sigma_2^2(z) =$ the second coordinate of $(\sigma^2)^{-1}(z)$.

(that is, if $\sigma^2(x,y) = z$ then $\sigma_2^2(z) = y$)

σ_1^2 and σ_2^2 are primitive recursive because

$$\sigma_1^2(z) = \mu x \leq z \left(\exists y \leq z \left(z = \sigma^2(x,y) \right) \right)$$

$$\sigma_2^2(z) = \mu y \leq z \left(\exists x \leq z \left(z = \sigma^2(x,y) \right) \right)$$

Observe that $\sigma_1^2(\sigma^2(x,y)) = x$

$$\sigma_2^2(\sigma^2(x,y)) = y$$

$$\sigma^2(\sigma_1^2(z), \sigma_2^2(z)) = z$$

1) Suppose that $n \geq 2$ and that there are a prim. rec. bijection $\sigma^n : \mathbb{N}^n \to \mathbb{N}$ and for every $1 \leq k \leq n$ prim. rec. injections $\sigma_k^n : \mathbb{N} \to \mathbb{N}$ such that

$$\sigma_k^n(\sigma^n(x_1 \ldots x_n)) = x_k$$

and $\sigma^n(\sigma_1^n(x), \ldots, \sigma_n^n(x)) = x$

Then we can define σ^{n+1} and σ_k^{n+1} for every $1 \leq k \leq n+1$ as follows

$$\sigma^{n+1}(x_1 \ldots x_{n+1}) = \sigma^2(\sigma^n(x_1 \ldots x_n), x_{n+1})$$

$$\sigma_{n+1}^{n+1}(x) = \sigma_2^2(x)$$

for $1 \leq k \leq n$

$$\sigma_k^{n+1}(x) = \sigma_k^n(\sigma_1^2(x))$$

Then it is easy to see that for every $1 \leq k \leq n+1$

$$\sigma_k^{n+1}(\sigma^{n+1}(x_1 \ldots x_{n+1})) = x_k$$

$$\sigma^{n+1}(\sigma_1^{n+1}(x), \ldots, \sigma_{n+1}^{n+1}(x)) = x$$

and that σ^{n+1} and σ_k^{n+1} are prim. rec.

Since (1) is true for $n=2$, we can, by induction, conclude that (1) holds for every n.

If we define a function $f(x_1 \dots x_n, y)$ in a way such that $f(x_1 \dots x_n, y+1)$ depends not only on $f(x_1 \dots x_n, y)$ but on several of the values $f(x_1 \dots x_n, 0), \dots, f(x_1 \dots x_n, y)$

then we say that f is defined by course of values recursion.

For any function $f(x_1 \dots x_n, y)$ let

$$f\#(x_1 \dots x_n, y) = \prod_{u<y} p(u)^{f(x_1 \dots x_n, u)}$$

Observe that $f(x_1 \dots x_n, y) = (f\#(x_1 \dots x_n, y+1))_y$

Proposition 1.7 If $h(x_1 \dots x_n, y, z)$ is primitive recursive (or recursive) and

$$f(x_1 \dots x_n, y) = h(x_1 \dots x_n, y, f\#(x_1 \dots x_n, y))$$

then f is primitive recursive (or recursive)

Proof. We have

$$f\#(x_1 \dots x_n, 0) = 1$$

$$f\#(x_1 \dots x_n, y+1) = f\#(x_1 \dots x_n, y) \cdot p(y)^{f(x_1 \dots x_n, y)}$$

$$= f\#(x_1 \dots x_n, y) \cdot p(y)^{h(x_1 \dots x_n, y, f\#(x_1 \dots x_n, y))}$$

so $f\#$ is prim. rec. (or rec.) and since

$$f(x_1 \dots x_n, y) = (f\#(x_1 \dots x_n, y+1))_y$$

it follows (by substitution (IV)) that f is prim. rec. (or rec.). $\square$

<u>Corollary 1.8</u> If $P(x_1 \ldots x_n, y, z)$ is a primitive recursive (or recursive) relation and the relation $R(x_1 \ldots x_n, y)$ holds iff and only if $P(x_1 \ldots x_n, y, C_R\#(x_1 \ldots x_n, y))$ holds (where C_R is the characteristic function of R) then R is primitive recursive (or recursive).

Proof. $C_R(x_1 \ldots x_n, y) = C_P(x_1 \ldots x_n, y, C_R\#(x_1 \ldots x_n, y))$

so by Prop. 1.7 C_R is prim. rec. (or rec.) $\square$

Example

Define $f(0) = f(1) = 1$, $f(x+2) = f(x) + f(x+1)$.

We show that f is primitive recursive.

Define $g(x) = \sigma^2(f(x), f(x+1))$.

Then $g(0) = \sigma^2(1,1)$

$$g(x+1) = \sigma^2(\sigma^2_2(g(x)), \sigma^2_1(g(x)) + \sigma^2_2(g(x)))$$

so g is primitive recursive.

Since $f(x) = \sigma^2_1(g(x))$ it follows that f is prim. rec.

Alternative solution:

Define $h(y,z) = \begin{cases} 1 & \text{if } y=0 \vee y=1. \\ (z)_{y-2} + (z)_{y-1} & \text{if } y \geq 2 \end{cases}$

Then h is prim. rec. by Proposition 1.6,
and $f(x) = h(x, f\#(x))$ so by Proposition 1.7
f is prim. rec.