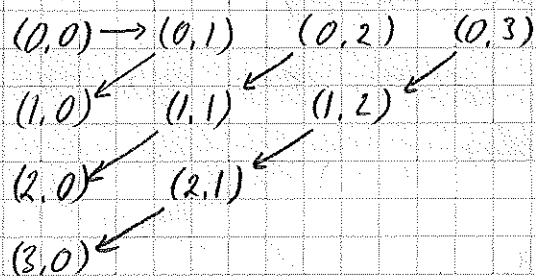


A bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$



$(0,0)$	$\mapsto 0$
$(0,1)$	$\mapsto 1$
$(1,0)$	$\mapsto 2$
$(0,2)$	$\mapsto 3$
$(1,1)$	$\mapsto 4$
$\vdots$	

Call this bijection σ^2 .

There are $z+1$ pairs (x,y) such that $x+y = z$.

$$\text{Hence } \sigma^2(x,y) = \left(\sum_{k=0}^{x+y} k \right) + x \quad \boxed{\sum_{k=0}^n k = \frac{n^2 + n}{2}}$$

$$= \frac{(x+y)^2 + (x+y)}{2} + x$$

$$= \frac{(x+y)^2 + 3x + y}{2}$$

We see that σ^2 is primitive recursive.

Let $\sigma_1^2(z) =$ the first coordinate of $(\sigma^2)^{-1}(z)$

(that is, if $\sigma^2(x,y) = z$ then $\sigma_1^2(z) = x$)

and $\sigma_2^2(z) =$ the second coordinate of $(\sigma^2)^{-1}(z)$.

(that is, if $\sigma^2(x,y) = z$ then $\sigma_2^2(z) = y$)

σ_1^2 and σ_2^2 are primitive recursive because

$$\sigma_1^2(z) = \mu x \leq z \left(\exists y \leq z \left(z = \sigma^2(x,y) \right) \right)$$

$$\sigma_2^2(z) = \mu y \leq z \left(\exists x \leq z \left(z = \sigma^2(x,y) \right) \right)$$

Observe that $\sigma_1^2(\sigma^2(x,y)) = x$

$$\sigma_2^2(\sigma^2(x,y)) = y$$

$$\sigma^2(\sigma_1^2(z), \sigma_2^2(z)) = z$$

1) Suppose that $n \geq 2$ and that there are a
 prim. rec. bijection $\sigma^n: \mathbb{N}^n \rightarrow \mathbb{N}$
 and for every $1 \leq k \leq n$ prim. rec. injections
 $\sigma_k^n: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\sigma_k^n(\sigma^n(x_1, \dots, x_n)) = x_k$$

 and
$$\sigma^n(\sigma_1^n(x), \dots, \sigma_n^n(x)) = x$$

Then we can define σ^{n+1} and σ_k^{n+1} for every $1 \leq k \leq n+1$
 as follows

$$\sigma^{n+1}(x_1, \dots, x_{n+1}) = \sigma^2(\sigma^n(x_1, \dots, x_n), x_{n+1})$$

$$\sigma_{n+1}^{n+1}(x) = \sigma_2^2(x)$$

for $1 \leq k \leq n$

$$\sigma_k^{n+1}(x) = \sigma_k^n(\sigma_1^2(x))$$

Then it is easy to see that for every $1 \leq k \leq n+1$

$$\sigma_k^{n+1}(\sigma^{n+1}(x_1, \dots, x_{n+1})) = x_k$$

$$\sigma^{n+1}(\sigma_1^{n+1}(x), \dots, \sigma_{n+1}^{n+1}(x)) = x$$

and that σ^{n+1} and σ_k^{n+1} are prim. rec.

Since (1) is true for $n=2$, we can, by
 induction, conclude that (1) holds for
 every n .

If we define a function $f(x_1, \dots, x_n, y)$ in a way such that $f(x_1, \dots, x_n, y+1)$ depends not only on $f(x_1, \dots, x_n, y)$ but on several of the values $f(x_1, \dots, x_n, 0), \dots, f(x_1, \dots, x_n, y)$.

then we say that f is defined by course of values recursion.

For any function $f(x_1, \dots, x_n, y)$ let

$$f^\#(x_1, \dots, x_n, y) = \prod_{u < y} p(u)^{f(x_1, \dots, x_n, u)}$$

Observe that $f(x_1, \dots, x_n, y) = (f^\#(x_1, \dots, x_n, y+1))_y$.

Proposition 1.7 If $h(x_1, \dots, x_n, y, z)$ is primitive recursive (or recursive) and

$$f(x_1, \dots, x_n, y) = h(x_1, \dots, x_n, y, f^\#(x_1, \dots, x_n, y))$$

then f is primitive recursive (or recursive)

Proof. We have

$$f^\#(x_1, \dots, x_n, 0) = 1$$

$$\begin{aligned} f^\#(x_1, \dots, x_n, y+1) &= f^\#(x_1, \dots, x_n, y) \cdot p(y)^{f(x_1, \dots, x_n, y)} \\ &= f^\#(x_1, \dots, x_n, y) \cdot p(y)^{h(x_1, \dots, x_n, y, f^\#(x_1, \dots, x_n, y))} \end{aligned}$$

so $f^\#$ is prim. rec. (or rec.) and since

$$f(x_1, \dots, x_n, y) = (f^\#(x_1, \dots, x_n, y+1))_y$$

it follows (by substitution (IV)) that

f is prim. rec. (or rec.). $\square$

Corollary 1.8 If $P(x_1, \dots, x_n, y, z)$ is a primitive recursive (or recursive) relation and the relation $R(x_1, \dots, x_n, y)$ holds if and only if $P(x_1, \dots, x_n, y, C_R \#(x_1, \dots, x_n, y))$ holds (where C_R is the characteristic function of R) then R is primitive recursive (or recursive).

Proof. $C_R(x_1, \dots, x_n, y) = C_P(x_1, \dots, x_n, y, C_R \#(x_1, \dots, x_n, y))$

so by Prop. 1.7 C_R is p.r. (or rec.) □

Example

Define $f(0) = f(1) = 1$, $f(x+2) = f(x) + f(x+1)$.

We show that f is primitive recursive.

Define $g(x) = \sigma^2(f(x), f(x+1))$.

Then $g(0) = \sigma^2(1, 1)$

$$g(x+1) = \sigma^2(\sigma_2^2(g(x)), \sigma_1^2(g(x)) + \sigma_2^2(g(x)))$$

so g is primitive recursive.

Since $f(x) = \sigma_1^2(g(x))$ it follows that f is prim. rec.

Alternative solution:

$$\text{Define } h(y, z) = \begin{cases} 1 & \text{if } y=0 \vee y=1 \\ (z)_{y-2} + (z)_{y-1} & \text{if } y \geq 2 \end{cases}$$

Then h is prim. rec. by Proposition 1.6;

and $f(x) = h(x, f\#(x))$ so by Proposition 1.7 f is prim. rec.

Example

1. Define the term "system" (10 marks)

2. Describe the components of a system (10 marks)

3. Explain the importance of a system (10 marks)

4. Discuss the role of a system (10 marks)

5. Summarize the key points (10 marks)

6. Conclude the report (10 marks)

7. Write a short paragraph (10 marks)

8. Draw a diagram (10 marks)

9. Calculate the value (10 marks)

10. Interpret the results (10 marks)

11. Present the findings (10 marks)

12. Discuss the implications (10 marks)

13. Summarize the key points (10 marks)

14. Conclude the report (10 marks)

15. Write a short paragraph (10 marks)

16. Draw a diagram (10 marks)

17. Calculate the value (10 marks)

18. Interpret the results (10 marks)

19. Present the findings (10 marks)

20. Discuss the implications (10 marks)

21. Summarize the key points (10 marks)

22. Conclude the report (10 marks)

23. Write a short paragraph (10 marks)

24. Draw a diagram (10 marks)