Recursion Theory

Let $k > 0$. If $A \subseteq \mathbb{N}^k$ and $f : A \to \mathbb{N}$ is a function with domain A, then we say that $f : \mathbb{N}^k \to \mathbb{N}$ is a *partial function*, and that A is the domain of f, abbreviated $\mathrm{dom}(f)$. The set $\{n \in \mathbb{N}\colon \exists n_1, \dots, n_k \in \mathrm{dom}(f)\ f(n_1, \dots, n_k) = n\}$ is called the range of f, abbreviated ran(f). If $(n_1, \dots, n_k) \notin A$ then we say that f is not defined for $(n_1, \dots, n_k)$ and we write $f(n_1, \dots, n_k) \uparrow$. Otherwise we say that f is defined for $(n_1, \dots, n_k)$ and we write $f(n_1, \dots, n_k) \downarrow$. Observe that every function $g : \mathbb{N}^k \to \mathbb{N}$ is a partial function.

Let $f, g : \mathbb{N}^k \to \mathbb{N}$ be partial functions and let $n_1, \dots, n_k \in \mathbb{N}$. We say that $f(n_1, \dots, n_k)$ and $g(n_1, \dots, n_k)$ are *strongly equal*, written

$$f(n_1, \dots, n_k) \simeq g(n_1, \dots, n_k)$$

if the following holds,
(i) $f(n_1, \dots, n_k) \downarrow$ if and only if $g(n_1, \dots, n_k) \downarrow$
(ii) if $f(n_1, \dots, n_k) \downarrow$ and $g(n_1, \dots, n_k) \downarrow$ then $f(n_1, \dots, n_k) = g(n_1, \dots, n_k)$

We say that f and g are equal if $f(n_1, \dots, n_k) \simeq g(n_1, \dots, n_k)$ for all $n_1, \dots, n_k \in \mathbb{N}$.

(VI') unrestricted μ-operator:
If $g(x_1, \dots, x_k, y)$ is a partial function and $f(x_1, \dots, x_k)$ is the partial function defined by

$$f(x_1, \dots, x_k) \simeq \begin{cases} \textit{least } y \textit{ such that } g(x_1, \dots, x_k, z) \downarrow \textit{ for all } z \leq y \textit{ and} \\ g(x_1, \dots, x_k, y) = 0, \quad \textit{if such } y \textit{ exists} \\ \\ \uparrow \textit{ otherwise} \end{cases}$$

then we say that f is defined from g by the unrestricted μ-operator, and we write

$$f(x_1, \dots, x_k) \simeq \mu y(g(x_1, \dots, x_k, y) = 0)$$

If $f(x_1, \dots, x_k)$ is defined from partial functions

$$g(y_1, \dots, y_m) \quad \text{and} \quad h_1(x_1, \dots, x_k), \dots, h_m(x_1, \dots, x_k)$$

by substitution (IV) then for all $n_1, \ldots, n_k \in \mathbb{N}$,

$$f(n_1, \ldots, n_k) \downarrow$$

if $h_i(n_1, \ldots, n_k) \downarrow$ for all $1 \leq i \leq m$ and

$$g(h_1(n_1, \ldots, n_k), \ldots, h_m(n_1, \ldots, n_k)) \downarrow .$$

If $f(x_1, \ldots, x_k, y)$ is defined from partial functions

$$g(x_1, \ldots, x_k) \text{ and } h(x_1, \ldots, x_k, x_{k+1}, x_{k+2})$$

by recursion (V) then for all $n_1, \ldots, n_k, m \in \mathbb{N}$, $f(n_1, \ldots, n_k, m+1) \downarrow$

$$\text{if } f(n_1, \ldots, n_k, m) \downarrow \text{ and } h(n_1, \ldots, n_k, m, f(n_1, \ldots, n_k, m)) \downarrow$$

$$, \text{and } f(n_1, \ldots, n_k, 0) \downarrow \text{ if } g(n_1, \ldots, n_k) \downarrow$$

If a partial function is defined from the initial functions by a finite number of applications of substitution (IV), recursion (V) and the unrestricted μ-operator (VI') then we say that the function is *partial recursive.*

Computable functions

We say that a partial function $f : \mathbb{N}^k \to \mathbb{N}$ is computable if there is an algorithm (a finite set of instructions) which tells us how to calculate $f(n_1, \ldots, n_k)$ for every $n_1, \ldots, n_k \in \mathbb{N}$ such that $f(n_1, \ldots, n_k) \downarrow$. But this definition of computable partial function is not precise since the notion of an algorithm is not mathematically precise. There are several proposals for how to make the notion of algorithm and computable partial function precise. Some of them are:

- Turing machines
- Herbrand-Gödel equational systems
- lambda calculus
- Markov algorithms

From each of these precise mathematical concepts we get a precise mathematical definition of what it means for a partial function to to be computable. We call these definitions of computability

- Turing computability

- Herbrand-Gödel computability
- lambda computability
- Markov computability

It turns out that all these different definitions (and others) of computability are equivalent to being partial recursive, that is, for any partial function f, f is partial recursive $\Leftrightarrow$ f is Turing computable $\Leftrightarrow$ f is Herbrand-Gödel computable $\Leftrightarrow$ f is lambda computable $\Leftrightarrow$ f is Markov computable

Hence the following assumption seems reasonable:
A partial function is computable if and only if it is partial recursive.
This assumption is called Church's thesis.
Hence, if we assume that Church thesis is true then the study of partial recursive functions is in fact the study of computable partial functions.

Recursively enumerable relations
For any k, a k-ary relation R is said to be *recursively enumerable* , abbreviated r.e. , if there is a recursive $(k+1)$-ary relation Q such that for any $n_1, \dots, n_k \in \mathbb{N}$,

$$R(n_1, \dots, n_k) \quad \Leftrightarrow \quad \exists x Q(n_1, \dots, n_k, x)$$

Observe that for any k a k-ary relation is a subset of $\mathbb{N}^k$ it follows that $A \subseteq \mathbb{N}$ is r.e. if there is a recursive binary relation Q such that $x \in A \Leftrightarrow \exists y Q(x, y)$

Example : By Theorem 4.2 it follows that the relation

$$\mathrm{Prov}_{\mathcal{P}}(x) \quad \Leftrightarrow \quad x \text{ is the Gödel number of a formula } \varphi \text{ such that } \mathcal{P} \vdash \varphi$$

is not recursive. But since the relation $\mathrm{Pf}_{\mathcal{P}}(y, x)$ is recursive and

$$\mathrm{Prov}_{\mathcal{P}}(x) \quad \Leftrightarrow \quad \exists y \mathrm{Pf}_{\mathcal{P}}(y, x)$$

it follows that $\mathrm{Prov}_{\mathcal{P}}(x)$ is recursively enumerable.

If f is a k-ary partial function then the graph of f, denoted by G_f, is the $(k+1)$-ary relation defined by,

$$G_f(x_1, \dots, x_k, x_{k+1}) \quad \Leftrightarrow \quad f(x_1, \dots, x_k)\downarrow \text{ and } f(x_1, \dots, x_k) = x_{k+1}$$

Lemma 5.0

(i) Every recursive relation is recursively enumerable.

(ii) For any function f, f is recursive if and only if G_f is recursive.

Proof. (i) If $R(x_1, \ldots, x_k)$ is recursive then $Q(x_1, \ldots, x_k, y) \Leftrightarrow R(x_1, \ldots, x_k)$ is recursive and $R(x_1, \ldots, x_k) \Leftrightarrow \exists y Q(x_1, \ldots, x_k, y)$.

(ii) Exercise. □

Lemma 5.1 For any $n, k \geq 1$, if $R(x_1 \ldots x_n, y_1 \ldots y_k)$ is a recursive relation and

$$Q(x_1 \ldots x_n) \iff \exists y_1 \ldots y_k\, R(x_1 \ldots x_n, y_1 \ldots y_k)$$

, then Q is r.e.

Proof. $\exists y_1 \ldots y_k\, R(x_1 \ldots x_n, y_1 \ldots y_k) \iff$

$$\exists y\, \underbrace{R(x_1 \ldots x_n, (y)_0, \ldots, (y)_{k-1})}_{R^*(x_1 \ldots x_n, y) \text{ recursive}} \qquad \square$$

Lemma 5.2

(a) If $R(x_1 \ldots x_n)$ is a r.e. relation then Q defined by $Q(x_1 \ldots x_n, y_1 \ldots y_k) \iff R(x_1 \ldots x_n)$, is r.e.

(b) If $R(x_1 \ldots x_n)$ and $Q(x_1 \ldots x_n)$ are r.e. then $R \wedge Q$ and $R \vee Q$ are r.e.

Proof. (a) Assume $R(x_1 \ldots x_n) \iff \exists z R^*(x_1 \ldots x_n, z)$ where R^* is recursive

then $Q(x_1 \ldots x_n y_1 \ldots y_k) \iff R(x_1 \ldots x_n) \wedge y_1 = y_1 \wedge \ldots \wedge y_k = y_k$

$$\iff \exists z \left(R^*(x_1 \ldots x_n, z) \wedge y_1 = y_1 \wedge \ldots \wedge y_k = y_k \right)$$

(b) Assume $Q(x_1 \ldots x_n) \iff \exists y\, Q^*(x_1 \ldots x_n)$ where Q^* is recursive.

Then $R(x_1 \ldots x_n) \wedge Q(x_1 \ldots x_n) \iff \exists u R^*(x_1 \ldots x_n, u)$

$$\wedge \exists v\, Q^*(x_1 \ldots x_n, v)$$

$$\iff \exists u v \left(R^*(x_1 \ldots x_n, u) \wedge Q^*(x_1 \ldots x_n, v) \right)$$

Similarly for $\vee$.

$\square$

Lemma 5.3

(a) If $R(x_1 .. x_n, x_{n+1})$ is a r.e. relation and Q is defined by

$$Q(x_1 .. x_n) \iff \exists z R(x_1 .. x_n, z)$$

then Q is r.e.

(b) If $R(x_1 .. x_n\ x_{n+1}\ x_{n+2})$ is a r.e. relation and Q is defined by

$$Q(x_1 .. x_n\ x_{n+1}) \iff \forall z < x_{n+1}\ R(x_1 .. x_n\ x_{n+1}\ z)$$

then Q is r.e.

Proof. (a) Assume $R(x_1 .. x_n\ x_{n+1}) \iff \exists y\ R^*(x_1 .. x_n\ x_{n+1}\ y)$ where R^* is rec.

Then $Q(x_1 .. x_n) \iff \exists z R(x_1 .. x_n, z) \iff \exists z \exists y\ R^*(x_1 .. x_n, z, y)$

Now use lemma 5.1.

(b) Assume $R(x_1 .. x_n\ x_{n+1}\ x_{n+2}) \iff \exists y\ R^*(x_1 .. x_n\ x_{n+1}\ x_{n+2}, y)$

then $Q(x_1 .. x_n\ x_{n+1}) \iff \forall z < x_{n+1}\ \exists y\ R^*(x_1 .. x_n\ x_{n+1}\ z, y)$

$$\iff \exists u\ \underbrace{\forall z < x_{n+1}\ R^*(x_1 .. x_n, x_{n+1}, z, (u)_z)}_{\text{recursive}}$$

$\square$

Lemma 5.4

(a) If $g(x_1 \ldots x_m)$ and $h_1(x_1 \ldots x_n), \ldots, h_m(x_1 \ldots x_n)$ are partial functions such that G_g and all $G_{h_1}, \ldots, G_{h_m}$ are r.e. and a partial function f is defined by

$$f(x_1 \ldots x_n) = g(h_1(x_1 \ldots x_n), \ldots, h_m(x_1 \ldots x_n))$$

, then G_f is r.e.

(b) If $g(x_1 \ldots x_n)$ and $h(x_1 \ldots x_n, x_{n+1}, x_{n+2})$ are partial functions such that G_g and G_h are r.e. and a partial function f is defined by

$$f(x_1 \ldots x_n, 0) = g(x_1 \ldots x_n)$$

$$f(x_1 \ldots x_n, y+1) = h(x_1 \ldots x_n, y, f(x_1 \ldots x_n, y))$$

, then G_f is r.e.

(c) If $g(x_1 \ldots x_n, x_{n+1})$ is a partial function such that G_g is r.e and a partial function f is defined by

$$f(x_1 \ldots x_n) = \mu y(g(x_1 \ldots x_n, y) = 0)$$

, then G_f is r.e.

Proof. (a) $G_f(x_1 .. x_n, y) \iff$

$$\underbrace{\exists z_1 .. z_m \left[\bigwedge_{i=1}^{m} G_{h_i}(x_1 .. x_n, z_i) \wedge G_g(z_1 \dots z_m, y) \right]}$$

r.e. by lemma 5.2 and lemma 5.1
and lemma 5.3

(b) $G_f(x_1 \dots x_n, y, z) \iff$

Gödel's β-function

$$\exists uv \Big[\exists w \big(\beta(u,v,0) = w \wedge G_g(x_1 .. x_n, w) \big)$$

$$\wedge \ \beta(u,v,y) = z$$

$$\wedge \ \forall w < y \, . \, \exists ab \big(\beta(u,v,w) = a \wedge \beta(u,v,s(w)) = b$$

$$\wedge \ G_h(x_1 .. x_n, w, a, b) \big) \Big]$$

so by lemmas 5.3, 5.2, 5.1 and 5.0 G_f is r.e.

(c) $G_f(x_1 .. x_n, y) \iff$

$$G_g(x_1 .. x_n, y, 0) \wedge \forall z < y \, \exists u \big(u \neq 0 \wedge G_g(x_1 .. x_n, z, u) \big)$$

so by lemmas 5.3, 5.2, 5.1 and 5.0 G_f is r.e.

□

<u>Proposition 5.5</u> For any partial function f, f is partial recursive if and only if the graph of f is recursively enumerable.

Proof. The initial functions are recursive so they have recursive (graphs), and hence r.e. graphs, by lemma 5.0. If f is partial recursive then it follows from the definition of partial recursive functions and lemma 5.4 that G_f is r.e.

Now suppose that G_f, the graph of $f(x_1 \ldots x_n)$ is r.e. Then there is a recursive relation $R(x_1 \ldots x_n x_{n+1}; y)$ such that

$$G_f(x_1 \ldots x_n x_{n+1}) \iff \exists y \, R(x_1 \ldots x_n x_{n+1} y).$$

Since $f(x_1 \ldots x_n) \simeq \left(\mu y \, R(x_1, \ldots, x_n, (y)_0, (y)_1)\right)_0$ it follows that f is partial recursive. $\square$

We will define prim. rec. functions sub_k for every $k = 1, 2, 3, \dots$ such that, if $u_1, \dots, u_k$ are Gödel numbers of terms $t_1, \dots, t_k$ over V_A and y is the Gödel number of an L_A-formula $\varphi(x_{i_1}, \dots, x_{i_k})$ where $x_{i_1}, \dots, x_{i_k}$ are (distinct) free variables in φ and for any other free variable x_ℓ in φ we have $\ell > i_1, \ell > i_2, \dots, \ell > i_k$

, then

$$sub_k(y, u_1, \dots, u_k) =$$

the Gödel number of $\varphi(t_1, \dots, t_k)$

Define $SUB(x, y, u) \iff \exists v < y [Fr(y, v) \wedge \forall w < v \neg Fr(y, w) \wedge x = Sub(y, u, v)] \vee [\neg \exists v < y\, Fr(y, v) \wedge x = y]$,

and for every $k > 0$ define a function $sub_k(y, u_1 \ldots u_k)$ by

$$sub_1(y, u) = \mu x < (p(u \cdot y)!)^{u \cdot y}\, SUB(x, y, u)$$

$$sub_{k+1}(y, u_1, \ldots, u_{k+1}) = sub_1(sub_k(y, u_1, \ldots, u_k), u_{k+1})$$

Then sub_k is a prim. rec. function for $k = 1, 2, 3, \ldots$.

For $k > 0$ define $T_k(e, n_1 \ldots n_k, y) \iff$

$$Pf_P\Big((y)_2,\ sub_{k+2}(e, Num(n_1), \ldots, Num(n_k), Num((y)_0), Num((y)_1))\Big)$$

Then T_k is a prim. rec. relation.

From now on, let's assume that P is consistent.

<u>Normal Form Theorem, 5.6</u> (Kleene)

Let $f(x_1 \ldots x_k)$ be a partial function.
Then f is partial recursive if and only if there exists a natural number e such that

$$f(n_1 \ldots n_k) \simeq (\mu y\, T_k(e, n_1, \ldots, n_k, y))_0 \quad \text{for all } n_1, \ldots, n_k \in \mathbb{N}.$$

Proof. Suppose that $f(n_1 .. n_k) \simeq (\mu y\, T_k(e, n_1 .. n_k, y))_0$ for all $n_1 .. n_k \in \mathbb{N}$. Then f is partial recursive by (VI') and the fact that T_k is prim. rec.

Suppose that f is partial recursive.
By prop. 5.5 G_f is r.e. so there is a recursive relation G_f^* such that $G_f(n_1 .. n_k, m) \Leftrightarrow \exists y\, G_f^*(n_1 .. n_k, m, y)$ for all $n_1 .. n_k, m \in \mathbb{N}$.

G_f^* is represented by some formula $\varphi(x_1 .. x_k\, x_{k+1}\, x_{k+2})$. Let e be the Gödel number of φ. If $f(n_1 .. n_k)\uparrow$ then $G_f^*(n_1 .. n_k, m, l)$ is false for all $m, l \in \mathbb{N}$, so $\mathcal{P} \vdash \neg\varphi(\bar{n}_1 .. \bar{n}_k, \bar{m}, \bar{l})$ for all $m, l \in \mathbb{N}$. Hence $T_k(e, n_1 .. n_k, y)$ is false for all $y \in \mathbb{N}$ and therefore $(\mu y\, T_k(e, n_1 .. n_k, y))_0\uparrow$.
If $f(n_1 .. n_k) = m$ then $\mathcal{P} \vdash \varphi(\bar{n}_1 .. \bar{n}_k, \bar{m}, \bar{l})$ for some $l \in \mathbb{N}$.
Let q be the Gödel number of a derivation of $\varphi(\bar{n}_1 .. \bar{n}_k, \bar{m}, \bar{l})$ from $\mathcal{P}$. Then

$Pf_{\mathcal{P}}(q, sub_{k+1}(e, Num(n_1), .., Num(n_k), Num(m), Num(l)))$ is true,
so $T_k(e, n_1 .. n_k, 2^m \cdot 3^l \cdot 5^q)$ is true. Since G_f is the graph of the function f, $G_f(n_1 .. n_k, i)$ is false for all $i \neq m$ and hence $G_f^*(n_1 .. n_k, i, j)$ is false for i, j with $i \neq m$. Then $\mathcal{P} \vdash \neg\varphi(\bar{n}_1 .. \bar{n}_k, \bar{i}, \bar{j})$ for all i, j with $i \neq m$. Since we assume that $\mathcal{P}$ is consistent we can not have $\mathcal{P} \vdash \varphi(\bar{n}_1 .. \bar{n}_k, \bar{i}, \bar{j})$ for any i, j such that $i \neq m$.

Hence $(\mu y\, T_k(e, n_1 .. n_k, y))_0 = m$.

This shows that for all $n_1 .. n_k \in \mathbb{N}$

$$f(n_1 .. n_k) \simeq (\mu y\, T_k(e, n_1 .. n_k, y))_0$$

$\square$

Let $f(x_1 \ldots x_k)$ be a partial recursive function. A number e given by the normal form thm. is called an <u>index</u> for f. Hence every partial recursive function has an index. (In fact every partial rec. function has infinitely many indices.).

For every $e \in \mathbb{N}$ by $\{e\}^k$ we denote the function

$$\{e\}^k(x_1 \ldots x_k) \simeq (\mu y\, T_k(e, x_1 \ldots x_k, y))_0$$

(we write $\{e\}$ for $\{e\}^1$)

Hence, $\{e\}^k$ is the k-ary partial function with index e.

<u>Corollary 5.7</u>

(i) There are infinitely but countably many partial recursive functions.

(ii) There are functions from $\mathbb{N}$ into $\mathbb{N}$ which are not recursive.

Proof. (i) follows from the normal form thm, since if $f \neq g$ and e_1, e_2 are indices for f, g respectively then $e_1 \neq e_2$.

There are uncountably many functions from $\mathbb{N}$ into $\mathbb{N}$ so from (i) we get (ii).

<u>Corollary 5.8</u> Every partial recursive function has infinitely many indices.

Proof. In the proof of the normal form theorem we can replace φ by $\varphi \wedge \underbrace{x_1 = x_1 \wedge \ldots \wedge x_1 = x_1}_{i \text{ times}}$, for any i, and hence obtain infinitely many indices. $\square$

s-m-n theorem 5.9

For all $m \geq 1$ and $n \geq 1$ there exists a $(m+1)$-ary primitive recursive function S^m_n such that for all $e, x_1 \cdots x_m, y_1 \cdots y_n \in \mathbb{N}$

$$\{S^m_n(e, x_1, \ldots, x_m)\}^n(y_1, \ldots, y_n) \simeq \{e\}^{m+n}(x_1, \ldots x_m, y_1, \ldots, y_n)$$

Explanation: Let f be the partial rec. function defined by $f(y_1, \ldots, y_n) \simeq \{e\}^{m+n}(x_1 \ldots x_n, y_1 \ldots y_m)$. Then $S^m_n(e, x_1 \ldots x_m)$ is an index for f.

Proof. Let $S^m_n(e, x_1, \ldots, x_m) = sub_m(e, Num(x_1), \ldots, Num(x_m))$

Then $\{S^m_n(e, x_1 \ldots x_m)\}^n(y_1, \ldots, y_n)$

$$\simeq (\mu z\, T_n(S^m_n(e, x_1 \ldots x_m), y_1 \cdots y_n, z))_0$$

$$\simeq (\mu z\, Pf_p((z)_1, sub_{n+1}(S^m_n(e, x_1 \ldots x_m), Num(y_1), \ldots, Num(y_n), Num((z)_0))))_0$$

$$\simeq (\mu z\, Pf_p((z)_1, sub_{n+1}(sub_m(e, Num(x_1), \ldots, Num(x_m)), Num(y_1), \ldots, Num(y_n), Num((z)_0))))_0$$

$$\simeq (\mu z\, Pf_p((z)_1, sub_{m+n+1}(e, Num(x_1), \ldots, Num(x_m), Num(y_1), \ldots, Num(y_n), Num((z)_0))))_0$$

$$\simeq (\mu z\, T_{m+n}(e, x_1 \cdots x_m, y_1 \cdots y_n, z))_0$$

$$\simeq \{e\}^{m+n}(x_1 \ldots x_m, y_1 \cdots y_n)$$

$\square$