Problems with solutions

The numberings of results used refer to my notes, available on the course home page.

(2.) Show that there exists a recursive function $h(x)$ such that for every $e \in \mathbb{N}$, if $\{e\}$ is total (i.e defined for all $n \in \mathbb{N}$) and if $\text{ran}(\{e\})$ is infinite then $\{h(e)\}$ is an injective total function such that $\text{ran}(\{e\}) = \text{ran}(\{h(e)\})$.

Solution: Define $f(e, x) \simeq \{e\}(x)$.

Then by lemma ~~2~~.11, $f(e,x)$ is partial recursive
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Now define $g(e,x) \simeq$

$$\simeq f\left(e, \left(\mu y\left[\forall u \le x \forall v \le x (u \neq v \to [(y)_u \neq (y)_v \wedge f(e,(y)_u) \neq f(e,(y)_v)])\right]\right)_x\right)$$

Then g is partial recursive so by Corollary ~~2~~5.12 there exists recursive function $h(x)$ such that $\{h(e)\}(x) \simeq g(e,x)$ for all $e, x \in \mathbb{N}$.
It follows from the definition of g that if $\{e\}$ is total and $\mathrm{ran}(\{e\})$ is infinite then $\{h(e)\}$ is total and $\mathrm{ran}(\{h(e)\}) = \mathrm{ran}(\{e\})$. $\square$

3. Show that the set

$A = \{e \in \mathbb{N} : W_e$ has at most 5 distinct elements$\}$

is not r.e.

Solution: First we show that $\bar{A} = \mathbb{N} - A$ is r.e but not recursive.

We have $\bar{A} = \{e \in \mathbb{N} : W_e$ has at least 5 distinct elements$\}$ so

$$e \in \bar{A} \iff \exists y_1 \ldots y_5 z_1 \ldots z_5 \Big[\underbrace{\bigwedge_{\substack{i,j \le 5 \\ i \neq j}} y_i \neq y_j \wedge \bigwedge_{i=1}^{5} T_1(e, y_i, z_i)}_{\text{recursive}} \Big]$$

and therefore $\bar{A}$ is r.e.

By applying Rice's theorem one can show that $\bar{A}$ is not recursive (exercise!)

By proposition ~~2~~5.16 it follows that if A would also be r.e then $\bar{A}$ (and A) would be recursive, which contradicts what we have just proved. Hence $\bar{A}$ is not r.e. $\square$

4. Show that the relation R defined by

$$R(x,y) \iff \{x\}(y) \simeq 0 \quad ,$$

is not recursive

Solution: Suppose that R is recursive.

Define
$$f(x,y) \simeq \begin{cases} 0 & \text{if } \{x\}(y)\downarrow \\ \uparrow & \text{if } \{x\}(y)\uparrow \end{cases}$$

Then f is partial recursive (because

$f(x,y) \simeq Z(h(x,y))$ where Z is the zero function and $h(x,y) \simeq \{x\}(y)$, where h is partial recursive by lemma ~~2~~5.11)

so by Corollary ~~2~~5.12 there is a recursive function $g(x)$ such that $\{g(x)\}(y) \simeq f(x,y)$ for all $x, y \in \mathbb{N}$. Now we get

$$K_0(x,y) \iff \{x\}(y)\downarrow \iff f(x,y) \simeq 0 \iff \{g(x)\}(y) \simeq 0$$

$\iff R(g(x),y)$ and since R and g are recursive it follows that K_0 is recursive, which contradicts proposition ~~2~~5.20. Hence R is not recursive. $\square$

(5.) Let f be an injective recursive function and let $A = ran(f)$ (so it follows that A is infinite) and let $B = \{x \in \mathbb{N} : \text{there is } y > x \text{ such that } f(y) < f(x)\}$

(i) Show that B is r.e

(ii) Show that if there is an infinite r.e. set $C \subseteq \mathbb{N}$ such that $B \cap C = \emptyset$ then A is recursive.

Solution: (i) Define $g(x) \simeq \mu y (y > x \wedge f(y) < f(x))$

Then g is partial recursive and $g(x)\downarrow$ if and only if $x \in B$, so $dom(g) = B$. Hence (by prop $\cancel{2}$.15) B is r.e.
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(ii) Supposet that $C \subseteq \mathbb{N}$ is infinite and r.e and that $B \cap C = \emptyset$. Then there is (by prop 2.15) a recursive function h such that $ran(h) = C$.

Define $H(x) \simeq (\mu y (h((y)_1) = (y)_0 \wedge (y)_0 > x))_0$

Then $H(x)$ is a total recursive function (because C is infinite), and $H(x) =$ the least n in C such that $n > x$.

It follows that there exists z such that $f(z) = x$ if and only if there exists $z < H(x)$ such that $f(z) = x$.

Then $x \in A \iff \exists z < H(x) (f(z) = x)$

and the right hand side is a recursive relation

Hence A is recursive. $\square$