1 Set Theory

1.1 Axioms of set theory

The axioms of Zermelo-Fraenkel axiomatic set theory, stated informaly, are the following (1)-(7):

(1) axiom of extensionality:
If X and Y are sets and for any element a, $a \in X \Leftrightarrow a \in Y$, then $X = Y$.

(2) axiom of pairing:
For any sets a and b there exists a set $\{a, b\}$ that contains exactly a and b.

(3) axiom of union:
If X is a set of sets then there exists a set Y such that

$$\forall a[\ a \in Y \Leftrightarrow \exists b(b \in X \text{ and } a \in b)\ ]$$

(we say that Y is the union of X and denote it by $\bigcup X$)

(4) axiom of power set:
If X is a set then there exists a set Y which contains exactly all subsets of X, i.e.

$$\forall a[\ a \in Y \Leftrightarrow \forall b(b \in a\ \Rightarrow b \in X)\ ]$$

(we call the set Y, the power set of X and denote it by $\mathcal{P}(X)$)

(5) axiom schema of separation:
If φ is a property and X is a set then

$$Y = \{a \in X :\ a \text{ has the property } \varphi\}$$

is a set.

(6) axiom schema of replacement:
If X is a set and $\varphi(x, y)$ is a property such that for every $a \in X$ there exists exactly one element b such that $\varphi(a, b)$ holds, then

$$Y = \{b :\ \text{there exists } a \in X \text{ such that } \varphi(a, b) \text{ holds}\}$$

is a set.

(7) axiom of infinity:
There exists a set X such that $\emptyset \in X$ and for every set a, if $a \in X$ then $a \cup \{a\} \in X$.
(Observe that if there exists any set at all then by the axiom of extensionality there will be a unique set which contains no elements at all.)

Zermelo-Fraenkel axiomatic set theory is usually abbreviated ZF. If we add the following axiom, called the axiom of choice then we get a theory which is usually called ZFC:

(8) axiom of choice (abbreviated AC):
For every set X of nonempty sets there exists a function f from X into $\bigcup X$ such that for every $a \in X$, $f(a) \in a$.
(Such a function f is called a choice function for X.)

The results in the following sections follow from the axioms of ZFC but we will not explicitly refer to them.

Many familiar objects such as ordered pairs, ordered tuples, relations and functions can be defined in terms of sets. For example, we define an ordered pair (a, b) to be the set $\{\{a\}, \{a, b\}\}$. For $k \geq 2$, ordered $k+1$-tuples can be defined as follows. Suppose that for elements $a_1, \dots, a_k$ we have already defined what the ordered k-tuple $(a_1, \dots, a_k)$ is. Then we define $(a_1, \dots, a_{k+1})$ to be $(a_1, (a_2, \dots, a_{k+1}))$. For any sets $A_1, \dots, A_k$, the cartesian product of $A_1, \dots, A_k$, denoted $A_1 \times \dots \times A_k$ is the set

$$\{(a_1, \dots a_k) \ : \ a_1 \in A_1, \dots, a_k \in A_k\}$$

If A is a set then the cartesian product

$$\underbrace{A \times \dots \times A}_{k \text{ times}}$$

is denoted by A^k. A k-ary relation on $A_1, \dots, A_k$ is a subset of $A_1 \times \dots \times A_k$. A k-ary relation on A is a subset of A^k. A function from a set A into a set B is a binary relation $R \subseteq A \times B$ such that for every $a \in A$ there exists exactly one $b \in B$ such that $(a, b) \in R$. So the function mentioned in the axiom of choice is just a particular kind of set. A sequence a_i, $i \in I$ of elements such that every $a_i \in A$ is a function from I into A. We will use the notation $A \subseteq B$ for "A is a subset of B" and the notation $A \subset B$ for "A is a proper subset of B" , i.e. "A is a subset of B and $A \neq B$".

1.2 Well orderings

Let A be a set. By definition, a *linear* (or *total*) ordering on A is a binary relation $<$ on A such that for all $a, b, c \in A$ the following holds (where we write $a < b$ for $(a, b) \in <$):
(i) $a \not< a$ ($x \not< y$ means not $x < y$)
(ii) if $a < b$ and $b < c$ then $a < c$
(iii) if $a \neq b$ then $a < b$ or $b < a$
We define $a \leq b$ to mean $a < b$ or $a = b$.

If $<$ is a linear ordering on A then we will say that $(A, <)$ is a linearly ordered set (or a linear ordering). Sometimes when it is evident which linear ordering we are refering to, we will just say that A is a linearly ordered set (or a linear ordering). Observe that if $(A, <)$ is a linear ordering and $B \subseteq A$ then $(B, <)$ is a linear ordering. The notation $B \subseteq A$ will mean that B is a subset of A and the notation $B \subset A$ will mean that B is a proper subset of A, that is, $B \subseteq A$ but not $B = A$. If $B \subseteq A$ then we say that B is an *initial segment* of A if for all $a, b \in A$, if $a < b$ and $b \in B$ then $a \in B$. If B is an initial segment of $(A, <)$ and $B \subset A$ then we say that B is a proper initial segment of $(A, <)$. Let $<$ be a linear ordering on the set A. If B is a nonempty subset of A then we say that $a \in B$ is a *least element* of B if for all $b \in B$, if $b \neq a$ then $a < b$. We say that $<$ is a *well ordering* on A (or that $(A, <)$ is a well ordering) if $(A, <)$ is a linear ordering and every nonempty subset of A has a least element. Observe that if $(A, <)$ is a well ordering and $B \subseteq A$ then $(B, <)$ is a well ordering.

Lemma 1.1 *If $(A, <)$ is a well ordering and $B \subset A$ is an initial segment of A then there exists $a \in A$ such that $B = \{b \in A : b < a\}$.*

Proof. Let a be the least element in $A - B$. Since B is an initial segment it follows that $B = \{b \in A : b < a\}$. $\square$

Let $(A_1, <_1)$ and $(A_2, <_2)$ be two linearly ordered sets. An *isomorphism* from $(A_1, <_1)$ onto $(A_2, <_2)$ is an injective function from A_1 onto A_2 such that for all $a, b \in A_1$, $a <_1 b$ if and only if $f(a) <_2 f(b)$. Note that if f is an isomorphism from $(A_1, <_1)$ onto $(A_2, <_2)$ and if g is an isomorphism from $(A_2, <_2)$ onto $(A_3, <_3)$ then the composition gf is an isomorphism from $(A_1, <_1)$ onto $(A_3, <_3)$. If there exists an isomorphism from $(A_1, <_1)$ onto $(A_2, <_2)$ then we say that $(A_1, <_1)$ and $(A_2, <_2)$ are *isomorphic.*

Lemma 1.2 *Let $(A_1, <_1)$ and $(A_2, <_2)$ be two well ordered sets.*
(i) *If B is an initial segment of $(A_2, <_2)$, and f is an isomorphism from*

$(A_1, <_1)$ *onto* $(A_2, <_2)$ *and* g *is an isomorphism from* $(A_1, <_1)$ *onto* $(B, <_2)$ *then* $A_2 = B$ *and* $f = g$.
(ii) *If* f *and* g *are two isomorphisms from* $(A_1, <_1)$ *onto* $(A_2, <_2)$ *then* $f = g$.
(iii) *If* f *is an isomorphism from* $(A_1, <_1)$ *onto* $(A_1, <_1)$ *then* f *is the identity function, i.e.* $f(a) = a$ *for all* $a \in A$.
(iv) $(A_1, <_1)$ *is not isomorphic to any proper initial segment of itself.*
(v) *If* A *is an initial segment of* A_1 *and* f *is an isomorphism from* $(A_1, <_1)$ *onto* $(A_2, <_2)$ *then* $f(A)$ *is an initial segment of* A_2.

Proof. (i) Suppose that there exists $a \in A_1$ such that $f(a) \neq g(a)$. Then the set $\{a \in A_1 : f(a) \neq g(a)\}$ has a least element, say c. Then for for all $a < c$, we have $f(a) = g(a)$. Let $C = \{a \in A : a < c\}$, and let d be the least element in $A_2 - f(C) = A_2 - g(C)$. Since f is an isomorphism from $(A_1, <_1)$ onto $(A_2, <_2)$ we must have $f(c) = d$, and since g is an isomorphism from $(A_1, <_1)$ onto $(B, <_2)$ we must have $g(c) = d$. Hence $f(c) = g(c)$ which contradicts the choice of c. Therefore there can not exist $a \in A_1$ such that $f(a) \neq g(a)$.

If $B \neq A_2$ then, since f is onto A_2 and g is onto B, there must be $a \in A_1$ such that $f(a) \neq g(a)$. But this contradicts what we just proved and therefore we must have $A_2 = B$. This concludes the proof of (i). (ii),(iii) and (iv) are special cases of (i). (v) follows easily from the assumption that f an isomorphism. □

1.3 Ordinals

We say that a set A is *transitive* if for every $a \in A$, we have $a \subseteq A$ (so all the members of A must be subsets of A). We say that a set is an *ordinal number* (or just an *ordinal*) if it is transitive and well ordered by $\in$. Ordinals will be denoted by $\alpha, \beta, \gamma, \ldots$. It follows directly that

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \quad \ldots$$

are ordinals.

Lemma 1.3 (i) *If* α *is an ordinal and* $a \in \alpha$ *then* a *is an ordinal.*
(ii) *If* α *and* β *are ordinals then* $\alpha \cap \beta$ *is an ordinal.*
(iii) *If* α *is an ordinal then* $\alpha \cup \{\alpha\}$ *is an ordinal.*
(iv) *If* α *is an ordinal then every initial segment of* α *is an ordinal.*
(v) *If* α *and* β *are ordinals and* $\alpha \subseteq \beta$ *then* α *is an initial segment of* β.

Proof. Follows easily from the definitions of ordinals and initial segments. (Use Lemma 1.1 to prove (iv).) □

Lemma 1.4 *We have $\alpha \notin \alpha$ for every ordinal α.*

Proof. If α is an ordinal it follows that $\in$ is a linear ordering on α, so for all $\beta \in \alpha$, $\beta \notin \beta$. So by Lemma 1.3 (iii) we can not have $\alpha \in \alpha$. $\square$

The ordinal $\emptyset$ will be denoted by 0, the ordinal $0 \cup \{0\}$ will be denoted by 1, the ordinal $1 \cup \{1\}$ will be denoted by 2, and so on. So in general we have $n + 1 = n \cup \{n\}$. If α is any ordinal then the ordinal $\alpha \cup \{\alpha\}$ will be denoted by $\alpha + 1$. If β is an ordinal and there exists an ordinal α such that $\beta = \alpha + 1$ then we say that β is the *successor* of α and we say that β is a *successor ordinal*. From Lemma 1.4 it follows that if β the successor of α then there is no ordinal γ such that $\alpha \in \gamma \in \beta$.

Lemma 1.5 *Let α and β be ordinals. Then :*
(i) *If $\alpha \subset \beta$ then $\alpha \in \beta$.*
(ii) *$\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*
(iii) *either $\alpha \subset \beta$ or $\alpha = \beta$ or $\beta \subset \alpha$.*
(iv) *either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$.*

Proof. (i) Suppose that $\alpha \subset \beta$. Since $\in$ is a linear ordering on β we have either $\gamma_1 \in \gamma_2$ or $\gamma_1 = \gamma_2$ or $\gamma_2 \in \gamma_1$ for all $\gamma_1, \gamma_2 \in \beta$.

Let γ be the least element (with respect to the well ordering $\in$ on β) of $\beta - \alpha$. We will show that $\alpha = \gamma$. Then, since $\gamma \in \beta$ it will follow that $\alpha \in \beta$.

Suppose that $\delta \in \alpha$. If $\delta = \gamma$ then $\gamma \in \alpha$, a contradiction. If $\gamma \in \delta$ then $\gamma \in \delta \in \alpha$ so by transitivity of α, $\gamma \in \alpha$, which contradicts the choice of γ. Hence we must have $\delta \in \gamma$. Since $\delta \in \alpha$ was arbitrary it follows that $\alpha \subseteq \gamma$.

Suppose that $\delta \in \gamma$. By transistivity of β we have $\delta \in \beta$. Since γ is the least element of $\beta - \alpha$ (with respect to $\in$) it follows that $\delta \notin \beta - \alpha$, and hence $\delta \in \alpha$. Since $\delta \in \gamma$ was arbitrary it follows that $\gamma \subseteq \alpha$. Hence we have proved that $\alpha = \gamma$.

(ii) By Lemma 1.3 $\gamma = \alpha \cap \beta$ is an ordinal. Clearly we have $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$. If $\gamma \neq \alpha$ and $\gamma \neq \beta$ then $\gamma \subset \alpha$ and $\gamma \subset \beta$ so by part (i) we get $\gamma \in \alpha$ and $\gamma \in \beta$. But then $\gamma \in \alpha \cap \beta = \gamma$ which contradicts Lemma 1.4. Hence we must have $\gamma = \alpha$ or $\gamma = \beta$, and therefore $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

(iii) By (ii), either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. If $\alpha \neq \beta$ then either $\alpha \subset \beta$ or $\beta \subset \alpha$.

(iv) By (iii), either $\alpha \subset \beta$ or $\alpha = \beta$ or $\beta \subset \alpha$, and by (i), either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$. $\square$

Lemma 1.6 *If A is a set of ordinals then $\bigcup A$ is an ordinal.*

Proof. Let $B = \bigcup A$. It is easy to see that B is a transitive set of ordinals, so we will only show that $\in$ is a well ordering on B. By Lemma 1.4, $\alpha \notin \alpha$ for any ordinal α. It follows from the definition of ordinals that if $\alpha, \beta, \gamma \in B$ and $\alpha \in \beta \in \gamma$ then $\alpha \in \gamma$. By Lemma 1.5, if $\alpha, \beta \in B$ and $\alpha \neq \beta$ then either $\alpha \in \beta$ or $\beta \in \alpha$. Now suppose that $C \subseteq B$ and $C \neq \emptyset$. We must show that C has a least element with respect to $\in$. Let $\alpha \in C$. If $\alpha \cap C = \emptyset$ then α is the least element of C because if $\beta \in C$ and $\beta \in \alpha$ then $\alpha \cap C \neq \emptyset$. Now suppose that $\alpha \cap C \neq \emptyset$. Then since α is well ordered by $\in$ there is a least element β (with respect to $\in$) in $\alpha \cap C$. If $\gamma \in C$ and $\gamma \in \beta$ then $\gamma \in \alpha$ by the transitivity of α, but this contradict the choice of β as the least element of $\alpha \cap C$. Hence β is the least element of C. $\square$

Proposition 1.7 *The collection of all ordinals is not a set.*

Proof. Suppose that the collection of all ordinals is a set, say A. Then by Lemma 1.6 $\beta = \bigcup A$ is an ordinal. Hence $\beta \cup \{\beta\}$ is an ordinal so $\beta \cup \{\beta\} \in A$ and therefore $\beta \in \bigcup A = \beta$ which contradicts Lemma 1.4. $\square$

A collection of objects which is not necessarily a set will be called a *class*. Observe that if α, β and γ are ordinals then
(i) $\alpha \notin \alpha$
(ii) if $\alpha \in \beta$ and $\beta \in \gamma$ then $\alpha \in \gamma$
(iii) if $\alpha \neq \beta$ then $\alpha \in \beta$ or $\beta \in \alpha$
Therefore we will say that the class of ordinals is linearly ordered by $\in$. If α and β are ordinals then will often use the notation $\alpha < \beta$ for $\alpha \in \beta$, and the notation $\alpha \leq \beta$ for "$\alpha \in \beta$ or $\alpha = \beta$". If A is a set of ordinals then $\bigcup A$ will also be denoted by $\sup A$ which we call the *supremum* of A. This makes sense since for every $\alpha \in A$ we have $\alpha \leq \sup A$, and if β is an ordinal such that for every $\alpha \in A$, $\alpha \leq \beta$, then $\sup A \leq \beta$ (so $\sup A$ is the least upper bound of A in the class of all ordinals).

Proposition 1.8 *If there exists an ordinal with the property* P *then there exists a least ordinal* α *with the property* P*, in the sense that if* β *is a different ordinal with the property* P *then* $\alpha < \beta$*.*

Proof. Let α be an ordinal with the property P. If α is not the least ordinal with the property P then $A = \{\gamma < \alpha : \ \gamma \text{ has the property P}\}$ is a nonempty subset of α so A has a least element β. Then if γ has the property P and $\gamma < \beta$ it follows that $\gamma \in A$ which contradicts the choice of β. Hence β is the least element with the property P. $\square$

We say that a set A is *inductive* if $\emptyset \in A$ and for every set a, if $a \in A$ then $a \cup \{a\} \in A$. An ordinal is called a *limit ordinal* if it is inductive.

Lemma 1.9 *Every ordinal is exactly one of the following* :
(i) *the empty set (also denoted* 0*)*
(ii) *a successor ordinal*
(iii) *a limit ordinal*

Proof. Suppose that $\alpha \neq 0$ and is not a limit ordinal. Then α is nonempty so $\emptyset$ is a proper subset of α and by Lemma 1.5 (i), $\emptyset \in \alpha$. Since α is not a limit ordinal there must be $\beta \in \alpha$ such that $\beta \cup \{\beta\} \notin \alpha$. But $\beta \cup \{\beta\}$ is an ordinal so (by Lemma 1.5 (iv)) either $\beta + 1 = \beta \cup \{\beta\} = \alpha$ or $\alpha \in \beta \cup \{\beta\}$. If $\alpha \in \beta \cup \{\beta\}$ then $\alpha = \beta$ or $\alpha \in \beta$ and by the assumption $\beta \in \alpha$ and the transitivity of α we get $\alpha \in \alpha$ in both cases, contradicting Lemma 1.4. Hence $\alpha \notin \beta \cup \{\beta\}$ and therefore $\beta \cup \{\beta\} = \alpha$ so α is a successor ordinal. To show that at most one of (i), (ii) and (iii) holds for any ordinal is left as an exercise. $\square$

The least limit ordinal will be called ω. The existence of ω follows from the existence of an inductive set (which is what the axiom of infinity asserts). We will say that an ordinal α is *finite* if for all ordinals $\beta \leq \alpha$, β is a successor ordinal or 0. Otherwise we say that α is an *infinite ordinal.* It follows that for any ordinal α, α is finite if and only if $\alpha < \omega$. Finite ordinals will often be denoted by $i, j, k, l, n, m, \ldots$.

Lemma 1.10 *Let* $(A, <)$ *be a well ordering. If every proper initial segment of* $(A, <)$ *is isomorphic to a unique ordinal then* $(A, <)$ *is isomorphic to a unique ordinal.*

Proof. If $A = \emptyset$ then $(A, <)$ is isomorphic to 0 and no other ordinal. Now suppose that $A \neq \emptyset$. For every $a \in A$ let α_a be the unique ordinal to which

$$A_a = \{x \in A : \ x < a\}$$

is isomorphic, and let f_a be the unique (by Lemma 1.2 (i)) isomorphism from A_a onto α_a. It follows from Lemma 1.5 (iii), Lemma 1.3 (v) and Lemma 1.2 (iv),(v) that for all $a, b, c \in A$ such that $a < b < c$, α_b is an initial segment of α_c and $f_b(a) = f_c(a)$. Let

$$A^- = \{a \in A : \ \exists b \in A, a < b\}.$$

For every $a \in A^-$, let a^+ be the least $b \in A$ such that $a < b$. Define $f(a) = f_{a^+}(a)$ for every $a \in A^-$. Then it f will be an isomorphism from $(A^-, <)$ onto the ordinal $\alpha = \bigcup\{\alpha_a : a \in A^-\}$. (It follows from the axiom of replacement that $\{\alpha_a : a \in A^-\}$ is a set.) Moreover, if h is an isomorphism from $(A^-, <)$ onto an ordinal $\gamma \neq \alpha$, then it follows that fh^{-1} is an isomorphism from γ onto α which contradicts Lemma 1.2 (iv) and the fact that one of α and γ must be a proper initial segment of the other. Hence $(A^-, <)$ is isomorphic to the unique ordinal α. If $A^- \neq A$ then $A - A^-$ contains exactly one element, say b. Define $g(b) = \alpha$ and $g(a) = f(a)$ for all $a \in A^-$. Then g is an isomorphism from $(A, <)$ onto the ordinal $\beta = \alpha \cup \{\alpha\}$. If h is an isomorphism from $(A, <)$ onto an ordinal $\gamma \neq \beta$, then gh^{-1} is an isomorphism from γ onto β which contradicts Lemma 1.2 (iv) and the fact that one of β and γ must be a proper initial segment of the other. Hence $(A, <)$ is isomorphic to the unique ordinal β. □

Theorem 1.11 *Every well ordering is isomorphic to a unique ordinal.*

Proof. Let $(A, <)$ be a well ordering. If $(A, <)$ is not isomorphic to a unique ordinal then by Lemma 1.10 there must be a proper initial segment of $(A, <)$ which is not isomorphic to a unique ordinal. By Lemma 1.1 every proper initial segment has the form

$$A_a = \{x \in A : x < a\}$$

for some $a \in A$. Hence there is a least element $a \in A$ such that $(A_a, <)$ is not isomorphic to a unique ordinal. If $A_a = \emptyset$ then $(A_a, <)$ is isomorphic to 0, a contradiction, so A_a must be nonempty. Then every proper initial segment of $(A_a, <)$ has the form $(A_b, <)$ for some $b < a$, and for every $b < a$, $(A_b, <)$ is isomorphic to a unique ordinal. Hence every proper initial segment of $(A_a, <)$ is isomorphic to a unique ordinal, so by Lemma 1.10 $(A_a, <)$ is isomorphic to a unique ordinal, a contradiction. We conclude that $(A, <)$ must be isomorphic to a unique ordinal. □

Corollary 1.12 *For any two well orderings, one of them is isomorphic to an initial segment of the other.*

Proof. Exercise. □

1.4 Equivalents of the axiom of choice

Recall that the axiom of choice is the following statement, which is abbreviated AC:

For any set X of nonempty sets there exists a function $f : X \to \bigcup X$ such that for every $a \in X$, $f(a) \in a$.
We call f a *choice function* for X. So far AC has not been needed (i.e all results obtained until now where proved from the axioms (1)-(7), although we did not emphasize this), but later on we will need AC to prove statements about cardinals and cardinalities (which will be defined later). We say that a statement S is equivalent to AC if from (1)-(7) and AC we can prove S, and from (1)-(7) and S we can prove AC. In this section we will prove that AC is equivalent to two other statements which are sometimes more usefull than AC itself.

Let A be a set. A *partial order* on A is a binary relation $<$ on A such that for all $a, b, c \in A$ the following holds:
(i) $a \not< a$ (i.e. not $a < a$)
(ii) if $a < b$ and $b < c$ then $a < c$
As usual we write $a \leq b$ for "$a < b$ or $a = b$". If $<$ is partial order on A then we will say that $(A, <)$ is a *partially ordered set* (or a *partial ordering*). Let $(A, <)$ be a partial ordering. A subset B of A is called a *chain* in A if for every $a, b \in B$ we have $a \leq b$ or $b \leq a$ (i.e $(B, <)$ is a linear ordering). Note that if $(A, <)$ is a linear ordering then every $B \subseteq A$ is a chain in A, and in particular A is a chain in A. An element $a \in A$ is called a *maximal element* if there exists no $b \in A$ such that $a < b$.

Theorem 1.13 *The following are equivalent:*
(i) AC
(ii) *Every set can be well ordered.*
(iii) *If $(A, <)$ is a nonempty partial ordering such that for every chain $B \subseteq A$ there exists $a \in A$ such for every $b \in B$, $b \leq a$, then $(A, <)$ has a maximal element.*
Condition (ii) is often called Zermelo's theorem, and condition (iii) is often called Zorn's lemma.

Proof. (i) $\Rightarrow$ (iii) Let $(A, <)$ be a partial ordering such that for every chain $B \subseteq A$ there exists $a \in A$ such that $b \leq a$ for all $b \in B$. We will show that $(A, <)$ has a maximal element. Let X be the set of chains in A. Then X is nonempty (because $\emptyset \in X$) and $\subset$ is a partial order on X. Suppose that C is a maximal element in X (with respect to $\subset$). Since $C \subseteq A$ is a chain there exists $a \in A$ such that for all $x \in C$, $x \leq a$. Then a must be a maximal element in A because if there would be $b \in A$ such that $a < b$ then $b \notin C$ and $C \cup \{b\}$ would be a chain , and this would contradict the maximality of C. Hence it suffices to show that X has a maximal element.

Observe that if $C \in X$ and $D \subseteq C$ then $D \in X$, and that if $\mathcal{C} \subseteq X$ is a chain then $\bigcup \mathcal{C}$ is a chain in A, and hence $\bigcup \mathcal{C} \in X$. By (i) let $f : \mathcal{P}(X) - \{\emptyset\} \to X$ be a function such that for that for every $Y \in \mathcal{P}(X) - \{\emptyset\}$, $f(Y) \in Y$. For every $C \in X$ let $C^* = \{x \in A : C \cup \{x\} \in X\}$. Note that $C \subseteq C^*$. We define a function $g : X \to X$ by:

$$g(C) = \begin{cases} C \cup \{f(C^* - C)\} & \text{if } C^* - C \neq \emptyset \\ C & \text{if } C^* - C = \emptyset \end{cases}$$

If $g(C) = C$ then there exists no $x \in A$ such that $C \cup \{x\}$ is a chain in A, and hence C is a maximal element in X. Therefore it suffices to find $C \in X$ such that $g(C) = C$.

We say that $Y \subseteq X$ is a *tower* if:
(I) $\emptyset \in Y$
(II) if $C \in Y$ then $g(C) \in Y$
(III) if $\mathcal{C} \subseteq Y$ is a chain then $\bigcup \mathcal{C} \in Y$
Observe that for example X itself is a tower. The intersection of the set of all towers $Y \subseteq X$ is itself a tower, call it Y_0. Now we will show that Y_0 is a chain.

We say that $C \in Y_0$ is *comparable* if for every $D \in Y_0$, $D \subseteq C$ or $C \subseteq D$. Note that $\emptyset \in Y_0$ is comparable. Suppose that $C \in Y_0$ is comparable. If $D \in Y_0$ and $D \subset C$ then $g(D) \in Y_0$ (because Y_0 is a tower) and if $C \subset g(D)$ then $D \subset C \subset g(D)$, which contradicts the fact that $g(D)$ contains at most one more element than D. Hence we have proved that,
$(*)$ if $D \in Y_0$ and $D \subset C$ then $g(D) \subseteq C$.
Let

$$Z_C = \{D \in Y_0 : D \subseteq C \text{ or } g(C) \subseteq D\}$$

We will show that Z_C is a tower. $\emptyset \subseteq C$ so $\emptyset \in Z_C$. Suppose that $\mathcal{C} \subseteq Z_C$ is a chain. Then for every $D \in \mathcal{C}$ we have $D \subseteq C$ or $g(C) \subseteq D$. If $D \subseteq C$ for every $D \in \mathcal{C}$ then $\bigcup \mathcal{C} \subseteq C$ so $\bigcup \mathcal{C} \in Z_C$, and if $g(C) \subset D$ for some $D \in \mathcal{C}$ then $g(C) \subseteq \bigcup \mathcal{C}$. Hence $\bigcup \mathcal{C} \in Z_C$. Let $D \in Z_C$. Then $D \subseteq C$ or $g(C) \subseteq D$. If $D \subset C$ then by $(*)$, $g(D) \subseteq C$, so $g(D) \in Z_C$. If $D = C$ then $g(D) = g(C)$ so $g(D) \in Z_C$. If $g(C) \subseteq D$ then $g(C) \subseteq g(D)$ so $g(D) \in Z_C$. Thus, we have proved that Z_C is a tower and since $Z_C \subseteq Y_0$ we must have $Z_C = Y_0$. So if $D \in Y_0$ then $D \in Z_C$ and hence $g(C) \subseteq D$ or $D \subseteq C$, and in the latter case $D \subseteq g(C)$. Hence, $g(C)$ is comparable.

Remember that C is assumed to be a comparable but otherwise arbitrary element of Y_0. It follows that for any $C \in Y_0$, if C is comparable then so is $g(C)$. Let $U = \{C \in Y_0 : C \text{ is comparable}\}$. Then, by what we have alredy shown and by the (easily verified) fact that if $\mathcal{C} \subseteq U$ is a chain then $\bigcup \mathcal{C}$ is comparable, it follows that U is a tower, and therefore $U = Y_0$. Hence, Y_0 is a chain in Y_0 so $C = \bigcup Y_0 \in Y_0$ (because Y_0 is a tower), and then $g(C) \in Y_0$. Then $g(C) \subseteq \bigcup Y_0 = C$, and since we also have $C \subseteq g(C)$ we get $g(C) = C$, and this is what we needed to prove.

(iii) $\Rightarrow$ (ii) Let A be any set. Let X be the set of pairs $(B, <)$ where $B \subseteq A$ and $<$ is a relation which is a well ordering on B. Then $(\emptyset, \emptyset) \in X$ so X is not empty. We define a relation $<$ on X as follows:
for $(B, <_1), (C, <_2) \in X$,

$$(B, <_1) < (C, <_2) \quad \text{if and only if}$$
$$B \subset C \text{ and for all } x, y \in B,\ x <_1 y \text{ if and only if } x <_2 y.$$

It is not difficult to see that $<$ is a partial ordering on X. Let $\mathcal{C} \subseteq X$ be a chain. Let $C = \bigcup\{B \subseteq A :\ (B, <) \in \mathcal{C}\}$. A well ordering $<_0$ on C can be defined as follows:
for all $x, y \in C$,

$$x <_0 y \text{ if and only if}$$
$$x <_1 y \text{ for some } (B, <_1) \in \mathcal{C} \text{ such that } x, y \in B.$$

Then (as the reader can verify) we have $(B, <_1) \leq (C, <_0)$ for every $(B, <_1) \in \mathcal{C}$. Since $\mathcal{C} \subseteq X$ was an arbitrary chain, by (iii) it follows that $(X, <)$ has a maximal element, say $(B, <_1)$. If $B \subset A$ then let $a \in A - B$ and let $<_2$ be the relation on $B \cup \{a\}$ which is defined by: for all $x, y \in B$, $x <_2 a$, and $x <_2 y$ if and only if $x <_1 y$. Then $(B, <_2) \in X$ and $(B, <_1) < (B \cup \{a\}, <_2)$, which contradicts that $(B, <_1)$ is a maximal element of X. Hence, we must have $B = A$ and then $(A, <_1)$ is a well ordering.

(ii) $\Rightarrow$ (i) Let A be a set of nonempty sets. By (ii) there is a well ordering $<$ on $\bigcup A$. Define $f : A \to \bigcup A$ by $f(a) =$ the least element in a. $\square$

1.5 Cardinals

Theorem 1.14 (Cantor-Bernstein) *Let A and B be two sets. If there are an injective function from A into B , and an injective function from B into A then there is a bijective function from A onto B.*

Proof. Suppose that $f : A \to B$ and $g : B \to A$ are injective functions. Let $C = g(B)$ and $D = g(f(A))$. Then $D \subseteq C \subseteq A$. We will show that there is an injective function F from A onto C. Then it will follow that $(g{\restriction}B)^{-1}F$ is an injective function from A onto B (where $g{\restriction}B$ is the restriction of g to

B). Let $h = gf$. Then h is an injective function from A onto D. For every $n < \omega$ we define A_n and A_n^* inductively by

$$A_0 = A \ , \ A_{n+1} = h(A_n) \qquad A_0^* = C \ , \ A_{n+1}^* = h(A_n^*)$$

Observe that for every $n < \omega$, $A_n^* \subseteq A_n$. Define F by

$$F(x) \ = \ \begin{cases} h(x) & \text{if } x \in A_n - A_n^* \text{ for some } n < \omega \\ x & \text{otherwise} \end{cases}$$

We will now show that F is an injective function from A onto C. Suppose that $F(a_0) = F(a_1)$. We have four cases which can occur:
(1) There exist $n < \omega$ such that $a_0 \in A_n - A_n^*$ but for all $m < \omega$, $a_1 \notin A_m - A_m^*$.
(2) There exist $n < \omega$ such that $a_1 \in A_n - A_n^*$ but for all $m < \omega$, $a_0 \notin A_m - A_m^*$.
(3) There are $n < \omega$ and $m < \omega$ such that $a_0 \in A_n - A_n^*$ and $a_1 \in A_m - A_m^*$.
(4) For all $n < \omega$, $a_0 \notin A_n - A_n^*$ and $a_1 \notin A_n - A_n^*$.

Suppose that (1) holds. Then $a_1 = F(a_1) = F(a_0) = h(a_0)$, so $a_1 \in h(A_n) = A_{n+1}$. By (1), $a_1 \notin A_{n+1} - A_{n+1}^*$ so we get $a_1 \in A_{n+1}^* = h(A_n^*)$. Then there exists $a_2 \in A_n^*$ such that $h(a_2) = a_1$, and since $a_0 \notin A_n^*$ we have $a_0 \neq a_2$. But then $h(a_0) = h(a_2)$ and $a_0 \neq a_2$ which contradicts that h is injective. Since the case (2) is symmetric to (1), we also get a contradiction from (2).

Hence the only cases that can really occur are (3) or (4). If (3) holds then $h(a_0) = F(a_0) = F(a_1) = h(a_1)$ so $a_0 = a_1$ by the injectivity of h , and if (4) holds then $a_0 = F(a_0) = F(a_1) = a_1$. So we conclude that F is injective.

Now we show that F is onto C. Let $c \in C$. If for all $n < \omega$, $c \notin A_n - A_n^*$ then $F(c) = c$. Suppose that for some $n < \omega$, $c \in A_n - A_n^*$. We can not have $n = 0$ because $A_0^* = C$, so $c \in A_{m+1} - A_{m+1}^*$ for some $m < \omega$. Then $c \in A_{m+1} - A_{m+1}^* = h(A_m) - h(A_m^*)$ so $c = h(a)$ for some $a \in A_m - A_m^*$, and by definition of F we get $F(a) = c$. Now we have proved that F is a bijective function from A onto C. $\square$

For every ordinal α there exists a least ordinal β for which there exists an injective function from α into β. By definition, a *cardinal number* (or just *cardinal*) is an ordinal α such that for every ordinal $\beta < \alpha$ there is no injective function from α into β.

Lemma 1.15 *If κ and λ are cardinals then $\kappa \leq \lambda$ if and only if there exists an injective function from κ into λ.*

Proof. If $\kappa \leq \lambda$ then $\kappa \subseteq \lambda$ and clearly the identity function on κ is an injective function from κ into λ. If there exists an injective function from κ into λ then we can not have $\lambda < \kappa$ because then κ would not be a cardinal. Hence $\kappa \leq \lambda$. $\square$

Proposition 1.16 *For every set A, there exists a unique cardinal κ such that there exists a bijective function from A onto κ.*

Proof By the axiom of choice and Theorem 1.13 there exists a binary relation, $<$, on A which is a well ordering, and by Theorem 1.11 there exists an isomorphism f from $(A, <)$ onto an ordinal α. Since f is an isomorphism, it is a bijective function from A onto α. Let κ be the least ordinal for which there exists an injective function from α into κ. By the definition of cardinals, κ must be a cardinal. Since $\kappa \subseteq \alpha$, there is an injective function from κ into α. By Theorem 1.14, there exists a bijective function $g : \alpha \to \kappa$. Since both $f : A \to \alpha$ and $g : \alpha \to \kappa$ are bijective, it follows that if $h = gf$, then $h : A \to \kappa$ is bijective.

If $\lambda \neq \kappa$ is another cardinal such that there exists a bijective function, say h_0, from A onto λ then hh_0^{-1} is a bijective function from λ onto κ, which contradicts Lemma 1.15. $\square$

If A is a set then the *cardinality* of A, denoted by $|A|$, is the the unique cardinal κ for which there exists an injective function from A onto κ. Observe that it follows that for any cardinal κ , $|\kappa| = \kappa$, and if $\alpha < \kappa$ is an ordinal then $|\alpha| < \kappa$. Also $|\alpha| \leq \alpha$ for every ordinal α.

Lemma 1.17 *If A and B are sets then :*
(i) $|A| \leq |B|$ *if and only if there exists an injective function from A* into *B.*
(ii) $|A| = |B|$ *if and only if there exists an bijective function from A* onto *B.*
(iii) $|A| = |B|$ *if and only if* $|A| \leq |B|$ *and* $|B| \leq |A|$.
(iv) $|A \times B| = ||A| \times |B||$.

Proof. Exercise. (To prove (ii), use (i) and Theorem 1.14.) $\square$

Theorem 1.18 (Cantor) *For every set A , there does not exist a function from A onto $\mathcal{P}(A)$.*

Proof. Let f be any function from A into $\mathcal{P}(A)$. Let $B = \{x \in A : x \notin f(x)\}$. If for some $a \in A$, $f(a) = B$, then

$$a \in B \Leftrightarrow a \notin f(a) \Leftrightarrow a \notin B,$$

a contradiction. Hence f is not onto $\mathcal{P}(A)$. $\square$

Proposition 1.19 *For every ordinal α there is a cardinal κ such that $\alpha < \kappa$.*

Proof. Let $\kappa = |\mathcal{P}(\alpha)|$. If $\kappa \leq \alpha$ then $\kappa \subseteq \alpha$ so there is a function from α onto κ and then there is also a function from α onto $\mathcal{P}(\alpha)$ which contradicts Theorem 1.18. Hence we must have $\alpha < \kappa$. $\square$

If κ is a cardinal then κ^+ denotes the least cardinal λ such that $\kappa < \lambda$.

Lemma 1.20 *If A is a set of cardinals then $\bigcup A$ is a cardinal.*

Proof. By Lemma 1.6 $\bigcup A$ is an ordinal. Let $\alpha = \bigcup A$. Suppose that α is not a cardinal. Then there exists an injective function f from α into an ordinal $\beta < \alpha$. But $\beta < \alpha$ means $\beta \in \alpha$ and since $\alpha = \bigcup A$ there exists a cardinal $\kappa \in A$ such that $\beta \in \kappa$. By transitivity we get $\beta \subset \kappa \subseteq \alpha$ so $f{\restriction}\kappa$ (the restriction of f to κ) is an injective function from κ *onto* $f{\restriction}\kappa(\kappa)$ and hence $\kappa = |\kappa| = |f{\restriction}\kappa(\kappa)|$ But $f{\restriction}\kappa(\kappa) \subseteq \beta$ so there is an injective function from $f{\restriction}\kappa(\kappa)$ into β (the identity function on $f{\restriction}\kappa(\kappa)$) and hence $|f{\restriction}\kappa(\kappa)| \leq |\beta| \leq \beta$, so we get $|f{\restriction}\kappa(\kappa)| \subseteq \beta$ and therefore $\kappa \subseteq \beta$. But we also have $\beta \in \kappa$ so we get $\beta \in \beta$, a contradiction. Therefore α must be a cardinal. $\square$

Proposition 1.21 *ω is a cardinal and every $\alpha < \omega$ is a cardinal.*

Proof. Exercise. $\square$

By definition, a *finite cardinal* is a cardinal which is finite as an ordinal, otherwise we call it an *infinite cardinal*. Let A be a set. We say that A is *finite* if $|A|$ is finite, otherwise we say that A is *infinite*. We say that A is *countable* (or *enumerable*) if $|A| \leq \omega$, otherwise we say that A is *uncountable*.

For any sets A and B , let the disjoint union of A and B, denoted $A \oplus B$, be the set

$$\{(0,a) \ : \ a \in A\} \cup \{(1,b) \ : \ b \in B\}$$

and let the set of functions from A into B be denoted by ${}^A B$. If κ and λ are cardinals then we define $\kappa + \lambda = |\kappa \oplus \lambda|$, and $\kappa \cdot \lambda = |\kappa \times \lambda|$, and $\kappa^\lambda = |{}^\lambda \kappa|$.

Lemma 1.22 *(a) $+$ and $\cdot$ are associative, commutative, and distributive.*
(b) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$ then $\kappa_1 + \lambda_1 \leq \kappa_2 + \lambda_2$ and $\kappa_1 \cdot \lambda_1 \leq \kappa_2 \cdot \lambda_2$.

(c) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
(d) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.
(e) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.
(f) *If* $\kappa \leq \lambda$ *then* $\kappa^\mu \leq \lambda^\mu$.
(g) *If* $0 < \lambda \leq \mu$ *then* $\kappa^\lambda \leq \kappa^\mu$.
(h) $\kappa^0 = 1$ *and* $1^\kappa = 1$.
(i) *If* $\kappa > 0$ *then* $0^\kappa = 0$.

Proof. One only needs to find the appropriate injective functions. □

In what follows we will identify the natural numbers with ω. If we define a function s from ω into ω by $s(n) = n + 1$ then one can show that the structure $(\omega, 0, s, +, \cdot, =)$ satisfies the axioms of arithmetic. Moreover, if we define a relation $<^*$ on ω by, $n <^* m \Leftrightarrow \exists x(x \neq 0 \wedge n + x = m)$ then $<^*$ coincides with the ordering $<$ on ω.

Theorem 1.23 *For every set* A *,* $|\mathcal{P}(A)| = 2^{|A|}$.

Proof. We will show that there exists an injective function from $\mathcal{P}(A)$ onto ${}^A2$. Then it is not difficult to see that there also exists an injective function from $|\mathcal{P}(A)|$ onto $2^{|A|}$, and therefore (by Lemma 1.17 (ii)) $|\mathcal{P}(A)| = 2^{|A|}$. For every $B \in \mathcal{P}(A)$ let f_B be the function from A into 2 which is defined by

$$f_B(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \notin B \end{cases}$$

Now define a function g from $\mathcal{P}(A)$ into ${}^A2$ by $g(B) = f_B$. Then it is easy to see that g is injective and onto ${}^A2$. □

Theorem 1.24 *If* λ *and* κ *are cardinals and* $\kappa \geq 2$ *, then* $\lambda < \kappa^\lambda$.

Proof. By the proof of Proposition 1.19 $\lambda < |\mathcal{P}(\lambda)|$ and by Theorem 1.23 $|\mathcal{P}(\lambda)| = 2^\lambda$, so we get $\lambda < 2^\lambda$. By Lemma 1.22 (f) we get $\lambda < \kappa^\lambda$ for all $\kappa \geq 2$. □

Proposition 1.25 *Let* $\mathbb{R}$ *be the set of real numbers. Then* $|\mathbb{R}| = 2^\omega$.

Proof. (*Sketch*) Integers, rational numbers and real numbers can be defined from the natural numbers (= elements of ω) and one can show that for every $r \in \mathbb{R}$ there exist a unique function $f_r \in {}^\omega 2$ and a unique $n_r \in \omega$ such that

$$r = (-1)^{f_r(0)} \cdot \sum_{n=0}^{n_r} f_r(2 \cdot (n+1)) \cdot 2^n + \sum_{n=0}^{\infty} \frac{f_r(2 \cdot n + 1)}{2^{n+1}}.$$

It follows that if $s, r \in \mathbb{R}$ and $s \neq r$ then $f_s \neq f_r$. Define a function $F : \mathbb{R} \to {}^{\omega}2$ by $F(r) = f_r$ where f_r is the unique function given above. Then F will be injective. Define a function $G : {}^{\omega}2 \to \mathbb{R}$ by

$$G(f) = \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}} \quad \text{(observe that this infinite sum is convergent for any } f\text{)}$$

Then G is injective. By Theorem 1.14 it follows that there exists an injective function from $\mathbb{R}$ onto ${}^{\omega}2$ and therefore $|\mathbb{R}| = |{}^{\omega}2| = 2^{\omega}$. □

Theorem 1.26 *If $\kappa > 0$ and $\lambda > 0$ are cardinals where at least one of κ and λ is infinite then : $\kappa + \lambda = \kappa \cdot \lambda = \sup\{\kappa, \lambda\}$.*

Proof. First note that for any cardinals μ_1, μ_2 we have $\mu_1 \leq \mu_1 + \mu_2$ and $\mu_1 \leq \mu_1 \cdot \mu_2$. Then by Lemma 1.22 (b) it is sufficient to prove that for any infinite cardinal κ , $\kappa + \kappa = \kappa \cdot \kappa = \kappa$. But $\kappa + \kappa = \kappa$ follows from $\kappa \cdot \lambda = \sup\{\kappa, \lambda\}$ and the fact that $\kappa + \kappa = |2 \times \kappa| = 2 \cdot \kappa$, so it is sufficient to prove that $\kappa \cdot \kappa = \kappa$. Since $\kappa \leq \kappa \cdot \kappa$ we will only prove $\kappa \cdot \kappa \leq \kappa$. We will prove this by showing that $\omega \cdot \omega \leq \omega$ and that if κ is and infinite cardinal and $\lambda \cdot \lambda \leq \lambda$ for all infinite cardinals $\lambda < \kappa$ then $\kappa \cdot \kappa \leq \kappa$. Then, by Proposition 1.8, it follows that $\kappa \cdot \kappa \leq \kappa$ for all infinite cardinals κ.

First we show that $\omega \cdot \omega \leq \omega$. We define a well ordering $\prec$ on $\omega \times \omega$ by, $(\alpha, \beta) \prec (\gamma, \delta)$ if and only if

$$\begin{aligned} &\sup\{\alpha, \beta\} < \sup\{\gamma, \delta\} \\ &\text{or } \sup\{\alpha, \beta\} = \sup\{\gamma, \delta\} \text{ and } \alpha < \gamma \\ &\text{or } \sup\{\alpha, \beta\} = \sup\{\gamma, \delta\} \text{ and } \alpha = \gamma \text{ and } \beta < \delta \end{aligned}$$

The reader can check that $\prec$ is a linear ordering (and we write $\preceq$ for "$\prec$ or $=$"). We will show that $\prec$ is a well ordering. Suppose that A is a nonempty subset of $\omega \times \omega$. Let α_1 be the least element of $\{\sup\{\alpha, \beta\} \ : \ (\alpha, \beta) \in A\}$, and let

$$A_1 = \{(\alpha, \beta) \in A : \ \sup\{\alpha, \beta\} = \alpha_1\}$$

Let α_2 be the least element of $\{\alpha \in \omega \ : \ \exists \beta, \ (\alpha, \beta) \in A_1\}$, and let

$$A_2 = \{(\alpha, \beta) \in A_1 : \ \alpha = \alpha_2\}$$

Let β_1 be the least element of $\{\beta \in \omega \ : \ (\alpha_2, \beta) \in A_2\}$. Then it follows from the definition of $\prec$ and the choice of α_2 and β_1 that (α_2, β_1) is the least element of A.

Now we will show that every proper initial segment of $(\omega \times \omega, \prec)$ is isomorphic to an ordinal $\alpha < \omega$. Then it will follow from the proof of Lemma 1.10 that $(\omega \times \omega, \prec)$ is isomorphic to an ordinal $\beta \leq \omega$, and hence we will have $|\omega \times \omega| \leq \omega$.

Suppose that some proper initial segment of $(\omega \times \omega, \prec)$ is not isomorphic to any ordinal $\alpha < \omega$. By Lemma 1.1 every proper initial segment can be written as

$$\{(\alpha, \beta) \in \omega \times \omega \ : \ (\alpha, \beta) \prec (\gamma, \delta)\}$$

for some $(\gamma, \delta) \in \omega \times \omega$. Let (γ, δ) be the least element in $\omega \times \omega$ such that

$$A = \{(\alpha, \beta) \in \omega \times \omega \ : \ (\alpha, \beta) \prec (\gamma, \delta)\}$$

is not isomorphic to any ordinal $\alpha < \omega$. If $\gamma = \delta = 0$ then $A = \emptyset$ and is isomorphic to 0, a contradiction, so $\gamma \neq 0$ or $\delta \neq 0$. We will show that both $\gamma \neq 0$ and $\delta \neq 0$ leads to contradictions and then we can conclude that every proper initial segment is isomorphic to an ordinal $\alpha < \omega$.

Suppose that $\delta \neq 0$. Since $\delta < \omega$ it follows that δ is a successor ordinal and hence $\delta = \delta_0 + 1$. Then $(\gamma, \delta_0) \prec (\gamma, \delta)$ and there does not exist $(\alpha, \beta) \in \omega \times \omega$ such that $(\gamma, \delta_0) \prec (\alpha, \beta) \prec (\gamma, \delta)$. By the choice of (γ, δ) it follows that

$$B = \{(\alpha, \beta) \in \omega \times \omega \ : \ (\alpha, \beta) \prec (\gamma, \delta_0)\}$$

is isomorphic to an ordinal $\alpha_0 < \omega$. Let f be the isomorphism from B onto α_0. Define g by $g((\alpha, \beta)) = f((\alpha, \beta))$ for all $(\alpha, \beta) \in B$ and $g((\gamma, \delta_0)) = \alpha_0$. Then g is an isomorphism from A onto the ordinal $\alpha_0 + 1 < \omega$, and this contradicts the choice of (γ, δ).

Now suppose that $\delta = 0$ but $\gamma \neq 0$. Since $\gamma < \omega$ it follows that γ is a successor ordinal and hence $\gamma = \gamma_0 + 1$. Then $(\gamma_0, \delta) \prec (\gamma, \delta)$ and there does not exist $(\alpha, \beta) \in \omega \times \omega$ such that $(\gamma_0, \delta) \prec (\alpha, \beta) \prec (\gamma, \delta)$. By the choice of (γ, δ) it follows that

$$C = \{(\alpha, \beta) \in \omega \times \omega \ : \ (\alpha, \beta) \prec (\gamma_0, \delta)\}$$

is isomorphic to an ordinal $\alpha_0 < \omega$. As in the previous case we can find an isomorphism from C onto $\alpha_0 + 1 < \omega$, which contradicts the choice of (γ, δ).

Now assume that $\lambda \cdot \lambda \leq \lambda$ for every cardinal $\lambda < \kappa$. We will show that $\kappa \cdot \kappa \leq \kappa$. Define a well ordering $\prec$ on $\kappa \times \kappa$ by, $(\alpha, \beta) \prec (\gamma, \delta)$ if and only if

$$\begin{aligned} &\sup\{\alpha, \beta\} < \sup\{\gamma, \delta\} \\ &\text{or } \sup\{\alpha, \beta\} = \sup\{\gamma, \delta\} \text{ and } \alpha < \gamma \\ &\text{or } \sup\{\alpha, \beta\} = \sup\{\gamma, \delta\} \text{ and } \alpha = \gamma \text{ and } \beta < \delta \end{aligned}$$

In the same way as before it can be proved that $\prec$ is a well ordering. Let α be the unique ordinal to which $(\kappa \times \kappa, \prec)$ is isomorphic. We will show that $\alpha \leq \kappa$. Then it will easily follow that $\kappa \cdot \kappa \leq \kappa$.

Suppose on the contrary that $\kappa < \alpha$. Let f be an isomorphism from α onto $(\kappa \times \kappa, \prec)$. Since $\kappa \in \alpha$ there is $(\beta, \gamma) \in \kappa \times \kappa$ such that $f(\kappa) = (\beta, \gamma)$. Let

$$B = \{(\alpha_1, \beta_1) \in \kappa \times \kappa \ : \ (\alpha_1, \beta_1) \prec (\beta, \gamma)\}$$

Then $f \restriction \kappa$ (the restriction of f to κ) is an injective function from κ onto B. Let $\delta = \sup\{\beta, \gamma\}$. Then $\delta < \kappa$ and since $|\delta| \leq \delta$ we have

$$|\delta| < \kappa$$

By the induction hypothesis we have

$$|\delta| \cdot |\delta| \leq |\delta|$$

By the choice of δ and the fact that B is an initial segment of $(\kappa \times \kappa, \prec)$ it follows that $B \subseteq \delta \times \delta$ and hence

$$|B| \leq |\delta \cdot \delta| \leq |\delta| \cdot |\delta|$$

If we now put the above inequalities together we get $|B| \leq |\delta \cdot \delta| \leq |\delta| \cdot |\delta| \leq |\delta| < \kappa$ which contradicts that $f \restriction \kappa$ is an injective function from κ onto $|B|$. $\square$

Corollary 1.27 *If A and B are nonempty sets and at least one of them is infinite then $|A \cup B| = |A \times B| = sup\{|A|, |B|\}$.*

Proof. Exercise. $\square$

Corollary 1.28 *If B is an infinite set and $A \subset B$ such that $|A| < |B|$, then $|B - A| = |B|$.*

Proof. If $|B - A| < |B|$ then $|B| = |A \cup (B - A)| = sup\{|A|, |B - A|\} < |B|$, a contradiction. Hence $|B - A| = |B|$. $\square$

Theorem 1.29 *If A is a nonempty set of sets and at least one $a \in A$ is infinite then*

$$|\cup A| \leq \sup\{|A|, \sup\{|a| \ : \ a \in A\}\}$$

Proof. Let $\kappa = \sup\{|A|, \sup\{|a| \ : \ a \in A\}\}$. For every $x \in \bigcup A$ let $B_x = \{a \in A \ : \ x \in a\}$ and let $B = \{B_x \ : \ x \in \bigcup A\}$. Clearly, B_x is nonempty for every $x \in \bigcup A$ so by the axiom of choice the exists a choice function f_1 from B into $\bigcup B = A$. Define a function g_1 from $\bigcup A$ into A by $g_1(x) = f_1(B_x)$. For every $a \in A$ let

$$C_a = \{h \ : \ h \text{ is an injective function from } a \text{ into } \kappa\}$$

and let $C = \{C_a \ : \ a \in A\}$. Since $|a| \leq \kappa$ for every $a \in A$, it follows that C_a is nonempty for every $a \in A$. By the axiom of choice there exists a choice function f_2 from C into $\bigcup C$. Define a function g_2 from A into $\bigcup C$, by $g_2(a) = f_2(C_a)$. Then, for every $a \in A$, $g_2(a)$ will be an injective function from a into κ. Since $|A| \leq \kappa$ there exists an injective function g_0 from A into κ. Now define a function G from $\bigcup A$ into $\kappa \times \kappa$ by,

$$G(x) = (g_0(g_1(x)), g_2(g_1(x))(x))$$

We show that G is injective. Suppose that $G(x) = G(y)$. Then

$$g_0(g_1(x)) = g_0(g_1(y)) \quad \text{and} \quad g_2(g_1(x))(x) = g_2(g_1(y))(y).$$

Since g_0 is injective we get $g_1(x) = g_1(y)$ so $g_2(g_1(x)) = g_2(g_1(y))$, and therefore $g_2(g_1(x))(x) = g_2(g_1(x))(y)$. But $g_2(g_1(x))$ is also injective so we get $x = y$. Now we have proved that $|\bigcup A| \leq \kappa \times \kappa$ and since κ is infinite it follows from Theorem 1.26 $\kappa \times \kappa = \kappa$, so we get $|\bigcup A| \leq \kappa$. $\square$

Corollary 1.30 *If A is a countable set of countable sets then $\bigcup A$ is countable.*

Proof. Follows immediately from Theorem 1.29. $\square$

We say that a cardinal κ is *regular* if the following holds:

$$\text{if } A \subset \kappa \text{ and } |A| < \kappa \text{ then } \sup A < \kappa$$

Otherwise we say that κ is *singular*.

Proposition 1.31 *For every cardinal κ , κ^+ is regular. (Hence there are arbitrarily large regular cardinals.)*

Proof. Suppose that $A \subset \kappa^+$ and $|A| < \kappa^+$. Then $|A| \leq \kappa$, and for every $\alpha \in A$, $|\alpha| < \kappa^+$, so $|\alpha| \leq \kappa$. Hence

$$\sup\{|\alpha| \ : \ \alpha \in A\} \leq \kappa$$

$$\text{and} \qquad \sup\{|A|, \sup\{|\alpha| \ : \ \alpha \in A\}\} \leq \kappa$$

and by Theorem 1.29 $\sup A \leq \sup\{|A|, \sup\{|\alpha| \ : \ \alpha \in A\}\}$, so we get $\sup A \leq \kappa < \kappa^+$. □

To every ordinal α we can associate a unique infinite cardinal, denoted $\aleph_\alpha$, in the following way. Let $\aleph_0$ be ω. If $\aleph_\alpha$ is defined then define $\aleph_{\alpha+1}$ to be $(\aleph_\alpha)^+$. If α is a limit ordinal and $\aleph_\beta$ is defined for every $\beta < \alpha$ then define $\aleph_\alpha$ to be $\sup\{\aleph_\beta : \ \beta < \alpha\}$ (by Lemma 1.20 $\sup\{\aleph_\beta : \ \beta < \alpha\}$ is a cardinal). This assignment from ordinals to cardinals is injective and onto, i.e. $\aleph_\alpha = \aleph_\beta$ if and only if $\alpha = \beta$ and for every cardinal κ there exists α such that $\kappa = \aleph_\alpha$. We also have $\aleph_\alpha < \aleph_\beta$ if and only if $\alpha < \beta$.

As an example of a singular cardinal we can now take $\aleph_\omega$. It is singular because if $A = \{\aleph_\alpha \ : \ \alpha < \omega\}$ then $A \subset \aleph_\omega$ and $|A| = \omega = \aleph_0 < \aleph_\omega$ and $\sup A = \aleph_\omega$.

We have proved that for any cardinal $\aleph_\alpha$, $\aleph_\alpha < 2^{\aleph_\alpha}$ so we must have $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$. For $\alpha = 0$ this gives $\aleph_1 \leq 2^{\aleph_0}$. The *continuum hypothesis*, abbreviated CH, is the statement $\aleph_1 = 2^{\aleph_0}$. The *generalized continuum hypothesis*, abbreviated GCH, is the statement

$$\text{for any ordinal } \alpha, \quad \aleph_{\alpha+1} = 2^{\aleph_\alpha}$$

It was proved by K. Gödel that if ZF is consistent then so is ZFC + GCH. Later it was proved by P. Cohen that if ZFC is consistent then neither CH nor its negation can be proved from ZFC.

The material in this chapter comes mainly from the following three books:

- R. Cori, D. Lascar, *Logique mathématique II*, Masson, 1993.
- T. Jech, *Set Theory*, Academic Press, 1978.
- P. Halmos, *Naive Set Theory*, D. Van Nostrand, 1960.

2 Model theory

2.1 Basics

Let L be a first order language and let V be the vocabulary of L. If $\varphi(x_1, \ldots, x_n)$ denotes an L-formula then we mean that no other variables than those among $x_1, \ldots, x_n$ occur free in that formula. By an *L-theory* we mean a set of L-sentences (where L-sentence means a closed L-formula). If T is an L-theory and φ an L-formula then $T \vdash \varphi$ means that φ is formally provable from T. We say that an L-theory T is *complete* if for every L-sentence φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$. An *L-structure* (or *L-model*) , is an object $\mathfrak{M}$ consisting of a set M , called the *universe* or *domain* of the structure, and for every symbol S in V an *interpretation in M*, denoted $S^{\mathfrak{M}}$, in such a way that:

- If $c \in V$ is a constant symbol then $c^{\mathfrak{M}}$ is an element of M.
- If $R \in V$ is an n-ary relation symbol then $R^{\mathfrak{M}}$ is an n-ary relation on M, i.e. $R^{\mathfrak{M}} \subseteq M^n$.
- If $f \in V$ is an n-ary function symbol then $f^{\mathfrak{M}}$ is a function from M^n into M.

We will always assume that the equality symbol is in the vocabulary of L and that the equality symbol is interpreted as the identity relation on M (i.e. $=^{\mathfrak{M}}$ is $\{(a, a) : a \in M\}$). Usually we will notationally identify the model $\mathfrak{M}$ with its universe M, so when we say that M is an L-structure then we mean that M is a set (the universe of the structure) together with an interpretation in M of every symbol in the vocabulary of L. Often, when the particular properties of the language L does not affect the discussion, we will just say that M is a structure (or model) without specifying the language.

Let φ be an L-sentence and let T be an L-theory and let M be an L-structure. By the notation $M \models \varphi$ we mean that φ is true in M (I assume that the reader knows the definition of a sentence being true in a model). We say that M is a *model of T*, written $M \models T$, if for every $\varphi \in T$, $M \models \varphi$. Let T and Γ be L-theories. We write $T \models \Gamma$ if for every L-structure M , if $M \models T$ then $M \models \Gamma$. If $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ then we will sometimes write $T \models \varphi_1, \ldots, \varphi_n$ instead of $T \models \Gamma$.

Recall the completeness theorem for first order logic.

Theorem 2.1 (Gödel's completeness theorem) *If T is a theory and φ is a sentence (in the same language), then $T \vdash \varphi$ if and only if $T \models \varphi$.*

¿From the completeness theorem it is easy to derive the model existence theorem.

Theorem 2.2 (Model existence) *A theory is consistent if and only if it has a model.*

Proof. Suppose that T has a model M. If T would be inconsistent then there would be a sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$. Then by the completeness theorem we would get $M \models \varphi$ and $M \models \neg\varphi$ which is impossible. Hence T must be consistent. Now suppose that T has no model. Then for any sentence ψ we have $T \models \psi$ and by the completeness theorem it follows that for every sentence ψ , $T \vdash \psi$, which means that T is inconsistent. $\square$

It is also easy to see that the completeness theorem follows from the model existence theorem. In fact, it is usually the case that one first proves the model existence theorem and then derives the completeness theorem from it. A fundamental tool in model theory is the compactness theorem.

Theorem 2.3 (Compactness theorem) *If T is a theory then* :
(i) *T is consistent if and only if every finite subset of T is consistent,*
(ii) *T is inconsistent if and only if some finite subset of T is inconsistent.*

Proof. It is easy to see that (i) and (ii) are equivalent so we will only prove (ii). If some finite subset of T is inconsistent then clearly T is inconsistent. Conversely, suppose that T is inconsistent. Then there exists a sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$. Since proofs are finite there are $\theta_1, \ldots, \theta_n \in T$ and $\sigma_1, \ldots, \sigma_m \in T$ such that $\theta_1, \ldots, \theta_n \vdash \varphi$ and $\sigma_1, \ldots, \sigma_m \vdash \neg\varphi$. Let $\Delta = \{\theta_1, \ldots, \theta_n, \sigma_1, \ldots, \sigma_m\}$. Then $\Delta \subseteq T$ is finite and $\Delta \vdash \varphi$ and $\Delta \vdash \neg\varphi$, so Δ is inconsistent. $\square$

Let $L \subseteq L'$ be two languages (so the vocabulary of L is included in the vocabulary of L'), and let $\mathfrak{M}$ be an L-structure and let $\mathfrak{N}$ be an L'-structure. We say that $\mathfrak{N}$ is an *expansion* of $\mathfrak{M}$ if $\mathfrak{N}$ and $\mathfrak{M}$ have the same universe and for every symbol S in the vocabulary of L , $S^{\mathfrak{M}} = S^{\mathfrak{N}}$. If $\mathfrak{N}$ is an expansion of $\mathfrak{M}$ then we say that $\mathfrak{M}$ is the *reduct* of $\mathfrak{N}$ to L (or the *L-reduct* of $\mathfrak{N}$). Observe that every L'-structure has a unique reduct to L (if $L \subseteq L'$). If $\mathfrak{N}$ is an L'-structure and $L \subseteq L'$ then the L-reduct of $\mathfrak{N}$ will be denoted by $\mathfrak{N}{\restriction}L$.

If $\mathfrak{M}$ is an L-structure with universe M and $a_1, \ldots, a_n \in M$ and $\varphi(x_1, \ldots, x_n) \in L$ then by $M \models \varphi(a_1, \ldots, a_n)$ we mean that if $c_1, \ldots, c_n$ are constant symbols which do not occur in the vocabulary of L and $\mathfrak{N}$ is an expansion of $\mathfrak{M}$ such that $c_i^{\mathfrak{N}} = a_i$, for $1 \leq i \leq n$, then $\mathfrak{N} \models \varphi(c_1, \ldots, c_n)$.

We say that a model is finite if it's universe is finite and infinite if it's universe is infinite. We say that a model $\mathfrak{M}$ has cardinality κ if it's universe has cardinality κ and in that case we write $|\mathfrak{M}| = \kappa$.

Example 2.4 If T is a theory with arbitrarily large finite models then T has an infinite model.

Proof. Assume that T is an L-theory which has arbitrarily large finite models, and let V be the vocabulary of L. Let $C = \{c_i : i < \omega\}$ be a set of constant symbols which are not in V, and let L' be the language with vocabulary $V \cup C$ (so $L \subset L'$). Let $T' = T \cup \{c_i \neq c_j : i, j < \omega, i \neq j\}$. If T' has a model then the L-reduct of this model will be an infinite model of T. We will show that T' has a model by showing that T' is consistent (and using the Model existence theorem). By the compactness theorem it is sufficient to show that every finite subset of T' is consistent. Let $\Delta \subseteq T'$ be finite. Then, for some $n < \omega$, $\Delta \subseteq T \cup \{c_i \neq c_j : i \leq n\}$. Since T has arbitrarily large finite models there exists a model $\mathfrak{M}$ of T such that the universe M of $\mathfrak{M}$ has at least n distinct elements. Let $a_1, \ldots, a_n \in M$ be distinct elements, and let $\mathfrak{N}$ be an expansion of $\mathfrak{M}$ in which $c_i^{\mathfrak{N}} = a_i$ for every $i \leq n$. Then $\mathfrak{N}$ is a model of Δ so Δ is consistent. $\square$

If M is an L-structure then $Th(M)$ is the set of all L-sentences that are true in M. It follows that $Th(M)$ is a complete L-theory. Let M and N be two L-structures. We say that M and N are *elementarily equivalent*, written $M \equiv N$, if $Th(M) = Th(N)$. By definition, an *embedding* from M into N is a function f from M into N such that:
(i) If c is a constant symbol in the vocabulary of L then $f(c^M) = c^N$.
(ii) If R is an n-ary relation symbol in the vocabulary of L then for all $a_1, \ldots, a_n \in M$,

$$(a_1, \ldots, a_n) \in R^M \text{ if and only if } (f(a_1), \ldots, f(a_n)) \in R^N$$

(iii) If g is an n-ary function symbol in the vocabulary of L then for all $a_1, \ldots, a_n \in M$,

$$f(\mathrm{g}^M(a_1, \ldots, a_n)) = \mathrm{g}^N(f(a_1), \ldots, f(a_n))$$

Since we always assume that the equality symbol $=$ is in the vocabulary of L and since we always assume that $=$ is interpreted as identity on the elements in the universe of a model it follows that an embedding is always injective. If an embedding f from M into N is surjective (onto) then we say that F

is an *isomorphism* from M onto N. It follows that if f is an isomorphism from M onto N then the inverse of f is an isomorphism from N onto M. We say that M and N are *isomorphic*, written $M \cong N$, if there exists an isomorphism from M onto N.

Lemma 2.5 *If $M \cong N$ then $M \equiv N$.*

Proof. Suppose that f is an isomorphism from M onto N. By induction on the complexity of formulas we will show that for every $n < \omega$ and all $a_1, \ldots, a_n \in M$ and every formula $\varphi(x_1, \ldots, x_n)$,

$$M \models \varphi(a_1, \ldots, a_n) \Leftrightarrow N \models \varphi(f(a_1), \ldots, f(a_n)). \qquad (*)$$

Then it follows that for every sentence ψ , $M \models \psi \Leftrightarrow N \models \psi$, so $M \equiv N$. We may assume that $\varphi(x_1, \ldots, x_n)$ contains only the connectives $\neg$ and $\wedge$ and the quantifier $\exists$ since $\vee$, $\leftarrow$, $\leftrightarrow$, $\forall$ are definable in terms of these. If $\varphi(x_1, \ldots, x_n)$ is an atomic formula then $(*)$ follows from the definition of isomorphism. If $\varphi(x_1, \ldots, x_n)$ has the form $\neg\psi(x_1, \ldots, x_n)$ then by the induction hypothesis we have

$$M \models \psi(a_1, \ldots, a_n) \Leftrightarrow N \models \psi(f(a_1), \ldots, f(a_n))$$

and then $(*)$ easily follows.

If $\varphi(x_1, \ldots, x_n)$ has the form $\psi(x_1, \ldots, x_n) \wedge \theta(x_1, \ldots, x_n)$ then by the induction hypothesis

$$M \models \psi(a_1, \ldots, a_n) \Leftrightarrow N \models \psi(f(a_1), \ldots, f(a_n))$$

$$\text{and } M \models \theta(a_1, \ldots, a_n) \Leftrightarrow N \models \theta(f(a_1), \ldots, f(a_n))$$

so $(*)$ easily follows.

Suppose that $\varphi(x_1, \ldots, x_n)$ has the form $\exists y \psi(x_1, \ldots, x_n, y)$. Then by the induction hypothesis

$$\text{for every } b \in M,\ M \models \psi(a_1, \ldots, a_n, b) \Leftrightarrow N \models \psi(f(a_1), \ldots, f(a_n), f(b))$$

If there exists $b \in M$ such that $M \models \psi(a_1, \ldots, a_n, b)$ then

$$N \models \psi(f(a_1), \ldots, f(a_n), f(b))$$

and if there exists $c \in N$ such that $N \models \psi(f(a_1), \ldots, f(a_n), c)$ then (since f is onto N) there is $b \in M$ such that $f(b) = c$ and we get

$$M \models \psi(a_1, \ldots, a_n, b).$$

Hence we have $(*)$. $\square$

2.2 Elementary substructures and extensions

Let M and N be L-structures. We say that M is a *substructure* (or *submodel*) of N if $M \subseteq N$ and
(i) if c is a constant symbol in the vocabulary of L then $c^N = c^M$, and
(ii) if R is an n-ary relation symbol in the vocabulary of L then $R^N \cap M^n = R^M$, and
(iii) if f is an n-ary function symbol in the vocabulary of L then for all $a_1, \ldots, a_n \in M$, $\mathrm{f}^N(a_1, \ldots, a_n) = \mathrm{f}^M(a_1, \ldots, a_n)$.
We say that M is an *elementary substructure* (or *elementary submodel*), of N, abbreviated $M \preccurlyeq N$, if M is a substructure of N and the following holds: For every $n < \omega$ and all $a_1, \ldots, a_n \in M$ and every L-formula $\varphi(x_1, \ldots, x_n)$ we have

$$M \models \varphi(a_1, \ldots, a_n) \text{ if and only if } N \models \varphi(a_1, \ldots, a_n)$$

When $n = 0$ we mean that for every sentence φ , $M \models \varphi$ if and only if $N \models \varphi$, so it follows that if $M \preccurlyeq N$ then $M \equiv N$. If $M \preccurlyeq N$ then we say that N is an *elementary extension* of M.

Proposition 2.6 (Tarski-Vaught test) *Let M be a substructure of N. Then M is an elementary substructure if and only if for every $n < \omega$ and every formula $\varphi(x_1, \ldots x_n, y)$ and all $a_1, \ldots, a_n \in M$,*

$$\textit{if } N \models \exists y \varphi(a_1, \ldots a_n, y) \textit{ then there is } b \in M \textit{ such that } N \models \varphi(a_1, \ldots a_n, b)$$

Proof. If $M \preccurlyeq N$ and $N \models \exists y \varphi(a_1, \ldots a_n, y)$, where $a_1, \ldots a_n \in M$ then $M \models \exists y \varphi(a_1, \ldots a_n, y)$ so there is $b \in M$ such that $M \models \varphi(a_1, \ldots a_n, b)$ and (since $M \preccurlyeq N$) we get $N \models \varphi(a_1, \ldots a_n, b)$.

Now suppose that for every $n < \omega$ and every formula $\varphi(x_1, \ldots x_n, y)$ and all $a_1, \ldots, a_n \in M$,

$$\text{if } N \models \exists y \varphi(a_1, \ldots a_n, y) \text{ then there is } b \in M \text{ such that } N \models \varphi(a_1, \ldots a_n, b)$$

We will show by induction on the complexity of formulas that for every $n < \omega$ and every formula $\varphi(x_1, \ldots x_n)$ and all $a_1, \ldots a_n \in M$,

$$M \models \varphi(a_1, \ldots a_n) \Leftrightarrow N \models \varphi(a_1, \ldots a_n) \qquad (*)$$

If $\varphi(x_1, \ldots x_n)$ is an atomic formula then $(*)$ follows from the definition of elementary substructure. If $\varphi(x_1, \ldots x_n)$ has the form $\neg\psi(x_1, \ldots x_n)$ then by the induction hypothesis that for all $a_1, \ldots a_n \in M$,

$$M \models \psi(a_1, \ldots a_n) \Leftrightarrow N \models \psi(a_1, \ldots a_n)$$

so it it is easy to see that ($*$) holds.

If $\varphi(x_1, \dots x_n)$ has the form $\psi(x_1, \dots x_n) \wedge \theta(x_1, \dots x_n)$ then by the induction hypothesis, for all $a_1, \dots a_n \in M$,

$$M \models \psi(a_1, \dots a_n) \Leftrightarrow N \models \psi(a_1, \dots a_n)$$

$$\text{and} \quad M \models \psi(a_1, \dots a_n) \Leftrightarrow N \models \psi(a_1, \dots a_n)$$

so it is easy to see that ($*$) holds.

Suppose that $\varphi(x_1, \dots x_n)$ has the form $\exists y \psi(x_1, \dots x_n, y)$. Then by the induction hypothesis , for all $a_1 \dots a_n, b \in M$,

$$M \models \psi(a_1, \dots a_n, b) \Leftrightarrow N \models \psi(a_1, \dots a_n, b) \qquad (**)$$

If $M \models \exists y \psi(a_1, \dots a_n, y)$ where $a_1 \dots a_n \in M$ then there exists $b \in M$ such that $M \models \psi(a_1, \dots a_n, b)$ so by ($**$) we get $N \models \psi(a_1, \dots a_n, b)$ and therefore $N \models \exists y \psi(a_1, \dots a_n, y)$. If $N \models \exists y \psi(a_1, \dots a_n, y)$ where $a_1 \dots a_n \in M$ then by the assumption there exists $b \in M$ such that $N \models \psi(a_1, \dots a_n, b)$ and by ($**$) we get $M \models \psi(a_1, \dots a_n, b)$, and hence $M \models \exists y \psi(a_1, \dots a_n, y)$. $\square$

Lemma 2.7 *Let N be an L-structure and suppose that $A \subseteq N$. Then there exists a substructure M of N such that $A \subseteq M$ and $|M| \leq \sup\{|A|, |L|\}$.*

Proof Let $\kappa = \sup\{|A|, |L|\}$ and let

$$C = \{\mathrm{c}^N : \text{ c is a constant symbol in the vocabulary of } L\}$$

Note that $\kappa \geq \aleph_0$ because $|L| \geq \aleph_0$. We will inductively define a sequence A_i, $i < \omega$ of subsets of N such that for every $i < \omega$, $A \subseteq A_i \subseteq A_{i+1}$, $|A_i| \leq \kappa$ and for every $n < \omega$ and n-ary function symbol f , if $a_1, \dots a_n \in A_i$, then $\mathrm{f}^N(a_1, \dots a_n) \in A_{i+1}$. Let $A_0 = A \cup C$. Then $|A_0| \leq \kappa$. Suppose that A_j has been defined for every $j \leq i$, $|A_j| \leq \kappa$ for $j \leq i$ and $A \subseteq A_j \subseteq A_{j+1}$ for $j < i$. Let X be the set of all pairs $(\mathrm{f}, \bar{a})$ such that f is a function symbol from the vocabulary of L and $\bar{a}$ is a sequence of elements from A_i such that$|\bar{a}| =$ the arity of f, where $|\bar{a}|$ is the length of $\bar{a}$. Then $|X| \leq \kappa$, and if we let $Y = \{\mathrm{f}^N(\bar{a}) : (\mathrm{f}, \bar{a}) \in X\}$ then $|Y| \leq \kappa$. If we now define $A_{i+1} = A_i \cup Y$ then it is easy to see that $|A_j| \leq \kappa$ for $j \leq i+1$ and $A \subseteq A_j \subseteq A_{j+1}$ for $j < i+1$.

Now let M be the L-structure with universe $M = \bigcup_{i<\omega} A_i$ and in which the symbols of the vocabulary of L are interpreted as follows. For every constant symbol c let $c^M = c^N$, and for every n-ary relation symbol R let $R^M = R^N \cap M^n$ and for every n-ary function symbol f and all $a_1, \dots a_n \in$,let $f^M(a_1, \dots a_n) = f^N(a_1, \dots a_n)$. Then M is well defined because for every constant symbol c, $c^N \in A_0 \subseteq M$ and for every function symbol f and elements $a_1, \dots a_n \in M$ we have $a_1, \dots a_n \in A_i$ for some $i < \omega$ and hence $f^N(a_1, \dots a_n) \in A_{i+1} \subseteq M$. Now it follows from the definition of M that M is a substructure of N and that $A \subseteq M$. Since $\kappa \geq \omega$ and $M = \bigcup_{i<\omega} A_i$ and $|A_i| \leq \kappa$ for all $i < \omega$ it follows that $|M| \leq \kappa$. □

Remark 2.8 In fact, the substructure M which is obtained in the proof of Lemma 2.7 is the smallest substructure which satisfies the same lemma (smallest in the sense that if M' also satisfies Lemma 2.7 then $M \subseteq M'$). We will call this smallest substructure that satifies Lemma 2.7 the substructure *generated* by A, and it will be denoted by $\langle A \rangle$.

Theorem 2.9 (Downward Löwenheim-Skolem theorem) *Let N be an L-structure such that $|N| \geq |L|$ and suppose that $A \subseteq N$. Then there exists an elementary substructure $M \preccurlyeq N$ such that $A \subseteq M$ and $|M| = \sup\{|A|, |L|\}$.*

Proof Let $\kappa = \sup\{|A|, |L|\}$. If $|A| < \kappa$ then let A' be a subset of N such that $A \subseteq A'$ and $|A'| = \kappa$, otherwise let $A' = A$.

We will inductively define a sequence A_i, $i < \omega$ of substructures of N such that $A' \subseteq A_i \subseteq A_{i+1}$ and $|A_i| \leq \kappa$. Let $A_0 = A'$. Now suppose that A_j is defined for every $j \leq i$, $|A_j| \leq \kappa$ for $j \leq i$ and $A' \subseteq A_j \subseteq A_{j+1}$ for every $j < i$. Let X be the set of all pairs

$$(\varphi(x_1, \dots, x_n, y), (a_1, \dots, a_n))$$

where $n < \omega$ and $\varphi(x_1, \dots, x_n, y)$ is a formula and $a_1, \dots, a_n \in A_i$ and

$$N \models \exists y \varphi(a_1, \dots, a_n, y).$$

Then $|X| \leq \kappa$. For every

$$(\varphi(x_1, \dots, x_n, y), (a_1, \dots, a_n)) \in X$$

let $a_{(\varphi,(a_1,\dots,a_n))} \in N$ be an element such that

$$N \models \varphi(a_1, \dots, a_n, a_{(\varphi,(a_1,\dots,a_n))}).$$

Let

$$B = A_i \cup \{a_{(\varphi,(a_1,\dots,a_n))} : (\varphi, (a_1, \dots, a_n)) \in X\}.$$

Then $|B| \leq \kappa$ so by Lemma 2.7 we can define A_{i+1} to be a substructure of N such that $B \subseteq A_{i+1}$ and $|A_{i+1}| \leq \kappa$. Then we have $|A_j| \leq \kappa$, for $j \leq i+1$, and $A' \subseteq A_j \subseteq A_{j+1}$ for $j < i+1$. Let M be the structure with universe $M = \bigcup_{i<\omega} A_i$ and where every symbol in the vocabulary of L is interpreted as it is in N. Then it is easy to see that M is a substructure of N, and we also have $|M| = \kappa$. Now we show that $M \preccurlyeq N$ by applying the Tarski-Vaught test (Proposition 2.6).

Let $\varphi(x_1, \ldots, x_n, y)$ be a formula and suppose that $N \models \exists y \varphi(a_1, \ldots, a_n, y)$ where $a_1, \ldots, a_n \in M$. Then $a_1, \ldots, a_n \in A_i$ for some i so by the definition of A_{i+1} it follows that there exists $b \in A_{i+1} \subseteq M$ such that $N \models \varphi(a_1, \ldots, a_n, b)$. $\square$

Corollary 2.10 *If T is a consistent L-theory then T has a model of cardinality less or equal to $|L|$.*

Proof. If T is consistent then T has model say M. If $|M| > |L|$ then we can use the downward Löwenhwim-Skolem theorem to get (by letting $A = \emptyset$) $N \preccurlyeq M$ with $|N| = \sup\{|\emptyset|, |L|\} = |L|$. $\square$

Corollary 2.11 *If the language L is countable and T is a consistent L-theory then T has a countable model.*

Proof. Follows immediately from corollary 2.10 $\square$

Let M be an L-structure with vocabulary V and let $A \subseteq M$. Then $L(A)$ is the language over the vocabulary $V(A) = V \cup \{\hat{a} : \ a \in A\}$ where for every $a \in A$, $\hat{a}$ is a *new* constant symbol which does not occur in V and $\hat{a} = \hat{b}$ if and only if $a = b$. By (M, A) (or $(M, a)_{a \in A}$) we mean the $L(A)$-structure which is obtained from M by interpreting every $\hat{a}$ as a, i.e. $\hat{a}^{(M,A)} = a$ for every $a \in A$. We will call the theory $Th((M, a)_{a \in M})$ the *elementary diagram* of M, and it will also be denoted by $D(M)$. We say that a function f from an L-structure M into an L-structure N is an *elementary embedding* if for any n, where $1 \leq n < \omega$, and any L-formula $\varphi(x_1, \ldots, x_n)$ and any $a_1, \ldots a_n \in M$ we have

$$M \models \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad N \models \varphi(f(a_1), \ldots, f(a_n)).$$

Observe that every elementary embedding is injective. If there exists an elementary embedding from M into N the we say that M is *elementarily embeddable* in N, abbreviated $M \precsim N$. Observe that if $M \precsim N$ then $M \equiv N$ (because if φ is a sentence and $f : M \to N$ is an elementary embedding then let $\psi(x)$ be $\varphi \wedge x = x$; then for any $a \in M$, $M \models \varphi \;\Leftrightarrow\; M \models \psi(a) \;\Leftrightarrow\; N \models \psi(f(a)) \;\Leftrightarrow\; N \models \varphi$.)

Proposition 2.12 *If M and N are L-structures then the following are equivalent :*
(i) *There exists an elementary embedding from M into N.*
(ii) *M is isomorphic to an elementary substructure of N.*
(iii) *N can be expanded to an $L(M)$-structure which is a model of the elementary diagram of M.*

Proof. (i) $\Rightarrow$ (ii) If $f : M \to N$ is an elementary embedding then f is an isomorphism from M onto $f(M)$ and $f(M) \preccurlyeq N$ (where $f(M)$ denotes the image of M under f).

(ii) $\Rightarrow$ (iii) Suppose that f is an isomorphism from M onto M_0 where $M_0 \preccurlyeq N$. Let $\mathfrak{N}$ be the expansion of N to $L(M)$ where, for every $a \in M$, $\hat{a}^{\mathfrak{N}} = f(a)$. We need to show that $\mathfrak{N} \models D(M)$. Let $\varphi \in D(M)$. Then φ has the form $\psi(\hat{a}_1, \ldots, \hat{a}_n)$ where $\psi(x_1, \ldots, x_n)$ is an L-formula and $a_1, \ldots, a_n \in M$, and $M \models \psi(a_1, \ldots a_n)$. It follows from the proof of Lemma 2.5 that $M_0 \models \psi(f(a_1), \ldots f(a_n))$ and since $M_0 \preccurlyeq N$ we get $N \models \psi(f(a_1), \ldots f(a_n))$, and since $\hat{a}_i^{\mathfrak{N}} = f(a_i)$, for $1 \le i \le n$, we get $\mathfrak{N} \models \psi(\hat{a}_1, \ldots \hat{a}_n)$, so $\mathfrak{N} \models \varphi$.

(iii) $\Rightarrow$ (i) Suppose that $\mathfrak{N}$ is an expansion of N and that $\mathfrak{N} \models D(M)$. Define a function $f : M \to N$ by $f(a) = \hat{a}^{\mathfrak{N}}$. We need to show that for any L-formula $\varphi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_n \in M$ we have

$$M \models \varphi(a_1, \ldots, a_n) \quad \Leftrightarrow \quad N \models \varphi(f(a_1), \ldots, f(a_n))$$

Let $\varphi(x_1, \ldots, x_n)$ be an L-formula and let $a_1, \ldots, a_n \in M$. If $M \models \varphi(a_1, \ldots, a_n)$ then $\varphi(\hat{a}_1, \ldots, \hat{a}_n) \in D(M)$ so $\mathfrak{N} \models \varphi(\hat{a}_1, \ldots, \hat{a}_n)$ and since $f(a_i) = \hat{a}_i^{\mathfrak{N}}$ for $1 \le i \le n$ we get $\mathfrak{N} \models \varphi(f(a_1), \ldots, f(a_n))$ and hence $N \models \varphi(f(a_1), \ldots, f(a_n))$. Suppose that $M \not\models \varphi(a_1, \ldots, a_n)$. Then $M \models \neg\varphi(a_1, \ldots, a_n)$ and in the same way as above we get $N \models \neg\varphi(f(a_1), \ldots, f(a_n))$ and hence we get $N \not\models \varphi(f(a_1), \ldots f(a_n))$ $\square$

Lemma 2.13 *For any structure M there exists a structure N such that $N \cong M$ and $N \cap M = \emptyset$.*

Proof. Let N be a set such that $N \cap M = \emptyset$ and $|N| = |M|$. Let f be an injective function from M onto N. Make N into a structure (in the same language as M) by interpreting symbols in the following way: (i) for every constant symbol c , $c^N = f(c^M)$
(ii) for every n-ary relation symbol R and all $a_1, \ldots a_n \in N$,

$$(a_1, \ldots, a_n) \in R^N \text{ if and only if } (f^{-1}(a_1), \ldots, f^{-1}(a_n)) \in R^M$$

(iii) for every n-ary function symbol h and all $a_1, \dots a_n \in N$,

$$\mathrm{h}^N(a_1, \dots, a_n) = f(\mathrm{h}^M(f^{-1}(a_1), \dots, f^{-1}(a_n)))$$

Then f is an isomorphism from M onto N. $\Box$

Lemma 2.14 *If $M \precsim N$ then there exists $N^* \succcurlyeq M$ such that $N^* \cong N$.*

Proof. Let f be an elementary embedding from M into N. By Lemma 2.13 there exists a structure N_0 such that $N_0 \cap M = \emptyset$ and $N_0 \cong N$. Let f_0 be an isomorphism from N onto N_0. Then $F = f_0 f$ is an elementary embedding from M into N_0, so in particular F is an injective function from M onto $F(M)$. Define a function $g : N_0 \to M \cup (N_0 - F(M))$, by

$$g(a) = \begin{cases} F^{-1}(a) & \text{if } a \in F(M) \\ a & \text{if } a \in N_0 - F(M) \end{cases}$$

Define N^* by $N^* = M \cup (N_0 - F(M))$ and ,
(i) for every constant symbol c , $c^{N^*} = g(c^{N_0})$
(ii) for every n-ary relation symbol R and all $a_1, \dots a_n \in N^*$,

$$(a_1, \dots, a_n) \in R^{N^*} \text{ if and only if } (g^{-1}(a_1), \dots, g^{-1}(a_n)) \in R^{N_0}$$

(iii) for every n-ary function symbol h and all $a_1, \dots a_n \in N^*$,

$$\mathrm{h}^{N^*}(a_1, \dots, a_n) = g(\mathrm{h}^{N_0}(g^{-1}(a_1), \dots, g^{-1}(a_n)))$$

Then $g f_0$ is an isomorphism from N onto N^* and $M \preccurlyeq N^*$. $\Box$

Theorem 2.15 (Upward Löwenheim-Skolem theorem) *Let M be an L-structure with $|M| = \kappa \geq |L|$. Then for every cardinal $\lambda > \kappa$ there exists an elementary extension $N \succcurlyeq M$ such that $|N| = \lambda$.*

Proof. Suppose that M is an L-structure with $|M| = \kappa \geq |L|$. Let $C = \{c_i : i < \lambda\}$ be a set of distinct constant symbols which do not occur in the vocabulary of $L(M)$. Let $T = D(M) \cup \{c_i \neq c_j : i < j < \lambda\}$.

First we will show that if T has a model then there exists N such that $N \succcurlyeq M$ and $|N| = \lambda$. Suppose that N_0 is a model of T. Then $|N_0| \geq \lambda$ and by Proposition 2.12 there exists an elementary embedding from M into $N_0 \upharpoonright L$. By Lemma 2.14 there exists a structure N^* such that $N^* \cong N_0$ and $M \preccurlyeq N^*$. Let A be a subset of N^* such that $M \subseteq A$ and $|A| = \lambda$.

Since $|N^*| = |N_0| \geq \lambda \geq \kappa \geq |L|$, it follows by the downward Löwenheim-Skolem theorem that there exists $N \preccurlyeq N^*$ such that $M \subseteq A \subseteq N$ and $|N| = \sup\{|A|, |L|\}$, and since $|A| \geq |M| = \kappa \geq |L|$ we get $|N| = |A| = \lambda$. It follows from $M \preccurlyeq N^*$ and $M \subseteq N \preccurlyeq N^*$ that $M \preccurlyeq N$.

Now it remains to show that T is consistent. By the model existence theorem and the compactness theorem it is sufficient to show that every finite subset of T has a model. Let Δ be a finite subset of T. Then, for some $n < \omega$,

$$\Delta \subset D(M) \cup \{c_i \neq c_j : i < j \leq n\}$$

and $(M, a)_{a \in M}$ is a model of $D(M)$. Since M is infinite (because $|L| \geq \aleph_0$ and $|M| \geq |L|$) there are $a_1, \ldots, a_n \in M$ such that $a_i \neq a_j$ for all $i < j \leq n$. It follows that if we expand $(M, a)_{a \in M}$ by interpreting c_i as a_i for every $i \leq n$ then the resulting structure is a model of $D(M) \cup \{c_i \neq c_j : i < j \leq n\}$ and hence also a model of Δ. □

Corollary 2.16 *If T is a consistent L-theory which has an infinite model then T has a model of cardinality κ for every $\kappa \geq |L|$. So in particular, if L is countable then T has a model of cardinality κ for every infinite cardinal κ.*

Proof. If T has an infinite model then by a compactness argument it follows that T has a model of cardinality $\geq |L|$, so by the downward Löwenheim-Skolem theorem T has a model M of cardinality $|L|$. By the upward Löwenheim-Skolem theorem, for every cardinal $\lambda \geq |L|$ there exists an elementary extension N of M with $|N| = \lambda$, and N is a model of T. □

Let $(\mathbb{N}, <, +, \cdot, s, 0)$ be the structure with universe $\mathbb{N}$ (the natural numbers) and where the symbols $=, <, +, \cdot, s, 0$ are interpreted as identity, the usual order on $\mathbb{N}$, addition, multiplication and the successor function on $\mathbb{N}$, and 0 is interpreted as the number 0. This structure is called the *standard model of arithmetic*. Any other model of the axioms of arithmetic is called a *nonstandard model of arithmetic*. By the previous corollary there are (many) nonstandard models of arithmetic (at least one in every uncountable cardinality).

Exercise 2.17 Show that there exists a countable nonstandard model of arithmetic.

If κ is a cardinal then we say that a theory T is *κ-categorical* if T has a model of cardinality κ and all models of cardinality κ are isomorphic. (This can be rephrased by saying that T has exactly one model of cardinality κ up to isomorphism.) If a theory is $\aleph_0$-categorical then we also say that it is *countably categorical.*

Theorem 2.18 (Vaught's theorem) *If T is an L-theory which has no finite models and is κ-categorical where $\kappa \geq |L|$, then T is complete.*

Proof. Suppose that T is κ-categorical but not complete where $\kappa \geq |L|$. Then there is a sentence $\varphi \in L$ such that $T \nvDash \varphi$ and $T \nvDash \neg\varphi$. Hence there exists a model M of $T \cup \{\varphi\}$, because if for every model M , $M \models T \Rightarrow M \models \neg\varphi$ then by the completeness theorem we get $T \vdash \neg\varphi$, which contradicts our assumption. By a similar argument there exists a model N of $T \cup \{\neg\varphi\}$. Since we assume that T has no finite models M and N are infinite, and by Corollary 2.16 we may assume that $|M| \geq |L|$ and $|N| \geq |L|$. If $|M| \geq \kappa$ then by the downward Löwenheim-Skolem theorem there exists $M_0 \preccurlyeq M$ with $|M_0| = \kappa$, and if $|M| < \kappa$ then by the upward Löwenheim-Skolem theorem there exists $M_0 \succcurlyeq M$ with $|M_0| = \kappa$. In both cases there exists $M_0 \equiv M$ with $|M_0| = \kappa$. By a similar argument there exists $N_0 \equiv N$ with $|N_0| = \kappa$. But then, since $M_0 \models T$ and $N_0 \models T$ and T is κ-categorical it follows that $M_0 \cong N_0$ which implies $M_0 \equiv N_0$. Then we have $M \equiv M_0 \equiv N_0 \equiv N$, so $M \equiv N$ which contradicts that $M \models \varphi$ and $N \models \neg\varphi$. □

Lemma 2.19 *Let T be an L-theory and let $c_1, \dots, c_n$ be a sequence of distinct constants which do not occur in L and let $\varphi(x_1, \dots, x_n)$ be a formula in L. If $T \models \varphi(c_1, \dots, c_n)$ then*

$$T \models \forall x_1, \dots x_n \varphi(x_1, \dots, x_n).$$

Proof. Suppose that

$$T \models \varphi(c_1, \dots, c_n). \qquad (*)$$

Let M be any L-structure which is a model of T. Let $a_1, \dots, a_n$ be any elements from M and let $\mathfrak{M}$ be the expansion of M which is obtained by interpreting c_i as a_i for all $1 \leq i \leq n$. Then $\mathfrak{M} \models T$ so by $(*)$ we get $\mathfrak{M} \models \varphi(c_1, \dots, c_n)$, and therefore $M \models \varphi(a_1, \dots, a_n)$. Since $a_1, \dots, a_n$ where arbitrary we get

$$M \models \forall x_1, \dots x_n \varphi(x_1, \dots, x_n)$$

and since M was an arbitrary model of T we get

$$T \models \forall x_1, \dots x_n \varphi(x_1, \dots, x_n).$$

□

Theorem 2.20 *If M_1 and M_2 are two L-structures and $M_1 \equiv M_2$ then there exists an L-structure N such that $M_1 \preccurlyeq N$ and $M_2 \precsim N$.*

Proof. Let $M_2' \cong M_2$ be such that $M_2' \cap M_1 = \emptyset$. Let $\Gamma = D(M_1) \cup D(M_2')$. If we can show that Γ has a model N' then by Proposition 2.12 we will have $M_1 \precsim N' \restriction L$ and $M_2' \precsim N' \restriction L$. By Lemma 2.14 there exists $N \cong N' \restriction L$ such that $M_1 \preccurlyeq N$. From $M_2 \cong M_2' \precsim N' \restriction L \cong N$ it follows that $M_2 \precsim N$. Hence it is sufficient to show that Γ is consistent.

Let $\Delta \subseteq \Gamma$ be finite, and Then $\Delta = \Delta_1 \cup \Delta_2$ where $\Delta_1 \subset D(M_1)$ and $\Delta_2 \subset D(M_2')$. The conjuction of all formulas in Δ_1 has the form $\varphi(\hat{a}_1, \ldots, \hat{a}_n)$ for some $\varphi(x_1, \ldots, x_n) \in L$ and some distinct elements a_1, $\ldots$, $a_n \in M_1$. It is sufficient to prove that

$$\Delta_2 \cup \{\varphi(\hat{a}_1, \ldots, \hat{a}_n)\}$$

is consistent. Suppose on the contrary that it is not consistent. Then (by the model existence theorem) every model of Δ_2 must be a model of $\neg\varphi(\hat{a}_1, \ldots, \hat{a}_n)$, so

$$\Delta_2 \models \neg\varphi(\hat{a}_1, \ldots, \hat{a}_n)$$

Since $M_2' \cap M_1 = \emptyset$ it follows that $\hat{a}_1, \ldots, \hat{a}_n$ do not occur in $L_{M_2'}$ and since Δ_2 is an $L_{M_2'}$-theory it follows from Lemma 2.19 that

$$\Delta_2 \models \forall x_1, \ldots, x_n \neg\varphi(x_1, \ldots, x_n)$$

Then, since $(M_2', b)_{b \in M_2'} \models D(M_2') \supseteq \Delta_2$ it follows that

$$(M_2', b)_{b \in M_2'} \models \forall x_1, \ldots, x_n \neg\varphi(x_1, \ldots, x_n)$$

and hence

$$M_2' \models \forall x_1, \ldots, x_n \neg\varphi(x_1, \ldots, x_n)$$

Since $(M_1, a)_{a \in M_1} \models D(M_1) \ni \varphi(\hat{a}_1, \ldots, \hat{a}_n)$ it follows that

$$M_1 \models \exists x_1, \ldots, x_n \varphi(x_1, \ldots, x_n)$$

but this contradicts that $M_1 \equiv M_2'$. Hence Δ must be consistent. □

2.3 Decidability

In this section suppose that the language L is countable, so in particlular the vocabulary V of L is countable and all variables can be listed as $x_i, i < \omega$. Let

$$W = V \cup \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \exists, \forall, (,)\} \cup \{x_i : i < \omega\}$$

and let W^* be the set of all finite sequences of symbols from W. There exists an injective function $\delta : W^* \to \omega$ (i.e. a coding, also called Gdel numbering) such that
(1) if $w_1 \in W^*$ is a subsequence of $w_2 \in W^*$ then $\delta(w_1) \leq \delta(w_2)$,
(2) $\delta(L)$ is a recursive set, and
(3) there are recursive functions $\mu_0(x,y), \mu_1(x,y,z), \mu_2(x,y,z), \mu_3(x,y,z)$ which, for any L-formulas φ, ψ and L-term t, satisfy:
$\mu_0(\delta(\neg), \delta(\varphi)) = \delta(\neg\varphi)$,
$\mu_1(\delta(\varphi), \delta(S), \delta(\psi)) = \delta(\varphi S \psi)$ if $S \in \{\wedge, \vee, \to, \leftrightarrow\}$,
$\mu_2(\delta(S), \delta(x_i), \delta(\varphi)) = \delta(S x_i \varphi)$ if $S \in \{\exists, \forall\}$,
$\mu_3(\delta(t), \delta(x_i), \delta(\varphi)) = \delta(\varphi[t/x_i])$, where $\varphi[t/x_i]$ is the result of substituting t for every free occurence of x_i in φ.

¿From (1), (2) and the above given property of μ_3 it follows that the set $\{\delta(\varphi) : \varphi \in L \text{ is a sentence}\}$ is recursive. We say that an L-theory T *is recursive* if the set $\{\delta(\varphi) : \varphi \in T\}$ is recursive. We say that an L-theory T *is decidable* if the set $\{\delta(\varphi) : \varphi \text{ is an } L\text{-sentence and } T \models \varphi\}$ is recursive.

Lemma 2.21 *If T is a recursive L-theory then $\{\delta(\varphi) : \varphi \in L$ and $T \vdash \varphi\}$ and
$\{\delta(\varphi) : \varphi \in L$ and $T \vdash \neg\varphi\}$ are recursively enumerable.*

Proof. By elementary computability theory there is an injective function $\lambda : \bigcup_{k<\omega} \omega^k \to \omega$ and recursive functions $\pi : \omega^2 \to \omega$, $a : \omega \to \omega$ such that the image of λ is a recursive set and if $n_0, \ldots, n_{k-1} < \omega$, $i < k < \omega$ then $a(\lambda((n_0, \ldots, n_{k-1})) = k$ and $\pi(i, \lambda((n_0, \ldots, n_{k-1})))$
$= n_i$. Now we describe an algorithm which given n halts if and only if $n = \delta(\varphi)$ for some sentence $\varphi \in L$ such that $T \vdash \varphi$; then it follows that

$$\{\delta(\varphi) : \ \varphi \in L \text{ is a sentence and } T \vdash \varphi\}$$

is recursively enumerable.

Algorithm: Let n be given. Set $j := 0$.
$(*)$ Check whether $j \in \mathrm{im}(\lambda)$. If $j \notin \mathrm{im}(\lambda)$ then set $j := j+1$ and go back to $(*)$. Otherwise set $k := a(j)$ and check if it is the case that for all $l < k$, $\pi(l,j) \in \delta(L)$. If it is not the case then set $j := j+1$ and go back to $(*)$. Otherwise check if it is the case that $n = \delta(\varphi)$ for some sentence $\varphi \in L$ and $\delta^{-1}(\pi(0,j)), \ldots, \delta^{-1}(\pi(k-1,j))$ is a proof of φ from T. To check the later part it is enough to check that $\pi(k-1,j) = n$ and that, for $l < k$, $\delta^{-1}(\pi(l,j))$ belongs to T or follows from some $\delta^{-1}(\pi(l',j))$ with $l' < l$ by a logical rule; here we use the assumption that T is recursive and the

properties of the recursive functions μ_i. If the answer is positive then halt; otherwise set $j := j + 1$ an go back to $(*)$.

In a similar way one shows that $\{\delta(\varphi) : \varphi \in L \text{ is a sentence and } T \vdash \neg\varphi\}$ is recursively enumerable. $\square$

Theorem 2.22 *If T is a recursive complete L-theory then T is decidable.*

Proof. Recall that by the completeness theorem $T \models \varphi$ if and only if $T \vdash \varphi$. If T is inconsistent then $T \vdash \varphi$ for every sentence φ so the theorem follows from the fact (following from (1), (2) and the properties of the recursive functions μ_i) that $\{\delta(\varphi) : \varphi \in L \text{ is a sentence}\}$ is recursive. Suppose that T is consistent. Then $\Phi = \{\delta(\varphi) : \varphi \in L \text{ is a sentence and } T \vdash \varphi\}$ and $\Psi = \{\delta(\varphi) : \varphi \in L \text{ is a sentence and } T \vdash \neg\varphi\}$ are disjoint. Since T is complete $\Phi \cup \Psi = \{\delta(\varphi) : \varphi \in L \text{ is a sentence}\}$. By Lemma 2.21 Φ and Ψ are recursively enumerable. Since $\{\delta(\varphi) : \varphi \in L \text{ is a sentence}\}$ is recursive it follows that Φ and Ψ are recursive which proves the theorem. $\square$

2.4 Axiomatisability

We say that a class $\mathcal{C}$ of L-structures is *axiomatised* by an L-theory T if for any L-structure M, $M \models T$ if and only if $M \in \mathcal{C}$. If there exists a theory T such that $\mathcal{C}$ is axiomatised by T then we say that $\mathcal{C}$ is *axiomatisable.* If $\mathcal{C}$ is axiomatisable by a finite theory then we say that $\mathcal{C}$ is *finitely axiomatisable.*

If T and T' are L-theories then we say that T is *axiomatized* by T' if for any L-structure M, $M \models T$ if and only if $M \models T'$. A theory T is said to be *finitely axiomatisable* if it is axiomatized by a finite theory. It follows from the definitions that if T is a theory and $\mathcal{C}$ is the class of models of T then $\mathcal{C}$ is finitely axiomatisable if and only if T is finitely axiomatisable.

Lemma 2.23 *Suppose $\mathcal{C}$ is a class of L-structures which is axiomatised by the L-theory T and that $\mathcal{C}$ is finitely axiomatisable. Then $\mathcal{C}$ is axiomatised by a finite subset of T.*

Proof. Suppose that $\mathcal{C}$ is axiomatised by the theory T and by the finite theory Δ. Let φ be the conjunction of all sentences in Δ. Then $M \models T \Leftrightarrow M \in \mathcal{C} \Leftrightarrow M \models \varphi$ for any L-structure M, so we get $T \models \varphi$ and $\varphi \models \psi$ for every $\psi \in T$. By the completeness theorem we get $T \vdash \varphi$, so (since proofs are finite) there are $\theta_1, \ldots, \theta_n \in T$ such that $\{\theta_1, \ldots, \theta_n\} \vdash \varphi$. By the completeness theorem we get $\{\theta_1, \ldots, \theta_n\} \models \varphi$ and since $\varphi \models \psi$ for every $\psi \in T$, we also have $\varphi \models \{\theta_1, \ldots, \theta_n\}$. Now we have $M \in \mathcal{C} \Leftrightarrow M \models \varphi \Leftrightarrow M \models \{\theta_1, \ldots, \theta_n\}$, so T is axiomatised by $\{\theta_1, \ldots, \theta_n\}$. $\square$

Theorem 2.24 *A class $\mathcal{C}$ of L-structures is finitely axiomatisable if and only if both $\mathcal{C}$ and the complement of $\mathcal{C}$ are axiomatisable (where the complement of $\mathcal{C}$ is the class of all L-structures which are not in $\mathcal{C}$.)*

Proof. First suppose that $\mathcal{C}$ is axiomatised by the finite L-theory $\{\theta_1, \ldots, \theta_n\}$. Then it is easy to see that the complement of $\mathcal{C}$ is axiomatised by $\{\neg(\theta_1 \wedge \ldots \wedge \theta_n)\}$.

Conversely, suppose that $\mathcal{C}$ is axiomatised by T_1 and that the complement is axiomatised by T_2. If $T_1 \cup T_2$ would be consistent then there would be a model M of $T_1 \cup T_2$ and this would imply that M belongs to $\mathcal{C}$ because $M \models T_1$ and that M belong to the complement of $\mathcal{C}$ because $M \models T_2$, but this is impossible so $T_1 \cup T_2$ is inconsistent. By the compactness theorem it follows that a finite subset Δ of $T_1 \cup T_2$ is inconsistent. Then Δ has the form

$$\Delta = \{\varphi_1, \ldots, \varphi_n\} \cup \{\psi_1, \ldots, \psi_m\}$$

where $\{\varphi_1, \ldots, \varphi_n\} \subseteq T_1$ and $\{\psi_1, \ldots, \psi_m\} \subseteq T_2$. Then we must have

$$\{\varphi_1, \ldots, \varphi_n\} \models \neg\psi_1 \vee \ldots \vee \neg\psi_m$$

Hence we get $M \models \{\varphi_1, \ldots, \varphi_n\} \;\Rightarrow\; M \models \neg\psi_1 \vee \ldots \vee \neg\psi_m \;\Rightarrow\; M$ is *not* in the complement of $\mathcal{C} \;\Rightarrow\; M \in \mathcal{C}$, and we get $M \in \mathcal{C} \;\Rightarrow\; M \models T_1 \;\Rightarrow\; M \models \{\varphi_1, \ldots, \varphi_n\}$. Therefore $\mathcal{C}$ is axiomatised by $\{\varphi_1, \ldots, \varphi_n\}$. $\square$

Example 2.25 Let V_F be the vocabulary $\{=, +, \cdot, 0, 1\}$ (where $0, 1$ are constant symbols and $+, \cdot$ are binary function symbols) and let L_F be the language over V_F. The usual axioms of fields can be expressed by a finite number of L_F-sentences. Hence the class $\mathcal{F}$ of of fields is finitely axiomatisable, by a finite L_F-theory, say T_F. The class $\mathcal{F}_0$ of fields of characteristic 0 is axiomatized by T_F and the following sentences

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} \neq 0$$

for every $n > 0$. so $\mathcal{F}_0$ is axiomatisable. Let $n \cdot 1 \neq 0$ be an abbreviation of the above given sentence. If $\mathcal{F}_0$ would be finitely axiomatisable then by Lemma 2.23 $\mathcal{F}_0$ would be axiomatisable by

$$T = T_F \cup \{n_i \cdot 1 \neq 0 : \; 0 < i < m\}$$

for some $m < \omega$. But if p is a prime that is bigger than all n_i for $0 < i < m$, then the $\mathbb{Z}/(p)$ is a model of T and $\mathbb{Z}/(p)$ does not have characteristic 0, a contradiction. Hence $\mathcal{F}_0$ is not finitely axiomatisable. If the class of fields of positive characteristic, call it $\mathcal{F}_+$, would be axiomatizable then it is easy to see (because $\mathcal{F}$ is finitely axiomatizable) that the complement of $\mathcal{F}_0$ would be axiomatizable and then by Theorem 2.24 it would follow that $\mathcal{F}_0$ is finitely axiomatizable, a contradiction. Hence $\mathcal{F}_+$ is not axiomatizable.

Example 2.26 Let $\mathcal{ACF}$ be the class of algebraically closed fields. $\mathcal{ACF}$ is axiomatised by the L_F-theory $ACF = T_F \cup \{\varphi_n : \ 0 < n < \omega\}$ where

$$\varphi_n = \forall y_0 \dots y_n \exists x (y_0 + y_1 \cdot x + y_2 \cdot x^2 + \dots + y_n \cdot x^n = 0)$$

and x^n is an abbreviation for

$$\underbrace{x \cdot \ldots \cdot x}_{n \text{ times}}$$

If $\mathcal{ACF}$ would be finitely axiomatisable then by Lemma 2.23 $\mathcal{ACF}$ would be axiomatised by a finite subset of ACF, but for every finite $\Delta \subset ACF$ one can find a model M of Δ such that some polynomial over M has no roots in M, so $M \notin \mathcal{ACF}$. Hence $\mathcal{ACF}$ is not finitely axiomatisable.

Let $\mathcal{ACF}_0$ be the class of algebraically closed fields of characteristic 0. $\mathcal{ACF}_0$ is axiomatised by

$$ACF_0 = ACF \cup \{n \cdot 1 \neq 0 : \ 0 < n < \omega\}.$$

The field $\mathbb{Q}$ can be embedded into any field of characteristic 0. A transcendence basis in a field is a maximal set A such that for any finite tuple $(a_1, \dots, a_n)$of elements from A there does not exist a nonzero polynomial over $\mathbb{Q}$ in n variables such that $a_1, \dots, a_n$ is a root of this polynomial. Classical results from field theory are that,
(i) every algebraically closed field of characteristic 0 has a transcendence basis, and that
(ii) any two transcendence bases have the same cardinality, called the transcendence rank, and that
(iii) for every cardinal κ there exists a unique algebraically closed field of characteristic 0 (up to isomorphism) of transcendence rank κ and the cardinality of this field is $\aleph_0 + \kappa$.
It follows that if $M \models ACF_0$ and $N \models ACF_0$ and $|M| = |N| = \kappa > \aleph_0$ then M and N have the same transcendence rank which is κ, and by (iii) M and N are isomorphic. Hence ACF_0 is κ-categorical for every uncountable cardinal κ. By Vaught's theorem (theorem 2.18) it follows that ACF_0 is a complete theory.

Exercise 2.27 Let $V_=$ be $\{=\}$ and let $L_=$ be the language over $V_=$. Show that the class of all infinite $L_=$-structures is axiomatisable but not finitely axiomatisable.

2.5 Interpolation and definability theorems

Theorem 2.28 (Craig interpolation theorem) *Let φ and ψ be sentences of some first order language such that $\varphi \models \psi$. Then there is a sentence θ (called interpolant) such that $\varphi \models \theta$ and $\theta \models \psi$ and every constant, relation or function symbol which occurs in θ occurs in both φ and ψ.*

Proof. Suppose that φ and ψ are sentences such that $\varphi \models \psi$. Let V_1 be the set of all constant, relation and function symbols which occur in φ and let V_2 be the set of all constant, relation and function symbols which occur in ψ. Let $V_0 = V_1 \cap V_2$ and $V_3 = V_1 \cup V_2$. For $i = 0, 1, 2, 3$ let L_i be the language over V_i (i.e. the set of first order formulas over V_i).

We want to show that there exists a sentence $\theta \in L_0$ such that $\varphi \models \theta$ and $\theta \models \psi$. Suppose that that no such interpolant θ exists. We will derive a contradiction by showing that then $\varphi \wedge \neg\psi$ has a model.

Let C be an infinite and countable set of constant symbols which do not occur in V_3 and, for $i = 0, 1, 2, 3$, let L'_i be the language over $V_i \cup C$. Observe that $|L'_i| \leq \omega$. Suppose that T is an L'_1-theory and that Γ is an L'_2-theory. We say that a sentence $\theta \in L'_0$ *separates* T and Γ if $T \models \theta$ and $\Gamma \models \neg\theta$. We say that T and Γ are *inseparable* if no sentence $\theta \in L'_0$ separates them.

Claim 1. $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable.

Suppose for a contradiction that a sentence $\theta \in L'_0$ separates $\{\varphi\}$ and $\{\neg\psi\}$. We may assume that θ has the form $\theta'(c_1, \ldots, c_n)$ where $\theta'(x_1, \ldots, x_n) \in L_0$ and $c_1, \ldots, c_n) \in C$. Let θ'' be the sentence $\forall x_1, \ldots, x_n \theta'(x_1, \ldots, x_n)$. Since $\varphi \models \theta'(c_1, \ldots, c_n)$ and $\neg\psi \models \neg\theta'(c_1, \ldots, c_n)$ we get (by Lemma 2.19) $\varphi \models \theta''$ and $\theta'' \models \psi$, which contradicts the assumption that no interpolant exists.

Let φ_i, $i < \omega$ be an enumeration of all sentences in L'_1 and let ψ_i, $i < \omega$ be an enumeration of all sentences in L'_2. For $i < \omega$ we can inductively construct finite theories $T_i \subseteq L'_1$ and $\Gamma_i \subseteq L'_2$ such that, for all $i < \omega$:

(1) $\{\varphi\} \subseteq T_i \subseteq T_{i+1}$ and $\{\neg\psi\} \subseteq \Gamma_i \subseteq \Gamma_{i+1}$.

(2) T_i and Γ_i are inseparable.

(3) If $T_i \cup \{\varphi_i\}$ and Γ_i are inseparable then $\varphi_i \in T_{i+1}$, and if T_{i+1} and $\Gamma_i \cup \{\psi_i\}$ are inseparable then $\psi_i \in \Gamma_{i+1}$.

(4) If φ_i has the form $\exists x \sigma(x)$ and $\varphi_i \in T_{i+1}$ then $\sigma(c) \in T_{i+1}$ for some $c \in C$, and
if ψ_i has the form $\exists x \sigma(x)$ and $\psi_i \in \Gamma_{i+1}$ then $\sigma(d) \in \Gamma_{i+1}$ for some $d \in C$.

It is left to the reader to do carry out the construction (start by letting $T_0 = \{\varphi\}$ and $\Gamma_0 = \{\neg\psi\}$; by Claim 1 T_0 and Γ_0 are insepareble). It should be clear how to take care of cases (3) and (4). By choosing c and d (in (4)) such that c and d do not occur in T_i, Γ_i, φ_i or ψ_i the inseparability is preserved.

Let $T = \bigcup_{i<\omega} T_i$ and $\Gamma = \bigcup_{i<\omega} \Gamma_i$. Then, by the Compactness theorem, it follows that T and Γ are inseparable, and therefore T and Γ are consistent.

Claim 2. For every sentence $\sigma \in L'_1$, either $\sigma \in T$ or $\neg\sigma \in T$, and for every sentence $\sigma \in L'_2$, either $\sigma \in \Gamma$ or $\neg\sigma \in \Gamma$.

Suppose for a contradiction that $\sigma \in L'_1$ is a sentence such that $\sigma \notin T$ and $\neg\sigma \notin T$. Then, for some $i < \omega$, $\sigma = \varphi_i$. It follows that $T_i \cup \{\varphi_i\}$ and Γ_i are not inseparable (because if they where inseparable then $\varphi_i \in T$) so there exists $\theta \in L'_0$ such that

$$T \models \varphi_i \to \theta \quad \text{and} \quad \Gamma \models \neg\theta.$$

By a similar argument there is $\theta' \in L'_0$ such that

$$T \models \neg\varphi_i \to \theta' \quad \text{and} \quad \Gamma \models \neg\theta'.$$

Then it follows that $T \models \theta \vee \theta'$ and $\Gamma \models \neg(\theta \vee \theta')$ which contradicts that T and Γ are inseparable. In a similar way we get a contradiction from the assumption there exists $\sigma \in L'_2$ such that $\sigma \notin \Gamma$ and $\neg\sigma \notin \Gamma$.

Claim 3. For every sentence $\sigma \in L'_0$, either $\sigma \in T \cap \Gamma$ or $\neg\sigma \in T \cap \Gamma$.

Let $\sigma \in L'_0$ be a sentence. By Claim 2, $\sigma \in T$ or $\neg\sigma \in T$, and $\sigma \in \Gamma$ or $\neg\sigma \in \Gamma$. Since T and Γ are inseparable we can not have $T \models \sigma$ and $\Gamma \models \neg\sigma$, or vice versa. Hence, either $\sigma \in T \cap \Gamma$ or $\neg\sigma \in T \cap \Gamma$.

Let $M_1 \models T$ (where M_1 is an L'_1-structure). Observe that for any constant symbol $e \in V_1$, any n-ary function symbol f and any $c_1, \ldots, c_n \in C$, $M_1 \models \exists x\, \mathrm{f}(c_1, \ldots, c_n) = x$ and $M_1 \models \exists x(e = x)$, so by Claim 2 $\exists x\, \mathrm{f}(c_1, \ldots, c_n) = x \in T$ and $\exists x(e = x) \in T$. This together with (4) implies that we can define a substructure N_1 of M_1 by letting $N_1 = \{c^{M_1} : c \in C\}$, $e^{N_1} = e^{M_1}$ for every constant symbol $e \in V_1 \cup C$, $R^{N_1} = R^{M_1} \cap N_1^n$ for every n-ary relation symbol $R \in V_1$ and $\mathrm{f}^{N_1}(a_1, \ldots a_n) = \mathrm{f}^{M_1}(a_1, \ldots a_n)$ for any n-ary function symbol $\mathrm{f} \in V_1$ and any $a_1, \ldots a_n \in N_1$.

If $\exists y \varphi(x_1, \ldots, x_n, y) \in L_1'$ and $M_1 \models \exists y \varphi(c_1, \ldots, c_n, y)$ where $c_1, \ldots c_n \in C$ then, by Claim 2, $\exists y \varphi(c_1, \ldots, c_n, y) \in T$ and, by (4), there exists $c \in C$ such that $\varphi(c_1, \ldots, c_n, c) \in T$ from which it follows that $M_1 \models \varphi(c_1, \ldots, c_n, c)$. Therefore, by the Tarski-Vaught test (Proposition 2.6), it follows that $N_1 \preccurlyeq M_1$, so in particular $N_1 \models T$.

Let $M_2 \models \Gamma$. In the same way as above we can show that there exists $N_2 \preccurlyeq M_2$ with $N_2 = \{c^{M_2} : c \in C\}$. In particular $N_2 \models \Gamma$. We now have $N_1 \restriction L_0' \models T \cap \Gamma$ and $N_2 \restriction L_0' \models T \cap \Gamma$. For every $\varphi(x_1, \ldots, x_n) \in L_0'$ and $c_1, \ldots c_n \in C$, if $N_1 \restriction L_0' \models \varphi(c_1, \ldots, c_n)$ then, by Claim 3, $\varphi(c_1, \ldots, c_n) \in T \cap \Gamma$ so $N_2 \restriction L_0' \models \varphi(c_1, \ldots, c_n)$, and vice versa. Therefore $N_1 \restriction L_0' \cong N_2 \restriction L_0'$, where an isomorphism is given by $c^{N_1} \mapsto c^{N_2}$ (remember that $c^{N_1} = c^{M_1}$ and $c^{N_2} = c^{M_2}$). Then there is an expansion $\mathfrak{N}_2$ of $N_2 \restriction L_2$ to the language $L_3 = L_1 \cup L_2$, such that $\mathfrak{N}_2 \models T$. But $N_2 \restriction L_2 \models \Gamma$ and hence $\mathfrak{N}_2 \models \Gamma$, and since $\varphi \in T$ and $\neg\psi \in \Gamma$ we get $\mathfrak{N}_2 \models \varphi \wedge \neg\psi$, the contradiction we are looking for. $\square$

Remark 2.29 By the completeness theorem we can replace $\models$ by $\vdash$ in the Craig interpolation theorem. Then the statement of the theorem is proof theoretical, and in fact, there is also an entirely proof theoretic proof of this theorem.

Let L be the language over a vocabulary V and suppose that P and P' be two n-ary relation symbols which are not in V. Let $L(P)$ and $L(P')$ be the languages over the vocabularies $V \cup \{P\}$ and $V \cup \{P'\}$, respectively. If T is an $L(P)$-theory then let T' be the theory obtained from T by replacing every occurence of P in T by P'. We say that T *defines* P *implicitly* if

$$T \cup T' \models \forall x_1, \ldots, x_n [P(x_1, \ldots, x_n) \leftrightarrow P'(x_1, \ldots, x_n)].$$

We say that T *defines* P *explicitly* if there exists a formula $\varphi(x_1, \ldots, x_n) \in L$ such that

$$T \models \forall x_1, \ldots, x_n [P(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)].$$

Theorem 2.30 (Beth's definability theorem) *T defines P implicitly if and only if T defines P explicitly.*

Proof. It is easy to see that if T defines P explicitly then T defines P implicitly, so we prove only the other direction. Suppose that T defines P implicitly. Let $c_1, \ldots, c_n$ be constant symbols which are not in V (the vocabulary of L). Then

$$T \cup T' \models P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n).$$

By the compactness theorem, there are finite subsets $\Delta \subseteq T$ and $\Delta' \subseteq T'$ such that

$$\Delta \cup \Delta' \models P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n).$$

For any $\theta \in L(P)$ let $\theta[P/P']$ denote the formula obtained by replacing every occurence of P in θ by P'. Let ψ be the conjunction of all $\theta \in T$ such that $\theta \in \Delta$ or $\theta[P/P'] \in \Delta'$. Then

$$\psi \wedge \psi[P/P'] \models P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n)$$

which implies

$$\psi \wedge P(c_1, \ldots, c_n) \models \psi[P/P'] \to P'(c_1, \ldots, c_n).$$

By the Craig interpolation theorem (Theorem 2.28) there is a formula

$$\varphi(x_1, \ldots, x_n) \in L$$

such that

$$\psi \wedge P(c_1, \ldots, c_n) \models \varphi(c_1, \ldots, c_n) \tag{1}$$

and

$$\varphi(c_1, \ldots, c_n) \models \psi[P/P'] \to P'(c_1, \ldots, c_n). \tag{2}$$

Clearly, (2) implies

$$\varphi(c_1, \ldots, c_n) \models \psi \to P(c_1, \ldots, c_n), \tag{3}$$

and now (1) and (3) gives

$$\psi \models P(c_1, \ldots, c_n) \leftrightarrow \varphi(c_1, \ldots, c_n).$$

Since $c_1, \ldots, c_n$ do not occur in ψ, it follows (from Lemma 2.19) that

$$\psi \models \forall x_1, \ldots, x_n[P(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)],$$

and since $\psi \in T$ it follows that T defines P explicitly. $\square$

Theorem 2.31 (Robinson's consistency theorem)
Let L_1 and L_2 be two languages and let $L = L_1 \cap L_2$. If T is a complete L-theory and $T_1 \subseteq L_1$ and $T_2 \subseteq L_2$ are consistent theories such that $T \subseteq T_1 \cap T_2$, then $T_1 \cup T_2$ is consistent.

Proof. Suppose for a contradiction that $T_1 \cup T_2$ is inconsistent. Then there are finite subsets Δ_1 and Δ_2 of T_1 and T_2, respectively, such that $\Delta_1 \cup \Delta_2$ is inconsistent. Let φ_1 be the conjuction of all formulas in Δ_1 and let φ_2 be the conjunction of all formulas in Δ_2. Then $\varphi_1 \models \neg\varphi_2$ and, by the Craig interpolation theorem, there exists a sentence $\theta \in L$ such that $\varphi_1 \models \theta$ and $\theta \models \neg\varphi_2$. Then $T_1 \models \theta$ and $T_2 \models \neg\theta$ and since T_1 and T_2 are consistent we get $T_1 \not\models \neg\theta$ and $T_2 \not\models \theta$. This implies that $T \not\models \neg\theta$ and $T \not\models \theta$, which contradicts the assumption that T is a complete L-theory. $\square$

2.6 Back and forth equivalence

We define the *quantifier rank* of formulas, abbreviated qr( ), in the following way:
(i) $\mathrm{qr}(\varphi) = 0$ if φ is atomic.
(ii) $\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$ and $\mathrm{qr}(\varphi \diamond \psi) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}$ if $\diamond$ is one of $\vee, \wedge, \rightarrow, \leftrightarrow$.
(iii) $\mathrm{qr}(\exists x\varphi) = \mathrm{qr}(\varphi) + 1$ and $\mathrm{qr}(\forall x\varphi) = \mathrm{qr}(\varphi) + 1$.
Now we introduce some notation. Finite sequences (of variables, or elements of some structure) will be denoted by $\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b}, \bar{c}$ etc. and the length of a sequence $\bar{a}$ is denoted by $|\bar{a}|$ (so if $\bar{a} = a_1, \ldots, a_n$ then $|\bar{a}| = n$). We will also consider the empty sequence, denoted () which contains no elements at all, and we have $|()| = 0$. If $\bar{a} = a_1, \ldots, a_n$ then $\bar{a}a$ denotes the sequence $\bar{a} = a_1, \ldots, a_n, a$, and if $k \leq n$ then $\bar{a}{\restriction}k$ denotes the sequence $a_1, \ldots, a_k$; if $k = 0$ then $\bar{a}{\restriction}k$ denotes (). If A is a set, then by $\bar{a} \in A$ we mean that every element in the sequence $\bar{a}$ is in A, so in particular this convention implies that $() \in A$ for any set A. Let $\bar{a} \in M$ (where M is a structure) and let $\bar{x}$ be a sequence of distinct variables. If $\bar{a} = a_1, \ldots, a_n$ and $\bar{x} = x_1, \ldots, x_n$ and $\varphi(x_1, \ldots, x_n)$ is a formula then $\varphi(x_1, \ldots, x_n)$ will also be denoted by $\varphi(\bar{x})$, and by $\varphi(\bar{a})$ we mean $\varphi(a_1, \ldots, a_n)$. In case we write $\varphi(\bar{x})$ or $\varphi(\bar{a})$ and $\bar{x} = ()$ and $\bar{a} = ()$ then we mean that φ is a sentence (i.e. has no free variables) in the language we are considering.

Let M and N be two L-structures. We say that two sequences $\bar{a} \in M$ and $\bar{b} \in N$ such that $|\bar{a}| = |\bar{b}|$ are *r-equivalent* if for every L-formula $\varphi(\bar{x})$ (where $\bar{x}$ is a sequence of distinct variables such that $|\bar{x}| = |\bar{a}| = |\bar{b}|$) with

$\mathrm{qr}(\varphi) \leq r$, we have

$$M \models \varphi(\bar{a}) \quad \text{if and only if} \quad N \models \varphi(\bar{b})$$

Note that it follows that () is r-equivalent to itself (with respect to M and N) if for every L-sentence φ with $\mathrm{qr}(\varphi) \leq r$, $M \models \varphi$ if and only if $N \models \varphi$.
A *back and forth system* for M and N is a set I of pairs $(\bar{a}, \bar{b})$ of sequences $\bar{a} \in M$, $\bar{b} \in N$ such that $|\bar{a}| = |\bar{b}|$ and :
(i) I is nonempty.
(ii) For all $(\bar{a}, \bar{b}) \in I$, $\bar{a}$ and $\bar{b}$ are 0-equivalent.
(iii) For all $(\bar{a}, \bar{b}) \in I$ and every $a \in M$ and every $b \in N$ there are $c \in M$ and $d \in N$ such that

$$(\bar{a}a, \bar{b}d), (\bar{a}c, \bar{b}b) \in I.$$

If there exists a back and forth system for M and N then we say that M and N are *back and forth equivalent*, denoted $M \sim N$.

Theorem 2.32 *If M and N are infinite countable structures (in the same language) and $M \sim N$ then $M \cong N$.*

Proof. Suppose that M and N are countable and let I be a back and forth system for M and N. Pick $(\bar{a}, \bar{b}) \in I$ and let $n = |\bar{a}|$ $(= |\bar{b}|)$. Let $\{a_k : n \leq k < \omega\} = M - \{a : a \text{ occurs in the sequence } \bar{a}\}$ and let $\{b_k : n \leq k < \omega\} = N - \{b : b \text{ occurs in the sequence } \bar{b}\}$, where we may assume that these enumerations contain no repetitions. We will define two sequences $(a_i^*)_{i<\omega}$ and $(b_i^*)_{i<\omega}$ of elements in M and N respectively, such that for all $i \geq |\bar{a}|$

$$(a_0^*, \ldots, a_i^*, b_0^*, \ldots, b_i^*) \in I.$$

If $\bar{a} = \bar{b} = ()$ then define $a_0^* = a_0$ and define $b_0^* = b_k$ where k is the least number such that $(a_0, b_k) \in I$. If $|\bar{a}| = |\bar{b}| = n > 0$ then there are $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ such that $\bar{a} = a_0, \ldots, a_{n-1}$ and $\bar{b} = b_0, \ldots, b_{n-1}$. Define $a_i^* = a_i$ and $b_i^* = b_i$ for $0 \leq i \leq n-1$. Suppose that a_j^* and b_j^* are defined for all $j \leq i$ and that i is even. Let k be the least number such that $a_k \neq a_j^*$ for all $j \leq i$ and let $d \in N$ be such that

$$(a_0^*, \ldots, a_i^* a_k, b_0^*, \ldots, b_i^*, d) \in I.$$

Define $a_{i+1}^* = a_k$ and define $b_{i+1}^* = d$. Now suppose that a_j^* and b_j^* are defined for all $j \leq i$ and that i is odd. Let k be the least number such that $b_k \neq b_j^*$ for all $j \leq i$ and let $c \in M$ be such that

$$(a_0^*, \ldots, a_i^* c, b_0^*, \ldots, b_i^*, b_k) \in I.$$

Define $a^*_{i+1} = c$ and $b^*_{i+1} = b_k$. Observe that for every $a \in M$ there is i such that $a = a^*_i$ and for every $b \in N$ there is i such that $b = b^*_i$. Moreover, for every i the sequences $a^*_1, \ldots, a^*_i$ and $b^*_1, \ldots, b^*_i$ are 0-equivalent. Therefore the function $f : M \to N$ defined by $f(a^*_i) = b^*_i$ is an isomorphism from M onto N. $\Box$

Example 2.33 Let T be the theory (in the language with vocabulary $\{=, <\}$) of dense linear order without endpoints, i.e T consists of the following sentences:

$$\forall x(x \not< x)$$
$$\forall x, y(x < y \vee x = y \vee y < x)$$
$$\forall x, y, z(x < y \wedge y < z \to x < z)$$
$$\forall x \exists y, z(y < x \wedge x < z)$$
$$\forall x, y[x < y \to \exists z(x < z \wedge z < y)].$$

Then T is countably categorical and hence complete (by Vaught's theorem 2.18) and decidable (by Theorem 2.22).
Proof. Let M and N be two countable models of T. By Theorem 2.32 it is enough to show that there exists a back and forth system for M and N. Let I be the set of all pairs $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ such that $a_1, \ldots, a_n \in M$ and $b_1, \ldots, b_n \in N$ are 0-equivalent. We will show that I is a back and forth system for M and N. Clearly, for any $a \in M$ and and any $b \in N$, a and b are 0-equivalent, so I is nonempty. Let $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in I$ and let $a \in M$. If $a = a_i$ for some i then

$$a_1, \ldots, a_n, a \text{ and } b_1, \ldots, b_n, b_i$$

are 0-equivalent and hence $(a_1, \ldots, a_n, a, b_1, \ldots, b_n, b_i) \in I$. If $a < a_i$ for all i then since N is a linear order without endpoints there exists $b \in N$ such that $b < b_i$ for all i, and then

$$a_1, \ldots, a_n, a \text{ and } b_1, \ldots, b_n, b$$

are 0-equivalent and hence $(a_1, \ldots, a_n, a, b_1, \ldots, b_n, b) \in I$. We can argue similarly if $a > a_i$ for all i. If none of the above cases hold then there is a permutation ρ of $\{1, \ldots, n\}$ such that for some $1 \leq k < n$

$$a_{\rho(1)} \leq \ldots \leq a_{\rho(k)} < a < a_{\rho(k+1)} \leq \ldots \leq a_{\rho(n)}$$

Since $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are 0-equivalent we must have

$$b_{\rho(1)} \leq \ldots \leq b_{\rho(k)} < b_{\rho(k+1)} \leq \ldots \leq b_{\rho(n)}$$

and since N is a dense linear order there exists $b \in N$ such that

$$b_{\rho(1)} \leq \ldots \leq b_{\rho(k)} < b < b_{\rho(k+1)} \leq \ldots \leq b_{\rho(n)}$$

and then

$$a_1, \ldots, a_n, a \text{ and } b_1, \ldots, b_n, b$$

are 0-equivalent and hence $(a_1, \ldots, a_n, a, b_1, \ldots, b_n, b) \in I$. If

$$(a_1, \ldots, a_n, b_1, \ldots, b_n) \in I \quad \text{and} \quad b \in N,$$

then in the same way we can show that there exists $a \in M$ such that

$$(a_1, \ldots, a_n, a, b_1, \ldots, b_n, b) \in I.$$

□

Let M and N be two L-structures. An *r-back and forth system* for M and N is a sequence $(I_0, \ldots I_r)$ where, for every $0 \leq s \leq r$, I_s is a set of pairs $(\bar{a}, \bar{b})$ of sequences $\bar{a} \in M$, $\bar{b} \in N$ of the same length, such that,
(i) I_r is nonempty,
(ii) for every $0 \leq s \leq r$ and every $(\bar{a}, \bar{b}) \in I_s$, $\bar{a}$ and $\bar{b}$ are 0-equivalent, and
(iii) for every $0 < s \leq r$, $(\bar{a}, \bar{b}) \in I_s$, $a \in M$ and $b \in N$ there are $c \in M$ and $d \in N$ such that

$$(\bar{a}a, \bar{b}d), (\bar{a}c, \bar{b}b) \in I_{s-1}.$$

If there exists an r-back and forth system for M and N then we say that M and N are *r-back and forth equivalent*, denoted $M \sim_r N$. Observe that if I is a back and forth system for M and N then for any r

$$(\underbrace{I, \ldots, I}_{r+1 \text{ times}})$$

is an r-back and forth system for M and N. Hence $M \sim N$ implies $M \sim_r N$ for every $r < \omega$.

Proposition 2.34 *For any $r < \omega$, if $(I_0, \ldots, I_r)$ in an r-back and forth system for M and N then for every $(\bar{a}, \bar{b}) \in I_r$, $\bar{a}$ and $\bar{b}$ are r-equivalent.*

Proof. Since every formula is equivalent to a formula with the same quantifier rank in which $\forall$ does not occur, it is sufficient to prove the proposition for formulas in which $\forall$ does not occur. We will do this by induction on

r. If $r = 0$ then the proposition follows directly from the definition of r-back and forth system. Now suppose that the proposition is true for r and suppose that $(I_0, \ldots, I_{r+1})$ is an $(r+1)$-back and forth system for M and N. We will show by induction on the complexity of formulas that for every $(\bar{a}, \bar{b}) \in I_{r+1}$ and every formula $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq r+1$ and $|\bar{x}| = |\bar{a}|$ where $\bar{x}$ is a sequence of distinct variables (which need not necessarily occur in φ)

$$M \models \varphi(\bar{a}) \quad \Leftrightarrow \quad N \models \varphi(\bar{b}). \qquad (*)$$

If $\varphi(\bar{x})$ is quantifier free then $(*)$ follows from the assumption that $(I_0, \ldots, I_{r+1})$ is an $(r+1)$-back and forth system. If $\varphi(\bar{x})$ has the form $\neg\psi(\bar{x})$ then by the induction hypothesis we have

$$M \models \psi(\bar{a}) \quad \Leftrightarrow \quad N \models \psi(\bar{b})$$

and from this $(*)$ follows. If $\varphi(\bar{x})$ has the form $\psi(\bar{x}) \diamond \theta(\bar{x})$ where $\diamond$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$ then, by using the induction hypothesis it is also easy to see that $(*)$ holds.

Now suppose that $\varphi(\bar{x})$ has the form $\exists y \psi(\bar{x}, y)$. If $M \models \exists y \psi(\bar{a}, y)$ then $M \models \psi(\bar{a}, a)$ for some $b \in M$ and, since $(I_0, \ldots, I_{r+1})$ is an $(r+1)$-back and forth system, there exists $b \in N$ such that $(\bar{a}, a, \bar{b}, b) \in I_r$, and since $(I_0, \ldots, I_r)$ is an r-back and forth system and $\mathrm{qr}(\psi) \leq r$, the induction hypothesis on r gives $N \models \psi(\bar{b}, b)$, and hence $N \models \exists y \psi(\bar{b}, y)$. In the same way one shows that if $N \models \exists y \psi(\bar{b}, y)$ then $M \models \exists y \psi(\bar{a}, y)$, so it follows that $(*)$ holds. This completes the induction step. $\square$

Let M and N be L-structures. We say that M and N are *r-elementarily equivalent* (or *elementarily equivalent up to r*) if for all L-sentences φ with $\mathrm{qr}(\varphi) \leq r$, $M \models \varphi$ if and only if $N \models \varphi$.

Theorem 2.35 (i) *If $M \sim_r N$ then $M \equiv_r N$.*
(ii) *If $M \sim N$ then $M \equiv N$.*

Proof. (i) Suppose that $M \sim_r N$. Then there exists an r-back and forth system $(I_0, \ldots, I_r)$ for M and N. Then I_r is nonempty so pick $(\bar{a}, \bar{b}) \in I_r$. By Proposition 2.34 we have

$$M \models \varphi(\bar{a}) \quad \Leftrightarrow \quad N \models \varphi(\bar{b})$$

for every formula $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq r$, so in particular $M \models \varphi \Leftrightarrow N \models \varphi$ for every sentence φ with $\mathrm{qr}(\varphi) \leq r$.
(ii) $M \sim N \Rightarrow M \sim_r N$ for every $r < \omega \Rightarrow M \equiv_r N$ for every $r < \omega \Rightarrow$

$M \equiv N$. $\square$

We say that two formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are *equivalent* if

$$\models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

If Φ is a set of formulas then we say that Φ is finite up to equivalence if there exists a finite subset Δ of Φ such that every $\varphi \in \Phi$ is equivalent to some $\psi \in \Delta$.

Let Φ be a set of formulas. The set of *boolean combinations* of Φ , denoted $B(\Phi)$, is defined inductively by:
(i) if $\theta \in \Phi$ then $\theta \in B(\Phi)$, and
(ii) if $\theta, \sigma \in \Psi$ then $\neg\theta \in B(\Phi)$ and if $\diamond$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$ then $\theta \diamond \sigma \in B(\Phi)$.

Lemma 2.36 *If Φ is a finite set of formulas then $B(\Phi)$ is finite up to equivalence.*

Proof. Exercise. $\square$

Lemma 2.37 *If the vocabulary of L is finite and contains no function symbols then for every $n \geq 1$ and $r \geq 0$, if $\bar{x}$ is a sequence of length n of distinct variables then there are only finitely many L-formulas $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq r$, up to equivalence.*

Proof. Since every formula is equivalent to a formula of the same quantifier rank in which the quantifier $\forall$ does not occur it is sufficient to prove the lemma only for formulas in which $\forall$ does not occur. We do this by induction on r.

Since the vocabulary is finite and contains no function symbols it follows that, for any $\bar{x}$, there are only finitely many atomic formulas in which only variables from $\bar{x}$ occur. By Lemma 2.36 there are, up to equivalence, only finitely many quantifier free formulas in which only variables from $\bar{x}$ occur. Hence the case $r = 0$ is proved.

Now for the induction step, suppose that for any $\bar{x}$ there are only finitely many formulas $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq r$, up to equivalence. Fix an arbitrary $\bar{x}$. We must show that there are only finitely many formulas $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq r + 1$, up to equivalence. By the induction hypothesis there are

$$\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})$$

$$\text{and } \psi_1(\bar{x}, x), \ldots, \psi_k(\bar{x}, x)$$

(where x does not occur in $\bar{x}$) such that every $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq r$ is equivalent to $\varphi_i(\bar{x})$ for some $1 \leq i \leq m$ and every $\psi(\bar{x}, x)$ with $\mathrm{qr}(\psi) \leq r$ is equivalent to $\psi_i(\bar{x}, x)$ for some $1 \leq i \leq k$. Let

$$\Psi = \{\exists x \psi_1(\bar{x}, x), \ldots, \exists x \psi_k(\bar{x}, x)\}$$

and let

$$\Delta = \{\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})\} \cup B(\Psi)$$

Then every formula in Δ has quantifier rank $\leq r + 1$, and by Lemma 2.36 Δ is finite up to equivalence. Hence it is sufficient to show that any $\chi(\bar{x})$ with $\mathrm{qr}(\chi) \leq r + 1$ is equivalent to a formula in Δ. If $\mathrm{qr}(\chi) \leq r$ then by the induction hypothesis $\chi(\bar{x})$ is equivalent to one of the $\varphi_i(\bar{x})$, so assume that $\mathrm{qr}(\chi) = r + 1$. If $\chi(\bar{x})$ has the form $\neg\chi_1(\bar{x})$ or $\chi_1(\bar{x}) \diamond \chi_2(\bar{x})$ where $\diamond$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$ then it is sufficient to show that, for $j = 1, 2$, χ_j is equivalent to a formula in Δ. Hence we may assume that $\chi(\bar{x})$ has the form $\exists y \theta(\bar{x}, y)$ where y does not occur in $\bar{x}$ (because $\bar{x}$ are free in χ). Then (since $\bar{x}, x$ are distinct) $\exists y \theta(\bar{x}, y)$ is equivalent to $\exists x \theta(\bar{x}, x)$, and by the induction hypothesis, $\theta(\bar{x}, x)$ is equivalent to $\psi_i(\bar{x}, x)$ for some $1 \leq i \leq k$. It follows that $\exists y \theta(\bar{x}, y)$ is equivalent to $\exists x \psi_i(\bar{x}, x)$ and $\exists x \psi_i(\bar{x}, x) \in \Delta$. □

Theorem 2.38 *Suppose that the vocabulary of L is finite and contains no function symbols. For every $r < \omega$, and any L-structures M and N , if $M \equiv_r N$ then $M \sim_r N$.*

Proof. Let $r < \omega$ be arbitrary and suppose that $M \equiv_r N$. For every $0 \leq s \leq r$, let

$$I_s = \{(\bar{a}, \bar{b}) : \ \bar{a} \in M \text{ and } \bar{b} \in N \text{ are } s\text{-equivalent}\}$$

We will show that $(I_0, \ldots, I_r)$ is an r-back and forth system for M and N.

Since $()$ is r-equivalent to $()$ (because $M \equiv_r N$) it follows that $((), ()) \in I_r$ so I_r is nonempty. By the definition of $(I_0, \ldots, I_r)$ it follows that for every $0 \leq s \leq r$ and every $(\bar{a}, \bar{b}) \in I_s$, $\bar{a}$ and $\bar{b}$ are 0-equivalent.

Now suppose that $(\bar{a}, \bar{b}) \in I_s$ where $0 < s \leq r$ and that $a \in M$ and $b \in N$. By Lemma 2.37 there is a finite set Γ of formulas with quantifier rank at most $s - 1$, such that any formula $\theta(\bar{x}, y)$ with $\mathrm{qr}(\theta) \leq s - 1$ is equivalent to a formula in Γ. Let $\varphi(\bar{x}, y)$ be the conjunction of all formulas $\psi(\bar{x}, y) \in \Gamma$ such that $M \models \psi(\bar{a}, a)$. Then $M \models \exists y \varphi(\bar{a}, y)$, and since $\bar{a}$ and $\bar{b}$ are s-equivalent it follows that $N \models \exists y \varphi(\bar{b}, y)$, and hence there exists $d \in N$ such that $N \models \varphi(\bar{b}, d)$. By the choice of $\varphi(\bar{x}, y)$ it follows that $\bar{a}, a$ and $\bar{b}, d$ are $s - 1$-equivalent so $(\bar{a}, a, \bar{b}, d) \in I_{s-1}$. In the same way one can show that there exists $c \in M$ such that $(\bar{a}, c, \bar{b}, b) \in I_r$. □

Corollary 2.39 (Fraïssé's theorem) *Suppose that the vocabulary of L is finite and contains no function symbols.*
(i) *For every $r < \omega$ and any L-structures M and N, $M \equiv_r N$ if and only if $M \sim_r N$.*
(ii) *For any L-structures M and N , $M \equiv N$ if and only if $M \sim_r N$, for every $r < \omega$.*

Proof. (i) follows from theorem 2.38 and Theorem 2.35. (ii) follows from (i). □

2.7 Ehrenfeucht-Fraïssé games

For any two L-structures M and N and any natural number r, we will define the *Ehrenfeucht-Fraïssé game of length r* , denoted $EF_r(M,N)$. $EF_r(M,N)$ is played by two players, called Spoiler and Duplicator, in the following way: If $r = 0$ then neither Spoiler nor Duplicator has to do anything and Duplicator wins the game if () is 0-equivalent to (), otherwise Spoiler wins the game. The pair ((), ()) is called a *play* of $EF_0(M,N)$ (this will make sense later). If $r > 0$ then each player makes r moves. Spoiler always makes his i:th move (for $1 \leq i \leq r$) first and then Duplicator makes his i:th move. Each move consists of choosing an element from one of the structures M and N. If in his i:th move Spoiler chooses an element a_i from M then Duplicator, in his i:th move, must choose an element b_i from N. If in his i:th move Spoiler chooses an element b_i from N then Duplicator, in his i:th move, must choose an element a_i from M. When both players have made their r moves we will have two sequences $a_1, \ldots, a_r \in M$ and $b_1, \ldots, b_r \in N$, such that for every $1 \leq i \leq r$, either a_i is Spoiler's choice in his i:th move and b_i is Duplicators choice in his i:th move, or b_i is Spoiler's choice in his i:th move and a_i is Duplicators choice in his i:th move. The pair $(a_1, \ldots, a_r, b_1, \ldots, b_r)$ is called a *play* of $EF_r(M,N)$, and for every $0 \leq i \leq r$ we say that $(a_1, \ldots, a_i, b_1, \ldots, b_i)$ is a *subplay* of $EF_r(M,N)$. Oberve that since both players are allowed to choose elements that where chosen in earlier moves, we can have $a_i = a_j$ or $b_i = b_j$ for $i \neq j$. If $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are 0-equivalent then Duplicator wins the game, otherwise Spoiler wins the game.

A *strategy* for $EF_r(M,N)$ is a function

$$f : \{(\bar{a}, \bar{b}) : \ \bar{a} \in M, \ \bar{b} \in N, \ |\bar{a}| = |\bar{b}| < r\} \times (M \cup N) \rightarrow M \cup N$$

such that we always have $f((\bar{a}, \bar{b}), c) \in N$ if $c \in M$ and $f((\bar{a}, \bar{b}), d) \in M$ if $d \in N$; we may assume that M and N are disjoint. We say that Duplicator *uses* the strategy f when he plays, if in his i:th move he chooses

$$f((a_1,\dots,a_{i-1},b_1,\dots,b_{i-1}),c) \text{ if Spoiler chose } c\in M \text{ in his } i\text{:th move}$$

$$\text{or } f((a_1,\dots,a_{i-1},b_1,\dots,b_{i-1}),d) \text{ if Spoiler chose } d\in N \text{ in his } i\text{:th move,}$$

where $a_1,\dots,a_{i-1}\in M$ and $b_1,\dots,b_{i-1}\in N$ are the elements chosen in the first $i-1$ moves.

If there exists a strategy f such that Duplicator always wins the game $EF_r(M,N)$ when he uses the strategy f then we say that Duplicator *has a winning strategy for* $EF_r(M,N)$.

Theorem 2.40 *For every $r<\omega$, $M\sim_r N$ if and only if Duplicator has a winning strategy for $EF_r(M,N)$.*

Proof. Suppose that Duplicator has a winning strategy, say f, for the game $EF_r(M,N)$. Let P be the set of all plays of $EF_r(M,N)$ where Duplicator uses f, so in particular all plays in P are won by Duplicator. For every $0\le i\le r$ let $I_{r-i}=\{(\bar a\restriction i,\bar b\restriction i):\ (\bar a,\bar b)\in P\}$ We will show that $(I_0,\dots,I_r)$ is an r-back and forth system for M and N.

I_r is nonempty because $((),())\in I_r$. For every $0\le i\le r$, and every $(\bar c,\bar d)\in I_i$, we have $\bar c=\bar a\restriction i$ and $\bar d=\bar b\restriction i$, for some $(\bar a,\bar b)\in P$. Then $(\bar a,\bar b)$ is a play which is won by duplicator so $\bar a$ and $\bar b$ are 0-equivalent, and therefore also $\bar c$ and $\bar d$ are 0-equivalent. Now suppose that $0<i\le r$, $(\bar a,\bar b)\in I_i$, $c\in M$ and $d\in N$. Then $(\bar a,\bar b)$ is a subplay of $EF_r(M,N)$ where Duplicator uses f and therefore also

$$(\bar a c,\bar b\, f((\bar a,\bar b),c)) \ \text{ and } \ (\bar a\, f((\bar a,\bar b),d),\bar b d)$$

are subplays of $EF_r(M,N)$ where Duplicator uses f and hence

$$(\bar a c,\bar b\, f((\bar a,\bar b),c))\in I_{i-1} \ \text{ and } \ (\bar a\, f((\bar a,\bar b),d),\bar b d)\in I_{i-1}.$$

Now suppose that $(I_0,\dots,I_r)$ is an r-back and forth system. For any $c\in M$, $d\in N$ and $\bar a\in M$, $\bar b\in N$ of length $<r$, such that for some $0<i\le r$, $\bar a'\in M$, $\bar b'\in N$ we have $(\bar a'\bar a,\bar b'\bar b)\in I_i$, there exists $d'\in N$ and $c'\in M$ such that

$$(\bar a'\bar a c,\bar b'\bar b d')\in I_{i-1} \quad (1)$$

$$\text{and} \quad (\bar a'\bar a c',\bar b'\bar b d)\in I_{i-1} \quad (2)$$

so we define $f((\bar a,\bar b),c)=d'$ where $d'\in N$ is an element such that (1) is satisfied and we define $f((\bar a,\bar b),d)=c'$ where $c'\in M$ is an element such that (2) is satisfied. If there are no $0 < i \leq r$, $\bar{a}' \in M$ and $\bar{b}' \in N$ such that $(\bar{a}'\bar{a}, \bar{b}'\bar{b}) \in I_i$ then let $f((\bar{a}, \bar{b}), c)$ be an arbitrary element of N and $f((\bar{a}, \bar{b}), d)$ an arbitrary element of M.

The definition of f implies that if Duplicator uses the strategy f then, if $0 \leq i \leq r$ and $(\bar{a}, \bar{b})$ is the subplay obtained after i moves, then there are $\bar{a}' \in M$ and $\bar{b}' \in N$ such that $(\bar{a}'\bar{a}, \bar{b}'\bar{b}) \in I_{r-i}$ which means that $\bar{a}$ and $\bar{b}$ are 0-equivalent. By taking $i = r$ it follows that Duplicator wins if he uses the strategy f. □

Corollary 2.41 *If the vocabulary of L is finite and contains no function symbols then for any L-structures M and N and any $r < \omega$ the following are equivalent* :
(i) $M \equiv_r N$.
(ii) $M \sim_r N$.
(iii) *Duplicator has a winning strategy for* $EF_r(M, N)$.

Proof. Follows from Theorem 2.40 and Theorem 2.39. □

Corollary 2.42 *If the vocabulary of L is finite and contains no function symbols then for any L-structures M and N and any $r < \omega$ the following are equivalent* :
(i) $M \equiv N$.
(ii) $M \sim_r N$ *for every* $r < \omega$.
(iii) *Duplicator has a winning strategy for* $EF_r(M, N)$*, for every* $r < \omega$.

Proof. Follows from corollary 2.41. □

The material of this booklet can be found in the following books:

• R. Cori, D. Lascar, *Logique mathématique II*, Masson, 1993.
• C.C. Chang, H.J. Kiesler, *Model Theory* (3rd edition), North-Holland, 1990.
• D. van Dalen, *Logic and Structure* (3rd edition), Springer, 1994.
• H.D. Ebbinghaus, J. Flum, W. Thomas, *Mathematical Logic*, Springer, 1984.
• W. Hodges, *Model Theory*, Cambridge Univ. Press, 1993.