UPPSALA UNIVERSITET
Matematiska institutionen
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PROV I MATEMATIK
Logik II / Logic II

In order to pass the course it is necessary to supply a correct proof, explanation or statement (according to the nature of the question) in every assignment that follows. In addition to this exam one also has to solve assignments, as explained on the home page of the course, and this will determine the grade, according to the rule explained on the home page.

1. (a) Give the definition of a theory being *consistent/inconsistent.*
(b) State the *model existence theorem.*
(c) Use the model existence theorem to prove the completeness theorem ($T \models \varphi \Longrightarrow T \vdash \varphi$). You may assume that, for every formula φ, if $T \cup \{\varphi\}$ is inconsistent then $T \vdash \neg\varphi$, and that if $T \vdash \neg\neg\varphi$ then $T \vdash \varphi$.

2. (a) Give an informal argument which explains the following fact: *If T is a recursive theory (that is, $\{\sharp\varphi : \varphi \in T\}$ is recursive), then the set $\{\sharp\varphi : T \vdash \varphi\}$ is recursively enumerable, where $\sharp\varphi$ is the code/Gödel number of φ.*
(b) Give an informal argument which explains the fact that: *If T is a recursive and complete theory, then the set*

$$\{\sharp\varphi : \ \varphi \textit{ is a sentence and } T \vdash \varphi\}$$

is recursive (in other words, T is decidable).
(c) A result (sometimes called Church's theorem) states that: *If T is a consistent theory such that $\mathcal{P} \subseteq T$, then T is undecidable.* Show that the (Gödel-Rosser) incompleteness theorem follows from this result.

3. (a) Give three different characterizations (including the definition) of a *recursively enumerable set.*
(b) By using the normal form theorem one can show that the relation $P(x, y) \iff \{x\}(y)\downarrow$ is recursively enumerable. Use this to show that the relation

$$R(x, y, z) \iff \{x\}(z)\downarrow \ \wedge \ \{y\}(z)\downarrow$$

is recursively enumerable.
(c) A consequence of the "s-m-n theorem" is that for every partial recursive function $f(x, y, z)$ there is a (total) recursive function $g(x, y)$ such that for all z, $\{g(x, y)\}(z) \simeq f(x, y, z)$. Use this (and previous parts) to prove that there is a recursive function $g(x, y)$ such that

$$\{g(x, y)\}(z)\downarrow \iff \{x\}(z)\downarrow \ \wedge \ \{y\}(z)\downarrow .$$

(d) Give an example of a set which can be proved to be *non*recursive by Rice's theorem, and motivate your answer.

4. (a) Define the following three notions: *ordinal, cardinal, cardinality.*
(b) State two conditions which are equivalent to the axiom of choice; one of them is usually called "Zorn's lemma" and the other speaks about well orderings.
(c) Show that every set has a unique cardinality. You may use any of the conditions from the previous part, the Cantor-Bernstein theorem, and the result that every well ordering is isomorphic to a unique ordinal.

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5. (a) State the *compactness theorem* and use the completeness and/or soundness theorems to prove it.
(b) Prove the following: *If $\mathfrak{M} \models T$ and M is infinite, then for every cardinal κ, T has a model of cardinality at least κ.*
(c) Prove that if T is a *complete* theory with an infinite model, then T has *only* infinite models.
(d) Define the notion *elementary substructure/extension.*
(e) Define the notion *κ-categoricity*, for cardinals κ.
(f) Prove Vaught's theorem in the case of a countable language: *Suppose that L is a countable language and that T is an L-theory. If T has no finite model and T is κ-categorical for some infinite κ, then T is complete.* You may use the upward and downward Löwenheim-Skolem theorems as well as the soundness-, completeness- and compactness theorems.

Good luck!