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Övningar/exempel på formella bevis, med kvantorer (lösningar på slutet)

Ge formella bevis för följande giltiga/valida slutledningar.

$$\begin{array}{l} a) \quad \left| \begin{array}{l} \forall x (P(x) \rightarrow Q(x)) \\ P(c) \end{array} \right. \\ \hline \quad \quad \quad \exists x Q(x) \end{array}$$

*-märkta upptäckas nog som svårare.

$$\begin{array}{l} b) \quad \left| \begin{array}{l} \forall x \forall y (A(x, y) \vee B(x, y)) \\ \neg A(c, d) \end{array} \right. \\ \hline \quad \quad \quad \exists x \exists y B(x, y) \end{array}$$

$$\begin{array}{l} c) \quad \left| \begin{array}{l} \forall x (P(x) \rightarrow Q(x)) \\ \forall x P(x) \end{array} \right. \\ \hline \quad \quad \quad \forall x Q(x) \end{array}$$

$$\begin{array}{l} d) \quad \left| \begin{array}{l} \forall x (\neg Q(x) \rightarrow \neg P(x)) \\ \forall x P(x) \end{array} \right. \\ \hline \quad \quad \quad \forall x Q(x) \end{array}$$

$$e) \begin{array}{|l} \forall x (P(x) \rightarrow R(x)) \\ \forall x (P(x) \vee Q(x)) \\ \hline \forall x (Q(x) \vee R(x)) \end{array}$$

$$f) \begin{array}{|l} \forall x (P(x) \vee Q(x)) \\ \hline \forall x (\neg P(x) \rightarrow Q(x)) \end{array}$$

$$g) \begin{array}{|l} \forall x \forall y A(x, y) \\ \hline \forall y \forall x A(y, x) \end{array}$$

$$h) \begin{array}{|l} \forall x \forall y A(x, y) \\ \hline \forall y \forall x A(x, y) \end{array}$$

$$i) \begin{array}{|l} \forall x \forall y ((P(x) \wedge Q(y)) \rightarrow R(x, y)) \\ \forall x \neg R(x, x) \\ \hline \forall x \neg (P(x) \wedge Q(x)) \end{array}$$

$$j) \begin{array}{|l} \forall x \forall y (R(x, y) \rightarrow P(x)) \\ \forall x (P(x) \leftrightarrow Q(x)) \\ \forall x \neg Q(x) \\ \hline \forall x \forall y \neg R(x, y) \end{array}$$

$$k)^* \left| \begin{array}{l} \neg \forall x A(x) \\ \hline \end{array} \right. \neg \exists x \neg A(x)$$

$$l)^* \left| \begin{array}{l} \exists x \neg A(x) \\ \hline \end{array} \right. \neg \forall x A(x)$$

$$m)^* \left| \begin{array}{l} \neg \forall x \neg A(x) \\ \hline \end{array} \right. \exists x A(x)$$

$$n) \left| \begin{array}{l} \forall x (A(x) \rightarrow B(x)) \\ \hline \exists x A(x) \\ \hline \end{array} \right. \exists x B(x)$$

$$o) \left| \begin{array}{l} \forall x A(x) \\ \hline \exists x (A(x) \rightarrow B(x)) \\ \hline \end{array} \right. \exists x B(x)$$

$$p) \left| \begin{array}{l} \forall x (A(x) \vee B(x)) \\ \hline \exists x \neg A(x) \\ \hline \end{array} \right. \exists x B(x)$$

$$q) \left| \begin{array}{l} \forall x (A(x) \leftrightarrow B(x)) \\ \hline \exists x \neg A(x) \\ \hline \end{array} \right. \exists x \neg B(x)$$

$$r) \left| \begin{array}{l} \forall x (A(x) \leftrightarrow B(x)) \\ \hline \exists x A(x) \wedge \exists x \neg B(x) \\ \hline \end{array} \right. \exists x B(x) \wedge \exists x \neg A(x)$$

$$s) \left| \begin{array}{l} \forall x (A(x) \leftrightarrow B(x)) \\ \exists x \neg B(x) \\ \hline \neg \forall x A(x) \end{array} \right.$$

$$t)^* \left| \begin{array}{l} \neg \exists x A(x) \\ \hline \forall x \neg A(x) \end{array} \right.$$

$$u) \left| \begin{array}{l} \forall x \neg A(x) \\ \hline \neg \exists x A(x) \end{array} \right.$$

$$v)^* \left| \begin{array}{l} \exists x B(x) \\ \exists x A(x) \rightarrow \forall x \neg B(x) \\ \hline \forall x \neg A(x) \end{array} \right.$$

$$w)^* \left| \begin{array}{l} \forall x \exists y \neg A(x, y) \\ \hline \neg \exists x \forall y A(x, y) \end{array} \right.$$

$$x)^* \left| \begin{array}{l} \neg \exists x \forall y A(x, y) \\ \hline \forall x \exists y \neg A(x, y) \end{array} \right.$$

$$y)^* \left| \begin{array}{l} \exists x \forall y A(x, y) \\ \hline \neg \forall x \exists y \neg A(x, y) \end{array} \right.$$

Formella bevis för a) – y).

I flera fall lämnas det som övning att fylla i vilka bevisregler som har använts, samt referenser till använda formler/rader.

a)	1		$\forall x (P(x) \rightarrow Q(x))$	
	2		$P(c)$	
	3		$P(c) \rightarrow Q(c)$	$\forall\text{Elim}, 1$
	4		$Q(c)$	$\rightarrow\text{Elim}, 2, 3$
	5		$\exists x Q(x)$	$\exists\text{Intro}, 4$

b)	1		$\forall x \forall y (A(x, y) \vee B(x, y))$	
	2		$\neg A(c, d)$	
	3		$\forall y (A(c, y) \vee B(c, y))$	$\forall\text{Elim}, 1$
	4		$A(c, d) \vee B(c, d)$	$\forall\text{Elim}, 3$
	5		$A(c, d)$	
	6		$\perp$	$\perp\text{Intro}, 2, 5$
	7		$B(c, d)$	$\perp\text{Elim}, 6$
	8		$B(c, d)$	
	9		$\neg B(c, d)$	$\text{Reit}, 8$
	10		$B(c, d)$	$\vee\text{Elim}, 4, 5-7, 8-9$
	11		$\exists y B(c, y)$	$\exists\text{Intro}, 10$
	12		$\exists x \exists y B(x, y)$	$\exists\text{Intro}, 11$

$$\begin{array}{c|l}
 c) \ 1 & \forall x (P(x) \rightarrow Q(x)) \\
 2 & \forall x P(x) \\
 \hline
 3 & [a] \\
 4 & P(a) \rightarrow Q(a) \quad \forall \text{Elim}, 1 \\
 5 & P(a) \quad \forall \text{Elim}, 2 \\
 6 & Q(a) \quad \rightarrow \text{Elim}, 4, 5 \\
 7 & \forall x Q(x) \quad \forall \text{Intro}, 3-6.
 \end{array}$$

$$\begin{array}{c|l}
 d) \ 1 & \forall x (\neg Q(x) \rightarrow \neg P(x)) \\
 2 & \forall x P(x) \\
 \hline
 3 & [a] \\
 4 & \neg Q(a) \rightarrow \neg P(a) \quad \forall \text{Elim}, 1 \\
 5 & P(a) \quad \forall \text{Elim}, 2 \\
 6 & \neg Q(a) \\
 7 & \neg P(a) \quad \rightarrow \text{Elim}, 4, 6 \\
 8 & \perp \quad \perp \text{Intro}, 5, 7 \\
 9 & \neg \neg Q(a) \quad \neg \text{Intro}, 6-8 \\
 10 & Q(a) \quad \neg \text{Elim}, 9 \\
 11 & \forall x Q(x) \quad \forall \text{Intro}, 3-10.
 \end{array}$$

$$\begin{array}{l}
 e) \quad \forall x (P(x) \rightarrow R(x)) \\
 \quad \forall x (P(x) \vee Q(x)) \\
 \quad \boxed{a} \\
 \quad P(a) \rightarrow R(a) \\
 \quad P(a) \vee Q(a) \\
 \quad | P(a) \\
 \quad | R(a) \\
 \quad | Q(a) \vee R(a) \\
 \quad | Q(a) \\
 \quad | Q(a) \vee R(a) \\
 \quad Q(a) \vee R(a) \\
 \forall x (Q(x) \vee R(x))
 \end{array}$$

$$\begin{array}{ll}
 f) \quad 1 & \forall x (P(x) \vee Q(x)) \\
 2 & \boxed{a} \\
 3 & \neg P(a) \\
 4 & P(a) \vee Q(a) \quad \forall\text{Elim}, 1 \\
 5 & | P(a) \\
 6 & \perp \quad \perp\text{Intro}, 3, 5 \\
 7 & | Q(a) \quad \perp\text{Elim}, 6 \\
 8 & | Q(a) \\
 9 & | Q(a) \quad \text{Reit}, 8 \\
 10 & Q(a) \quad \vee\text{Elim}, 4, 5-7, 8-9 \\
 11 & \neg P(a) \rightarrow Q(a) \quad \rightarrow\text{Intro}, 3-10 \\
 12 & \forall x (\neg P(x) \rightarrow Q(x)) \quad \forall\text{Intro}, 2-11.
 \end{array}$$

g)

1		$\forall x \forall y A(x, y)$	
2		$[c]$	
3		$[d]$	
4		$\forall y A(c, y)$	$\forall E \text{ line } 1$
5		$A(c, d)$	$\forall E \text{ line } 4$
6		$\forall x A(c, x)$	$\forall \text{Intro, } 3-5$
7		$\forall y \forall x A(y, x)$	$\forall \text{Intro, } 2-6$

h)

1		$\forall x \forall y A(x, y)$	
2		$[c]$	
3		$[d]$	
4		$\forall y A(d, y)$	$\forall E \text{ line } 1$
5		$A(d, c)$	$\forall E \text{ line } 4$
6		$\forall x A(x, c)$	$\forall \text{Intro, } 3-5$
7		$\forall y \forall x A(x, y)$	$\forall \text{Intro, } 2-6$

mer



$$i) \quad \forall x \forall y ((P(x) \wedge Q(y)) \rightarrow R(x, y))$$

$$\forall x \neg R(x, x)$$

[a]

$$P(a) \wedge Q(a)$$

$$\forall y ((P(a) \wedge Q(y)) \rightarrow R(a, y))$$

$$(P(a) \wedge Q(a)) \rightarrow R(a, a)$$

$$\neg R(a, a)$$

$\forall E_{lim}$

$$R(a, a)$$

$\rightarrow E_{lim}$

$\perp$

$$\neg (P(a) \wedge Q(a))$$

$$\forall x \neg (P(x) \wedge Q(x))$$

j)

$$\forall x \forall y (R(x, y) \rightarrow P(x))$$

$$\forall x (P(x) \leftrightarrow Q(x))$$

$$\forall x \neg Q(x)$$

[a]

[b]

$$R(a, b)$$

$$\forall y (R(a, y) \rightarrow P(a))$$

$\forall E_{lim}$

$$R(a, b) \rightarrow P(a)$$

$\forall E_{lim}$

$$P(a) \leftrightarrow Q(a)$$

$\forall E_{lim}$

$$P(a)$$

$$Q(a)$$

$$\neg Q(a)$$

$\forall E_{lim}$

$\perp$

$$\neg R(a, b)$$

$$\forall y \neg R(a, y)$$

$$\forall x \forall y \neg R(x, y)$$

k)

$$\begin{array}{l}
 \neg \forall x A(x) \\
 \neg \exists x \neg A(x) \\
 \boxed{c} \\
 \neg A(c) \\
 \exists x \neg A(x) \\
 \perp \\
 \neg \neg A(c) \\
 A(c) \\
 \forall x A(x) \quad \forall \text{Intro} \\
 \perp \\
 \neg \neg \exists x \neg A(x) \\
 \exists x \neg A(x)
 \end{array}$$

l)

$$\begin{array}{l}
 \exists x \neg A(x) \\
 \forall x A(x) \\
 \boxed{c} \neg A(c) \\
 A(c) \\
 \perp \\
 \neg \forall x A(x) \\
 \neg \forall x A(x) \quad \exists \text{Elim} \\
 \perp \\
 \neg \forall x A(x)
 \end{array}$$

m)

$$\begin{array}{l}
 \neg \forall x \neg A(x) \\
 \neg \exists x A(x) \\
 \boxed{c} \\
 A(c) \\
 \exists x A(x) \\
 \perp \\
 \neg A(c) \\
 \forall x \neg A(x) \quad \forall \text{Intro} \\
 \perp \\
 \neg \neg \exists x A(x) \\
 \exists x A(x)
 \end{array}$$

n)

$$\begin{array}{l}
 \forall x (A(x) \rightarrow B(x)) \\
 \exists x A(x) \\
 \boxed{c} A(c) \\
 A(c) \rightarrow B(c) \\
 B(c) \\
 \exists x B(x) \\
 \exists x B(x) \quad \exists \text{Elim}
 \end{array}$$

o)

$$\begin{array}{l}
 \forall x A(x) \\
 \exists x (A(x) \rightarrow B(x)) \\
 \boxed{c} A(c) \rightarrow B(c) \\
 A(c) \\
 B(c) \\
 \exists x B(x) \\
 \exists x B(x)
 \end{array}$$

P) $\forall x (A(x) \vee B(x))$
 $\exists x \neg A(x)$

$\boxed{c} \neg A(c)$
 $A(c) \vee B(c)$
 $A(c)$
 $\perp$
 $\exists x B(x)$
 $B(c)$
 $\exists x B(x)$
 $\exists x B(x) \quad \vee E \text{lim}$
 $\exists x B(x) \quad \exists E \text{lim}$

q) $\forall x (A(x) \leftrightarrow B(x))$
 $\exists x \neg A(x)$

$\boxed{c} \neg A(c)$
 $A(c) \leftrightarrow B(c)$
 $B(c)$
 $A(c)$
 $\perp$
 $\neg B(c)$
 $\exists x \neg B(x)$
 $\exists x \neg B(x) \quad \exists E \text{lim}$

r) $\forall x (A(x) \leftrightarrow B(x))$
 $\exists x A(x) \wedge \exists x \neg B(x)$

$\exists x A(x)$
 $\exists x \neg B(x)$
 $\boxed{c} A(c)$
 $A(c) \leftrightarrow B(c)$
 $B(c)$
 $\exists x B(x)$
 $\exists x B(x)$
 $\boxed{d} \neg B(d)$
 $A(d) \leftrightarrow B(d)$
 $A(d)$
 $B(d)$
 $\perp$
 $\neg A(d)$
 $\exists x \neg A(d)$
 $\exists x \neg A(x)$
 $\exists x B(x) \wedge \exists x \neg A(x)$

s) $\forall x (A(x) \leftrightarrow B(x))$
 $\exists x \neg B(x)$

$\forall x A(x)$
 $\boxed{c} \neg B(c)$
 $A(c) \leftrightarrow B(c)$
 $A(c)$
 $B(c)$
 $\perp$
 $\perp \quad \exists E \text{lim}$
 $\neg \forall x A(x)$

t) $\neg \exists x A(x)$
 $\neg \forall x \neg A(x)$

$\boxed{c}$
 $A(c)$
 $\exists x A(x)$
 $\perp$
 $\neg A(c)$
 $\forall x \neg A(x)$
 $\perp$
 $\neg \neg \forall x \neg A(x)$
 $\forall x \neg A(x)$

u) $\forall x \neg A(x)$
 $\neg \exists x A(x)$
 $\boxed{c} A(c)$
 $\neg A(c)$
 $\perp$
 $\perp \quad \exists E \text{lim}$
 $\neg \exists x A(x)$

v) $\exists x B(x)$
 $\exists x A(x) \rightarrow \forall x \neg B(x)$
 $\boxed{c}$
 $A(c)$
 $\exists x A(x)$
 $\forall x \neg B(x)$
 $\boxed{d} B(d)$
 $\neg B(d) \quad \forall E \text{lim}$
 $\perp$
 $\perp \quad \exists E \text{lim}$
 $\neg A(c)$
 $\forall x \neg A(x)$

w) $\forall x \exists y \neg A(x, y)$
 $\neg \exists x \forall y A(x, y)$
 $\boxed{c} \forall y A(c, y)$
 $\exists y \neg A(c, y) \quad \forall E \text{lim}$
 $\boxed{d} \neg A(c, d)$
 $A(c, d) \quad \forall E \text{lim}$
 $\perp$
 $\perp \quad \exists E \text{lim}$
 $\perp \quad \exists E \text{lim}$
 $\neg \exists x \forall y A(x, y)$

x) $\neg \exists x \forall y A(x, y)$

[c]

$\forall y A(c, y)$

$\exists x \forall y A(x, y)$

$\perp$

$\neg \forall y A(c, y)$

$\neg \exists y \neg A(c, y)$

[d]

$\neg A(c, d)$

$\exists y \neg A(c, y)$

$\perp$

$\neg \neg A(c, d)$

$A(c, d)$

$\forall y A(c, y)$

$\perp$

$\neg \neg \exists y \neg A(c, y)$

$\exists y \neg A(c, y)$

$\forall x \exists y \neg A(x, y)$

y) $\exists x \forall y A(x, y)$

$\forall x \exists y \neg A(x, y)$

[c] $\forall y A(c, y)$

$\exists y \neg A(c, y)$

[d] $\neg A(c, d)$

$A(c, d)$

$\perp$

$\perp$ $\exists E$ elim

$\perp$ $\exists E$ elim (på premissen)

$\neg \forall x \exists y \neg A(x, y)$

som beviset av k)
med ' $\forall y A(c, y)$ ',
och ' $\exists y \neg A(c, y)$ ',
i stället för ' $\neg \forall x A(x)$ ',
och ' $\exists x \neg A(x)$ '.